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Global Finite-time Stabilization for Stochastic Nonlinear Systems via Output-feedback ⋆

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Abstract: In this paper, global finite-time stabilization via output-feedback is investigated for a class of stochastic nonlinear systems. By introducing a homogeneous observer, we design an output-feedback control law by adding one power integrator technique and homogeneous domination approach. Based on stochastic finite-time stability theorem, it is proved that the closed-loop system is globally finite-time stable in probability. Moreover, a simulation example is presented to demonstrate the effectiveness of the proposed design procedure.

Keywords: Stochastic nonlinear systems, adding one power integrator, output-feedback, homogeneous domination, finite-time stability

1. INTRODUCTION

In this paper, we consider the problem of finite-time stabilization via output-feedback for a class of stochastic nonlinear systems described by

$$dx(t) = Ax(t)dt + Bu(t)dt + f(x(t))dt + g^T(x(t))d\omega,$$
$$y(t) = Cx(t) \tag{1}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = [\, 1 \;\; 0 \;\; \cdots \;\; 0 \,],$$

$$f(x(t)) = \begin{bmatrix} f_1(\bar{x}_1(t)) \\ f_2(\bar{x}_2(t)) \\ \vdots \\ f_{n-1}(\bar{x}_{n-1}(t)) \\ f_n(\bar{x}_n(t)) \end{bmatrix},$$

$$g(x(t)) = [g_1(\bar{x}_1(t)),\ g_2(\bar{x}_2(t)),\ \cdots,\ g_n(\bar{x}_n(t))] \tag{2}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T \in \mathbb{R}^n$ is the system states, $u(t), y(t) \in \mathbb{R}$ are the control input and output, respectively. $\bar{x}_i(t) = (x_1(t), \cdots, x_i(t))$, $i = 1, \cdots, n$, are the state vectors. $\omega(t)$ is an $r-$dimensional standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with Ω being a sample space, $\mathcal{F}$ being a $\sigma-$field, $\mathcal{F}_t$ being a filtration and P being a probability measure. The drift terms $f_i(\cdot) : \mathbb{R}^i \to \mathbb{R}$ and the diffusion terms $g_i(\cdot) : \mathbb{R}^i \to \mathbb{R}^r$, $i = 1, \cdots, n$, are Borel measurable, continuous in system states and satisfy $f_i(0) = 0$ and $g_i(0) = 0$.

In deterministic systems, finite-time stability and stabilization have been an acute subject of research recently, since finite-time stable systems might have not only faster convergence but also better robustness and disturbance rejection properties. Haimo gave a sufficient condition for finite-time stability of continuous systems in Haimo (1986). Based on the Lyapunov finite-time stability theorem proposed in Bhat and Bernstein (2000), some conditions for finite-time stability have been presented in Moulay and Perruquetti (2006, 2008) and several problems of finite-time stabilization have been discussed in Hong et al. (2001); Li and Qian (2005); Zhai and Qian (2012); Zhai (2014); Zhai et al. (2013). Specifically, the finite-time stabilization for the double integrator systems was achieved by coupling a finite-time convergent observer with a finite-time control law in Hong et al. (2001). By adopting homogeneous domination approach, the work Li and Qian (2005) has constructed an output-feedback controller to render the closed-loop system globally finite-time stable.

On the other hand, stochastic modeling has come to play an important role in many branches of science and industry. Due to the existence of the Hessian term, the control problem for stochastic nonlinear systems is more complex. In the literature, there exist several results on global output-feedback stabilization in probability for s-tochastic nonlinear systems, for example, Deng and Krstić (1999); Zhai (2013); Zha et al. (2014) and the references therein. However, the above-mentioned results require that the closed-loop system satisfy the local Lipschitz condition

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in order to achieve asymptotic stabilization. According to Mao (1997), it is known that only if at least one coefficient does not satisfy the local Lipschitz condition, it is possible to consider the finite-time stability in probability for a class of stochastic nonlinear systems. Recently, based on the stochastic finite-time stability theorem in Yin et al. (2011), the work Khoo et al. (2013) has designed a continuous finite-time *state-feedback* controller for stochastic nonlinear systems in strict-feedback form.

Motivated by Li and Qian (2005) and Khoo et al. (2013), we consider the finite-time stabilization problem for a class of stochastic nonlinear systems via *output-feedback*. To tackle this problem, we first adopt adding one power integrator technique and homogeneous domination approach to design an output-feedback controller. Then, a theorem is presented to analyze the existence and the finite-time stability in probability of the solution to the stochastic nonlinear systems.

Notations: $\mathbb{R}_+$ denotes the set of all nonnegative real numbers, and $\mathbb{R}^n$ denotes the real $n-$dimensional space. $\mathbb{R}^+_{odd} =: \{q \in \mathbb{R} : q \geq 0$ is a ratio of two odd integers$\}$. For a given vector or matrix X, X^T represents its transpose; $Tr\{X\}$ represents its trace when X is square; $\|\cdot\|$ denotes the Euclidean norm of a vector X or the Frobenius norm of a matrix X. $\mathcal{C}^i$ denotes the set of all functions with continuous ith partial derivatives; $\mathcal{K}$ denotes the set of all functions, $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanishing at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded; $a \wedge b$ means the minimum of a and b.

2. PRELIMINARY RESULTS

Consider the following stochastic nonlinear system

$$dx(t) = f(x(t))dt + g^T(x(t))d\omega(t), \quad x(0) = x_0 \in \mathbb{R}^n \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the system state and $\omega(t)$ is an $r-$dimensional standard Wiener process defined on a probability space (Ω, F, F_t, P). The Borel measurable functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g^T : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are continuous in x that satisfy $f(0) = 0$ and $g(0) = 0$. In what follows, some useful definitions and lemmas are presented which play very important roles in this paper.

Lemma 1. (Skorokhod (1965)) Suppose that, $f(x(t))$ and $g(x(t))$ are continuous with respect to their variables and satisfy the linear growth condition:

$$\|f(x(t))\|^2 + \|g(x(t))\|^2 \leq K(1 + \|x(t)\|^2)$$

for $K > 0$. Then given any x_0 independent of $\omega(t)$, (3) has a continuous solution with probability one.

Definition 2. (Khoo et al. (2013)) The trivial solution of (3) is said to be finite-time stable in probability if the solution exists for any initial value $x_0 \in \mathbb{R}^n$, denoted by $x(t; x_0)$. Moreover, the following statements hold:
(i) Finite-time attractiveness in probability: For every initial value $x_0 \in \mathbb{R}^n\backslash\{0\}$, the first hitting time $\tau_{x_0} = \inf\{t; x(t; x_0) = 0\}$, which is called the stochastic settling time, is finite almost surely, that is, $P\{\tau_{x_0} < \infty\} = 1$;
(ii) Stability in probability: For every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r) > 0$ such that $P\{\|x(t; x_0)\| < r, \forall t \geq 0\} \geq 1 - \varepsilon$, whenever $\|x_0\| < \delta$;
(iii) The solution $x((t + \tau_{x_0}); x_0)$ is unique for $t \geq 0$.

Definition 3. (Florchinger (1995)) For any given $V(x(t)) \in \mathcal{C}^2$ associated with stochastic system (3), the infinitesimal generator $\mathcal{L}$ is defined as $\mathcal{L}V(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2}Tr\{g(x)\frac{\partial^2 V}{\partial x^2} g^T(x)\}$, where $\frac{1}{2}Tr\{g(x)\frac{\partial^2 V}{\partial x^2} g^T(x)\}$ is called as the Hessian term of $\mathcal{L}$.

Lemma 4. (Khoo et al. (2013)) For system (3), if there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$, $\mathcal{K}_\infty$ class functions μ_1 and μ_2, positive real numbers $c > 0$ and $0 < \gamma < 1$, such that for all $x \in \mathbb{R}^n$ and $t \geq 0$,

$$\begin{aligned} \mu_1(\|x\|) \leq V(x) \leq \mu_2(\|x\|), \\ \mathcal{L}V(x) \leq -c \cdot (V(x))^\gamma \end{aligned} \quad (4)$$

then the trivial solution of (3) is finite-time attractive and stable in probability.

Lemma 5. For $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $p \geq 1$, then

$$|x + y|^p \leq 2^{p-1}|x^p + y^p|,$$
$$\big(|x| + |y|\big)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}\big(|x| + |y|\big)^{\frac{1}{p}}.$$

Moreover, if $p \geq 1$ is an odd integer or a ratio of two odd integers, then

$$|x - y|^p \leq 2^{p-1}|x^p - y^p|,$$
$$|x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leq 2^{1-\frac{1}{p}}|x - y|^{\frac{1}{p}}.$$

Lemma 6. Suppose c and d are two positive real numbers. Given any positive number $\gamma > 0$, then

$$|x|^c|y|^d \leq \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}.$$

3. MAIN RESULTS

In this section, an output-feedback controller is designed for system (1) under the following assumption.

Assumption 7. There exist two positive constants a_1 and a_2 such that for $i = 1, \cdots, n$

$$\begin{aligned} |f_i(\bar{x}_i)| &\leq a_1(|x_1|^{\frac{r_i+\tau}{r_1}} + \cdots + |x_i|^{\frac{r_i+\tau}{r_i}}), \\ \|g_i(\bar{x}_i)\| &\leq a_2(|x_1|^{\frac{2r_i+\tau}{2r_1}} + \cdots + |x_i|^{\frac{2r_i+\tau}{2r_i}}) \end{aligned} \quad (5)$$

with $\tau \in (-\frac{1}{n}, 0)$ and a series of parameters

$$r_i = 1 + (i-1)\tau, \quad i = 1, \cdots, n+1. \quad (6)$$

For simplicity, we assume $\tau = -p/q$ with p being an even integer and q being an odd integer. Based on this, r_i will be odd in both denominator and numerator.

Remark 8. In deterministic cases, the works Li and Qian (2005); Zhai and Qian (2012) have dealt with the finite-time output-feedback stabilization problem for a class of nonlinear systems whose nonlinearities satisfy Assumption 7 with $a_2 = 0$. Taking stochastic factors into consideration, we impose lower-triangular homogeneous growth conditions on diffusion terms as well, and therefore, Assumption 7 is a more general condition. However, the appearance of Hessian term will bring much more nonlinearities. In what follows, we will present how to handle these terms skillfully.

We can divide the design procedure into two steps. First of all, by adopting adding one power integrator technique, we can design an output-feedback controller for the nominal system without perturbing nonlinearities. Next, based on homogeneous domination approach, a scaled output-feedback controller is constructed for system (1) and the stability analysis is given to show the finite-time stabilization in probability of the closed-loop system.

3.1 Output-feedback Controller Design

In this subsection, an output-feedback stabilizer is designed for the following nominal system in an iterative way:

$$dz = Azdt + Bvdt,$$
$$y = Cz \tag{7}$$

where the matrices A, B, C are defined in (2), $z = (z_1, \cdots, z_n)^T \in \mathbb{R}^n$ and $v \in \mathbb{R}$ are the system states and control input, respectively.

Initial Step: Define $\xi_1 = z_1^{\mu/r_1}$, $\mu \geq 2\max_{1\leq j\leq n}\{r_j\}$, $\mu \in \mathbb{R}^+_{odd}$, and choose the Lyapunov function $V_1(z_1) = \frac{r_1}{4\mu} z_1^{4\mu/r_1}$. According to Definition 3, the infinitesimal generator $\mathcal{L}$ of V_1 along the trajectory of (7) is

$$\mathcal{L}V_1(z_1) = z_1^{\frac{4\mu - r_1}{r_1}} z_2 = \xi_1^{\frac{4\mu - r_1}{\mu}} z_2. \tag{8}$$

Then, with the virtual controller defined as $z_2^* = -\beta_1 \xi_1^{r_2/\mu}$, $\beta_1 \geq n$, it yields

$$\mathcal{L}V_1(z_1) \leq -n\xi_1^{\frac{4\mu+\tau}{\mu}} + \xi_1^{\frac{4\mu - r_1}{\mu}}(z_2 - z_2^*). \tag{9}$$

Inductive Step: Suppose at step $k-1$, there exists a Lyapunov function $V_{k-1}(z_1, \cdots, z_{k-1}) : \mathbb{R}^{k-1} \to \mathbb{R}$, and a set of virtual controllers $z_1^*, \cdots, z_k^*$ defined by

$$\begin{aligned} z_1^* &= 0, & \xi_1 &= z_1^{\mu/r_1} - z_1^{*\mu/r_1}, \\ z_i^* &= -\beta_{i-1}\xi_{i-1}^{r_i/\mu}, & \xi_i &= z_i^{\mu/r_i} - z_i^{*\mu/r_i}, \quad i = 2, \cdots, k \end{aligned} \tag{10}$$

where $\beta_1, \cdots, \beta_{k-1}$ are positive constants, such that

$$\begin{aligned} \mathcal{L}V_{k-1} \leq &-(n-k+2)(\xi_1^{\frac{4\mu+\tau}{\mu}} + \cdots + \xi_{k-1}^{\frac{4\mu+\tau}{\mu}}) \\ &+ \xi_{k-1}^{\frac{4\mu - r_{k-1}}{\mu}}(z_k - z_k^*). \end{aligned} \tag{11}$$

Next, we will claim that (11) also holds at step k. To prove this, one can construct the kth Lyapunov function $V_k(z_1, \cdots, z_k) : \mathbb{R}^k \to \mathbb{R}$, as

$$V_k(z_1, \cdots, z_k) = V_{k-1}(z_1, \cdots, z_{k-1}) + W_k(z_1, \cdots, z_k) \tag{12}$$

with $W_k(z_1, \cdots, z_k) = \int_{z_k^*}^{z_k} (s^{\mu/r_k} - z_k^{*\mu/r_k})^{\frac{4\mu - r_k}{\mu}} ds$. The infinitesimal generator $\mathcal{L}$ of V_k along the trajectory of (7) is

$$\begin{aligned} \mathcal{L}V_k \leq &-(n-k+2)\sum_{i=1}^{k-1}\xi_i^{\frac{4\mu+\tau}{\mu}} + \xi_{k-1}^{\frac{4\mu - r_{k-1}}{\mu}}(z_k - z_k^*) \\ &+ \xi_k^{\frac{4\mu - r_k}{\mu}} z_{k+1} + \sum_{i=1}^{k-1}\frac{\partial W_k}{\partial z_i} z_{i+1} \\ \leq &-(n-k+2)\sum_{i=1}^{k-1}\xi_i^{\frac{4\mu+\tau}{\mu}} + \xi_{k-1}^{\frac{4\mu - r_{k-1}}{\mu}}(z_k - z_k^*) \\ &+ \sum_{i=1}^{k-1}\frac{\partial W_k}{\partial z_i} z_{i+1} + \xi_k^{\frac{4\mu - r_k}{\mu}} z_{k+1}^* \\ &+ \xi_k^{\frac{4\mu - r_k}{\mu}}(z_{k+1} - z_{k+1}^*). \end{aligned} \tag{13}$$

It follows from Lemmas 5 and 6 that

$$\begin{aligned} |\xi_{k-1}^{\frac{4\mu - r_{k-1}}{\mu}}(z_k - z_k^*)| &\leq 2^{1-\frac{r_k}{\mu}}|\xi_{k-1}|^{\frac{4\mu - r_{k-1}}{\mu}}|\xi_k|^{\frac{r_k}{\mu}} \\ &\leq \frac{1}{2}\xi_{k-1}^{\frac{4\mu+\tau}{\mu}} + c_1\xi_k^{\frac{4\mu+\tau}{\mu}} \end{aligned} \tag{14}$$

where c_1 is a positive constant. To estimate the third term in (13), the following proposition is introduced whose proof is similar to the one in Li and Qian (2005) and therefore is omitted here.

Proposition 9. There exists a positive constant c_2 such that

$$\Big|\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial z_i} z_{i+1}\Big| \leq \sum_{i=1}^{k-2}\xi_i^{\frac{4\mu+\tau}{\mu}} + \frac{1}{2}\xi_{k-1}^{\frac{4\mu+\tau}{\mu}} + c_2\xi_k^{\frac{4\mu+\tau}{\mu}}. \tag{15}$$

Substituting (14) and (15) into (13), it yields

$$\begin{aligned} \mathcal{L}V_k \leq &-(n-k+1)\sum_{i=1}^{k}\xi_i^{\frac{4\mu+\tau}{\mu}} + (c_1 + c_2)\xi_k^{\frac{4\mu+\tau}{\mu}} \\ &+ \xi_k^{\frac{4\mu - r_k}{\mu}}(z_{k+1} - z_{k+1}^*) + \xi_k^{\frac{4\mu - r_k}{\mu}} z_{k+1}^*. \end{aligned} \tag{16}$$

Obviously, the virtual controller of the form

$$z_{k+1}^* = -\beta_k \xi_k^{r_{k+1}/\mu}, \ \beta_k \geq n-k+1+c_1+c_2 \tag{17}$$

leads to

$$\mathcal{L}V_k \leq -(n-k+1)\sum_{i=1}^{k}\xi_i^{\frac{4\mu+\tau}{\mu}} + \xi_k^{\frac{4\mu - r_k}{\mu}}(z_{k+1} - z_{k+1}^*). \tag{18}$$

This completes the inductive proof.

Last Step: Based on the inductive argument above, one can conclude that (18) holds for $k = n$ with a series of virtual controllers defined as (10). Hence, by choosing the Lyapunov function $V_n(z)$

$$\begin{aligned} V_n(z) &= V_{n-1}(z_1, \cdots, z_{n-1}) + W_n(z), \\ W_n(z) &= \int_{z_n^*}^{z_n}(s^{\mu/r_n} - z_n^{*\mu/r_n})^{\frac{4\mu - r_n}{\mu}} ds \end{aligned} \tag{19}$$

and a virtual controller

$$\begin{aligned} z_{n+1}^* = &-\beta_n\Big(z_n^{\frac{\mu}{r_n}} + \beta_{n-1}^{\frac{\mu}{r_n}}\big(z_{n-1}^{\frac{\mu}{r_{n-1}}} + \cdots + \beta_2^{\frac{\mu}{r_3}}(z_2^{\frac{\mu}{r_2}} \\ &+ \beta_1^{\frac{\mu}{r_2}} z_1^{\mu})\big)\Big)^{\frac{r_{n+1}}{\mu}} \end{aligned} \tag{20}$$

one can deduce that

$$\mathcal{L}V_n \leq -\sum_{k=1}^{n}\xi_k^{\frac{4\mu+\tau}{\mu}} + \xi_n^{\frac{4\mu - r_n}{\mu}}(v - z_{n+1}^*) \tag{21}$$

with a positive constant β_n.

Since $z_2, \cdots, z_n$ are unmeasurable, a homogeneous observer is constructed for system (7), which is generalized in the following Lemma.

Lemma 10. For system (7), there exist positive constants $l_1, \cdots, l_{n-1}$, such that the following homogeneous output-feedback stabilizer

$$\begin{aligned} \dot{\hat{\eta}}_k &= -l_{k-1}\hat{z}_k, \\ \hat{z}_k &= (\hat{\eta}_k + l_{k-1}\hat{z}_{k-1})^{\frac{r_k}{r_{k-1}}}, \ k = 2, \cdots, n, \end{aligned} \tag{22}$$

$$\begin{aligned} v(\hat{z}) = &-\beta_n\Big(\hat{z}_n^{\frac{\mu}{r_n}} + \beta_{n-1}^{\frac{\mu}{r_n}}\big(\hat{z}_{n-1}^{\frac{\mu}{r_{n-1}}} + \cdots + \beta_2^{\frac{\mu}{r_3}}(\hat{z}_2^{\frac{\mu}{r_2}} \\ &+ \beta_1^{\frac{\mu}{r_2}} z_1^{\mu})\big)\Big)^{\frac{r_{n+1}}{\mu}} \end{aligned} \tag{23}$$

with $\hat{z}_1 = z_1$ and $\hat{z} = (z_1, \hat{z}_2, \cdots, \hat{z}_n)^T$, renders the closed-loop system (7)-(22)-(23) globally finite-time stable.

Proof: For $k = 2, \cdots, n$, let $e_k = (z_k - \hat{z}_k)^{\frac{\mu}{r_k}}$ and choose

$$T_k = \int_{\gamma_k^{(4\mu-r_{k-1})/r_{k-1}}}^{z_k^{(4\mu-r_{k-1})/r_k}} \left(s^{\frac{r_{k-1}}{4\mu-r_{k-1}}} - \gamma_k\right) ds \quad (24)$$

where $\gamma_k = \hat{\eta}_k + l_{k-1}z_{k-1}$. Hence, along the trajectory of (7)-(22)-(23), one has

$$\begin{aligned}\mathcal{L}T_k =& \frac{\partial T_k}{\partial z_{k-1}} z_k + \frac{\partial T_k}{\partial z_k} z_{k+1} - \frac{\partial T_k}{\partial \hat{\eta}_k} l_{k-1}\hat{z}_k \\ =& \frac{4\mu - r_{k-1}}{r_k} z_k^{\frac{4\mu-r_{k-1}-r_k}{r_k}} \left(z_k^{\frac{r_{k-1}}{r_k}} - \gamma_k\right) z_{k+1} \\ & - l_{k-1} e_k^{\frac{r_k}{\mu}} \left(z_k^{\frac{4\mu-r_{k-1}}{r_k}} - \hat{z}_k^{\frac{4\mu-r_{k-1}}{r_k}}\right) \\ & - l_{k-1} e_k^{\frac{r_k}{\mu}} \left(\hat{z}_k^{\frac{4\mu-r_{k-1}}{r_k}} - \gamma_k^{\frac{4\mu-r_{k-1}}{r_{k-1}}}\right) \end{aligned} \quad (25)$$

where $z_{n+1} = v(\hat{z})$. By choosing a Lyapunov function $U = V_n + \sum_{k=2}^n T_k$, one can obtain the global finite-time stabilization result for (7), whose proof is similar to the one introduced in Zha et al. (2014) with some modifications. For the sake of space, the detailed proof is omitted here.

From the construction of U, it can be verified that U is positive definite and proper with respect to $Z := (z_1, \cdots, z_n, \hat{\eta}_2, \cdots, \hat{\eta}_n)^T$. First, we consider the following two cases to prove that W_k is positive definite and proper, where $V_1 = W_1 = \int_0^{z_1} s^{4\mu-1} ds$.

Case 1: If $z_k^* \le z_k$, with $\mu \ge 2\max_{1\le i\le n}\{r_i\}$, $\mu \in \mathbb{R}^+_{odd}$ and by Lemma 5, one gets

$$\begin{aligned} W_k =& \int_{z_k^*}^{z_k} (s^{\mu/r_k} - z_k^{*\mu/r_k})^{\frac{4\mu-r_k}{\mu}} ds \\ \ge& (2^{1-\frac{\mu}{r_k}})^{\frac{4\mu-r_k}{\mu}} \int_{z_k^*}^{z_k} (s - z_k^*)^{\frac{4\mu-r_k}{r_k}} ds \\ =& (2^{1-\frac{\mu}{r_k}})^{\frac{4\mu-r_k}{\mu}} \frac{r_k}{4\mu} (z_k - z_k^*)^{\frac{4\mu}{r_k}}. \end{aligned} \quad (26)$$

Case 2: If $z_k^* \ge z_k$, (26) can be proved similarly.

Next, we aim to prove that T_k is positive definite and proper. With $\sigma = s^{\frac{r_{k-1}}{4\mu-r_{k-1}}}$, there exists a positive constant $c_0 \in [\underline{c}, \bar{c}]$, such that

$$\begin{aligned} V_k =& \int_{\gamma_k^{(4\mu-r_{k-1})/r_{k-1}}}^{z_k^{(4\mu-r_{k-1})/r_k}} \left(s^{\frac{r_{k-1}}{4\mu-r_{k-1}}} - \gamma_k\right) ds \\ =& \frac{4\mu - r_{k-1}}{r_{k-1}} \int_{\gamma_k}^{z_k^{r_{k-1}/r_k}} (\sigma - \gamma_k)\sigma^{\frac{4\mu-2r_{k-1}}{r_{k-1}}} d\sigma \\ =& \frac{4\mu - r_{k-1}}{r_{k-1}} c_0 \int_{\gamma_k}^{z_k^{r_{k-1}/r_k}} (\sigma - \gamma_k) d\sigma \\ =& \frac{4\mu - r_{k-1}}{2r_{k-1}} c_0 (z_k^{\frac{r_{k-1}}{r_k}} - \gamma_k)^2 \end{aligned} \quad (27)$$

with $\underline{c}$ and $\bar{c}$ represent the infimum and supremum of $\sigma^{\frac{4\mu-2r_{k-1}}{r_{k-1}}}$ in $[\gamma_k, z_k^{r_{k-1}/r_k}]$ (or $[z_k^{r_{k-1}/r_k}, \gamma_k]$, if $\gamma_k \ge z_k^{r_{k-1}/r_k}$), respectively.

Therefore, $U = \sum_{k=1}^n W_k + \sum_{k=2}^n T_k \ge \sum_{k=1}^n (2^{1-\frac{\mu}{r_k}})^{\frac{4\mu-r_k}{\mu}} \frac{r_k}{4\mu}(z_k - z_k^*)^{\frac{4\mu}{r_k}} + \sum_{k=2}^n \frac{4\mu-r_{k-1}}{2r_{k-1}} c_0 (z_k^{\frac{r_{k-1}}{r_k}} - \gamma_k)^2$ is definite positive and proper with respect to Z. Denoting the dilation weight

$$\Delta = (\underbrace{r_1, \cdots, r_n}_{for\ z_1,\cdots,z_n}, \underbrace{r_1, \cdots, r_{n-1}}_{for\ \hat{\eta}_2,\cdots,\hat{\eta}_n}), \quad (28)$$

the closed-loop system (7)-(22)-(23) which can be rewritten as

$$dZ = E(Z)dt = (z_2, \cdots, z_n, v(\hat{z}), \dot{\hat{\eta}}_2, \cdots, \dot{\hat{\eta}}_n)^T dt \quad (29)$$

is homogeneous of degree τ. Moreover, it can be shown that U is homogeneous of degree 4μ with respect to Δ, under which, there exist a positive constant α_1, such that

$$\frac{\partial U(Z)}{\partial Z} E(Z) \le -\alpha_1 \|Z\|_\Delta^{4\mu+\tau}. \quad (30)$$

3.2 Stability Analysis

Under the new coordinates

$$z_i = \frac{x_i}{L^{i-1}},\ i = 1, \cdots, n,\ v = \frac{u}{L^n} \quad (31)$$

with $L \ge 1$ a constant to be determined later, system (1) can be rewritten as

$$\begin{aligned} dz &= LAzdt + LBvdt + \bar{f}(z)dt + \bar{g}^T(z)d\omega, \\ y &= Cz \end{aligned} \quad (32)$$

with $\bar{f}(z) = (f_1(\cdot), \frac{f_2(\cdot)}{L}, \cdots, \frac{f_n(\cdot)}{L^{n-1}})^T$ and $\bar{g}(z) = (g_1(\cdot), \frac{g_2(\cdot)}{L}, \cdots, \frac{g_n(\cdot)}{L^{n-1}})$.

We construct an observer with the scaling gain L

$$\begin{aligned} &\dot{\hat{\eta}}_k = -Ll_{k-1}\hat{z}_k, \quad \hat{z}_k = (\hat{\eta}_k + l_{k-1}\hat{z}_{k-1})^{\frac{r_k}{r_{k-1}}}, \\ &k = 2, \cdots, n \end{aligned} \quad (33)$$

where l_k, $k = 1, \cdots, n-1$, are observer gains. With $E(Z)$ defined in (29), it is straightforward to verify that the closed-loop system (32)-(33)-(23) can be represented as

$$dZ = (LE(Z) + F(Z))dt + G^T(Z)d\omega \quad (34)$$

where $F(Z) = (f_1(\cdot), \frac{f_2(\cdot)}{L}, \cdots, \frac{f_n(\cdot)}{L^{n-1}}, 0, \cdots, 0)^T$ and $G(Z) = (g_1(\cdot), \frac{g_2(\cdot)}{L}, \cdots, \frac{g_n(\cdot)}{L^{n-1}}, 0, \cdots, 0)$.

Based on the observer and controller design, the following theorem gives the finite-time stabilization result of the closed-loop system (34) via output-feedback.

Theorem 11. Under Assumption 7, there exists an output-feedback controller rendering system (1) finite-time stable in probability.

Proof. According to Definition 2, the proof of Theorem 11 is divided into four steps.

Step 1: We consider the existence of the solution to the closed-loop system (34). The construction of the observer and controller indicates that the closed-loop system is continuous with respect to its variables. Since $\tau \in (-\frac{1}{n}, 0)$, $f_i(\cdot)$ and $g_i(\cdot)$ satisfy lower-order growth conditions. In addition, if $\|Z\| \ge 1$, with $0 < \frac{r_i+\tau}{r_j} < 1$, $j = 1, \cdots, i$, there exists a positive constant K_1 such that $\|F(Z)\|^2 \le K_1\|Z\|^2$. If $\|Z\| < 1$, one has $\|F(Z)\|^2 \le K_2$, with $K_2 \ge 0$. Therefore, by choosing $K_3 = \max\{K_1, K_2\}$, one has

$$\|F(Z)\|^2 \le K_3(1 + \|Z\|^2) \quad (35)$$

which implies that $F(Z)$ satisfies the linear growth condition. Similarly, one can obtain that there is a positive constant K such that

$$\|LE(Z) + F(Z)\|^2 + \|G(Z)\|^2 \le K(1 + \|Z\|^2). \quad (36)$$

By Lemma 1, there exists a continuous solution $Z(t)$ with probability one that can be written as

$$\begin{aligned}Z(t) =&Z_0 + \int_0^t (LE(Z(s)) + F(Z(s)))ds \\ &+ \int_0^t G^T(Z(s))d\omega(s) \end{aligned} \tag{37}$$

with the initial value Z_0.

Step 2: Under Assumption 7 and the new coordinates (31), it can be deduced that, for $i = 1, \cdots, n$

$$\begin{aligned}\Big|\frac{f_i(\cdot)}{L^{i-1}}\Big| \leq& \frac{a_1}{L^{i-1}}(|z_1|^{\frac{r_i+\tau}{r_1}} + |Lz_2|^{\frac{r_i+\tau}{r_2}} + \cdots + |L^{i-1}z_i|^{\frac{r_i+\tau}{r_i}}) \\ \leq& a_1 L^{1-\frac{1}{r_i}}(|z_1|^{\frac{r_i+\tau}{r_1}} + \cdots + |z_i|^{\frac{r_i+\tau}{r_i}}) \\ \leq& \bar{a}_1 L^{1-\frac{1}{r_i}} \|Z\|_\Delta^{r_i+\tau} \end{aligned} \tag{38}$$

with a constant $\bar{a}_1 \geq 0$. Recall that for $i = 1, \cdots, n$, $\frac{\partial U(Z)}{\partial Z_i}$ is homogeneous of degree $4\mu - r_i$, then

$$\begin{aligned}\Big|\frac{\partial U(Z)}{\partial Z}F(Z)\Big| \leq& \sum_{i=1}^n \Big|\frac{\partial U(Z)}{\partial Z_i}\Big|\Big|\frac{f_i(\cdot)}{L^{i-1}}\Big| \\ \leq& \sum_{i=1}^n L^{1-\frac{1}{r_i}}\rho_{1i}\|Z\|_\Delta^{4\mu+\tau} \\ \leq& \rho_1 \|Z\|_\Delta^{4\mu+\tau} \end{aligned} \tag{39}$$

where ρ_{1i}, $i = 1, \cdots, n$ and ρ_1 are positive constants. In a similar way, for $i = 1, \cdots, n$,

$$\begin{aligned}\Big\|\frac{g_i(\cdot)}{L^{i-1}}\Big\| \leq& a_2 L^{\frac{1}{2}-\frac{1}{2r_i}}(|z_1|^{\frac{2r_i+\tau}{2r_1}} + \cdots + |z_i|^{\frac{2r_i+\tau}{2r_i}}) \\ \leq& \bar{a}_2 L^{\frac{1}{2}-\frac{1}{2r_i}} \|Z\|_\Delta^{r_i+\frac{\tau}{2}} \end{aligned} \tag{40}$$

with $\bar{a}_2 \geq 0$, which indicates

$$\begin{aligned}&\frac{1}{2}Tr\Big\{G(Z)\frac{\partial^2 U(Z)}{\partial Z^2}G^T(Z)\Big\} \\ \leq& \frac{1}{2}r\sqrt{r}\sum_{i,j=1}^n \Big|\frac{\partial^2 U(Z)}{\partial Z_i \partial Z_j}\Big|\Big\|\frac{g_i^T(\cdot)}{L^{i-1}}\Big\|\Big\|\frac{g_j(\cdot)}{L^{j-1}}\Big\| \\ \leq& \rho_2 \|Z\|_\Delta^{4\mu+\tau} \end{aligned} \tag{41}$$

for $\rho_2 \geq 0$. According to Definition 3, the infinitesimal generator $\mathcal{L}$ of U along the trajectory of (34) is

$$\begin{aligned}\mathcal{L}U(Z) \leq& \frac{\partial U(Z)}{\partial Z}LE(Z) + \frac{\partial U(Z)}{\partial Z}F(Z) \\ &+ \frac{1}{2}Tr\Big\{G(Z)\frac{\partial^2 U(Z)}{\partial Z^2}G^T(Z)\Big\} \\ \leq& -(L\alpha_1 - \rho_1 - \rho_2)\|Z\|_\Delta^{4\mu+\tau}. \end{aligned} \tag{42}$$

By choosing $L > \max\{1, \frac{\rho_1+\rho_2}{\alpha_1}\}$, there exist positive constants α_2 and α_3, such that

$$\mathcal{L}U(Z) \leq -\alpha_2 \|Z\|_\Delta^{4\mu+\tau} \leq -\alpha_3 U^{\frac{4\mu+\tau}{4\mu}}. \tag{43}$$

By Lemma 4, it can be obtained that the solution of the closed-loop system (34) is finite-time attractive and stable in probability.

Step 3: In this step, we will prove that after the first hitting time τ_{Z_0}, the solution $Z(t + \tau_{Z_0})$ remains zero almost surely, $\forall t \geq 0$. Define the stopping time $\tau_m = \inf\{t \geq \tau_{Z_0}; \|Z(t; Z_0)\| \geq m, m > 0\}$. It is clear that τ_m is an increasing time sequence. Applying Itô's formula, one has, $\forall t \geq 0$

$$\begin{aligned}&U(Z((t + \tau_{Z_0}) \wedge \tau_m)) \\ =& U(Z(\tau_{Z_0} \wedge \tau_m)) + \int_{\tau_{Z_0}\wedge\tau_m}^{(t+\tau_{Z_0})\wedge\tau_m} \mathcal{L}U(Z(s))ds \\ &+ \int_{\tau_{Z_0}\wedge\tau_m}^{(t+\tau_{Z_0})\wedge\tau_m} \frac{\partial U(Z(s))}{\partial Z}G^T(Z(s))d\omega(s) \end{aligned} \tag{44}$$

which indicates

$$\begin{aligned}&EU(Z((t + \tau_{Z_0}) \wedge \tau_m)) \\ =& EU(Z(\tau_{Z_0})) + E\int_{\tau_{Z_0}}^{(t+\tau_{Z_0})\wedge\tau_m} \mathcal{L}U(Z(s))ds \leq 0. \end{aligned} \tag{45}$$

Since $U(Z)$ is positive definite, one can obtain

$$EU(Z((t + \tau_{Z_0}) \wedge \tau_m)) = 0 \tag{46}$$

which implies that

$$U(Z((t + \tau_{Z_0}) \wedge \tau_m)) = 0, \ \forall t \geq 0 \ a.s. \tag{47}$$

Letting $m \to +\infty$, we get $Z(t + \tau_{Z_0}) = 0$ almost surely, $\forall t \geq 0$. Therefore, the result of finite-time stability in probability for the closed-loop system (34) is achieved according to Definition 2.

Step 4: Since coordinate transformation (31) does not change the properties of the system, then the solution of stochastic nonlinear system (1) is finite-time stable in probability. □

4. AN ILLUSTRATIVE EXAMPLE

In what follows, we use an example to illustrate the effectiveness of the proposed output-feedback controller.

Example 12. Consider the following stochastic nonlinear system

$$\begin{aligned}dx_1 =& x_2 dt + \frac{1}{2}x_1^{\frac{9}{11}}\cos x_1 dt + \frac{1}{4}x_1^{\frac{10}{11}}d\omega, \\ dx_2 =& udt + \frac{1}{3}x_2^{\frac{7}{9}}dt + \frac{1}{5}x_1^{\frac{3}{11}}x_2^{\frac{5}{9}}d\omega. \end{aligned} \tag{48}$$

In the simulation, we choose $\tau = -2/11$, $r_1 = 1$, $r_2 = 9/11$ and $\mu = 3$. It is easy to verify that

$$\begin{aligned}&|f_1| \leq \frac{1}{2}|x_1|^{\frac{9}{11}}, \quad |f_2| \leq \frac{1}{2}|x_2|^{\frac{7}{9}}, \\ &\|g_1\| \leq \frac{1}{4}|x_1|^{\frac{10}{11}}, \quad \|g_2\| \leq \frac{1}{4}(|x_1|^{\frac{8}{11}} + |x_2|^{\frac{8}{9}}) \end{aligned}$$

satisfy Assumption 7 with $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{4}$. Therefore, according to Theorem 11, there exists an output-feedback controller rendering the closed-loop system finite-time stable in probability. Specifically, the output-feedback controller can be constructed as follows:

$$\begin{aligned}&\dot{\hat{\eta}}_2 = -Ll_1\hat{z}_2, \quad \hat{z}_2 = (\hat{\eta}_2 + l_1 z_1)^{\frac{9}{11}}, \\ &u = -L^2\beta_2(\hat{z}_2^{\frac{11}{3}} + \beta_1^{\frac{11}{3}}z_1^3)^{\frac{7}{33}} \end{aligned} \tag{49}$$

where $l_1 = 3.2$, $\beta_1 = 2$, $\beta_2 = 3$ and $L = 2$. With initial values $x_1(0) = 1$, $x_2(0) = -1.2$ and $\hat{\eta}_2(0) = 0$, Fig.1 demonstrates the finite-time stability in probability of the closed-loop system (48) and (49).

5. CONCLUSION

In this paper, the problem of global finite-time stabilization has been solved for a class of stochastic lower-triangular nonlinear systems via output-feedback. By employing adding one power integrator technique, homogeneous domination approach and stochastic finite-time

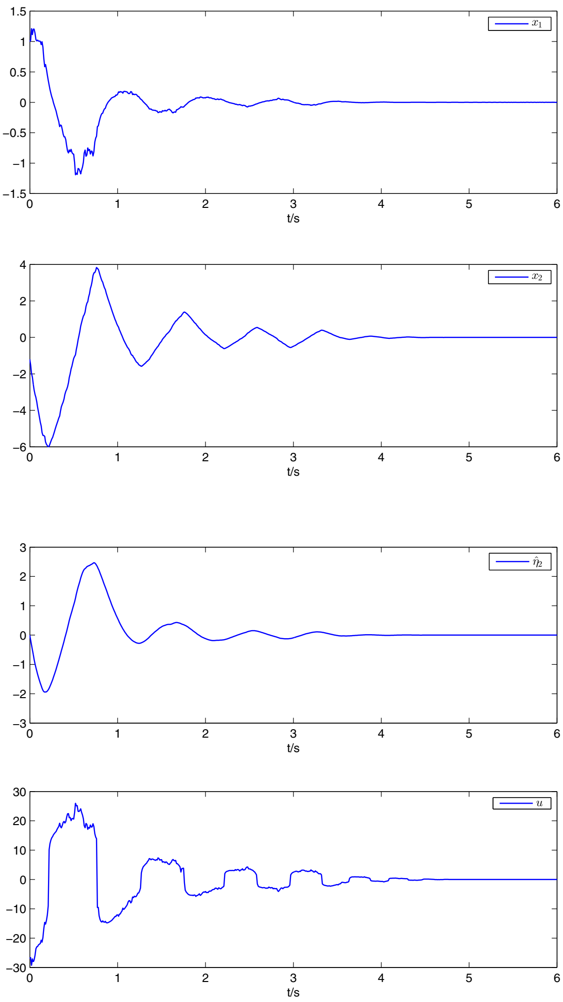

Fig. 1. The responses of the closed-loop system (48) and (49)

stability theorem, a systematic design method has been presented to ensure that the solution of the closed-loop system will converge to the origin in finite time and stay at the origin thereafter with probability one.

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