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Model matching problems for switching linear systems

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Abstract: This paper investigates the problem of designing a feedback compensator to force the response of a plant modeled by a switching linear system to match that of a prescribed, switching linear model, for any choice of the switching law. The problem is stated by considering both the situation in which the state of the model is measurable and that in which it is not. Accordingly, static compensators or, alternatively, dynamic ones will be sought. The additional requirement of asymptotic stability of the compensated system is introduced by reasonably restricting the class of admissible switching laws. Using geometric methods, that extend classic ones to the framework of switching systems, a complete solution, in terms of necessary and sufficient conditions that are algorithmically checkable, is given for matching without stability and for matching with asymptotic stability for a mildly restricted class of plants.

Keywords: Model matching problem; switching systems; geometric methods; stability and regulation.

1. INTRODUCTION

Given two dynamical systems with the same output space, respectively the model M and the plant P, the model matching problem, in a general formulation, consists in searching for a feedback compensator such that the forced response of the compensated plant equals that of the model. Stability and asymptotic matching of the global responses, for all initial conditions, can be viewed as additional requirements.

The model matching problem has been introduced in [21] in the early 70's and then studied by many authors, using different approaches, in several contexts, including, in particular, those of linear systems, nonlinear systems, time-delay systems (see in particular [20], [18], [12], [19], [6], [13], [7], [4]).

Here, we investigate the problem of matching, in a suitable sense, a given model, in the case in which both the model and the plant are switching systems. Namely, they are dynamical structures that consist of an indexed family $\Sigma = \{\Sigma_i\}_{i \in I}$, where I is a finite set, of linear systems, called modes, having the same input, output and state space, and of a supervisory law, described by a map $\sigma : \mathbb{R}^+ \to I$ and called switching rule, that defines the switching from one mode to another (see [10], [11], [17]).

Input/output behavior, free response and qualitative properties, like stability, of switching systems depend both on the modes and on the choice of the switching law.

Taking into account this aspect, the problems we deal with can be stated by requiring matching of the input/output behavior, or equivalently of the forced response, for any choice of the switching rule or by adding to this requirement, at least for switching rules in a restricted class, that of asymptotic stability of the compensated system, so to assure also asymptotic matching of the free responses for all initial conditions. In designing the compensator that achieves matching, one can then choose to consider static feedback if the state of the model is accessible, or dynamic feedback if it is not.

The approach we follow is based on tools and methods which allow extension of the classic geometric approach described in [2] and [22] to the framework of switching systems. The same approach has already been used for investigating decoupling problems and regulation problems involving switching systems in [15], [5], [16], [23], [24].

The results we obtain are represented, first, by a geometric, structural, necessary and sufficient condition for the existence of solutions to the model matching problem by static feedback, if the state of the model is accessible, or, alternatively, by dynamic feedback, if the state of the model is not accessible. Then, we give a necessary and sufficient condition for the existence of solutions to the model matching problem with asymptotic stability of the compensated system, for a restricted class of switching rules, by static and by dynamic feedback.

All conditions can be practically checked and solutions, if existing, can be practically constructed by means of geometric algorithms.

The paper is organized in the following way. In Section 2, we recall some notions and results of the geometric approach. In Section 3, we formally state the matching problems we consider and, in Section 4, we provide necessary and sufficient conditions for their solution. An illustrative example is presented in Section 5. Section 6 contains some concluding remarks.

2. PRELIMINARIES

Let $\mathbb{R}$ denote the field of real numbers. By a continuous-time switching linear system Σ_σ we mean a dynamical system defined by the equations

$$\Sigma_\sigma \equiv \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t) \end{cases} \quad (1)$$

where $t \in \mathbb{R}^+$ is the time variable, $x \in \mathcal{X} = \mathbb{R}^n$ is the state, $u \in \mathcal{U} = \mathbb{R}^m$ is the input, $y \in \mathcal{Y} = \mathbb{R}^p$ is the output, σ is a function that takes values in the set $I = \{1, \ldots, N\}$ and that is assumed to depend on time only, that is $\sigma : \mathbb{R}^+ \to I$, and, finally, for any value $i \in I$ taken by σ, A_i, B_i, C_i are matrices of suitable dimensions with real coefficients.
In other terms, a continuous-time switching system Σ_σ consists of an indexed family $\Sigma = \{\Sigma_i\}_{i \in I}$ of continuous-time, time-invariant, linear systems of the form

$$\Sigma_i \equiv \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases} \quad \text{for } i = 1, \ldots, N, \quad (2)$$

called *modes* of Σ_σ, and of a supervisory *switching rule* σ, whose value $\sigma(t)$ specifies the mode which is active at time t.
A standard assumption on σ is that it generates only a finite number of switches in any time interval of finite length, so to exclude chattering phenomena.
According to the envisaged applications, interest can be in studying properties of Σ_σ which hold for any choice of the switching rule σ, as well as in investigating the existence of (restricted classes of) switching rules that guarantee the fulfillment of specific requirements (see e.g. [11]).
Both structural and qualitative properties can, in particular, be investigated by extending to the class of switching systems tools and notion of the classic geometric approach ([2], [22]).
In order to follow this approach, let us recall a number of geometric notions and results that will be used in the sequel.
For a linear system Σ_i, defined by (2), a subspace $\mathcal{V}_i \subseteq \mathcal{X}$ is said to be a *controlled invariant subspace*, or an (A_i, B_i)-*invariant subspace*, if

$$A_i \mathcal{V}_i \subseteq \mathcal{V}_i + Im\ B_i.$$

Controlled invariance of $\mathcal{V}_i$ is equivalent to the existence of a feedback map, called *friend* of $\mathcal{V}_i$, $F_i : \mathcal{X} \to \mathcal{U}$ such that

$$(A_i + B_i F_i)\mathcal{V}_i \subseteq \mathcal{V}_i.$$

The set $V(A_i, B_i, \mathcal{K})$ of all controlled invariant subspaces for Σ_i contained in a given subspace $\mathcal{K} \subseteq \mathcal{X}$ has a maximum element that is denoted by $\mathcal{V}_i{}^*(\mathcal{K})$.
The notion of controlled invariant subspace is extended to the framework of switching system as follows.

Definition 1. ([1], [3]) Given a family $\Sigma = \{\Sigma_i\}_{i \in I}$ of linear systems of the form (2), a subspace $\mathcal{V}_R \subseteq \mathcal{X}$ is called a *robust controlled invariant subspace* for Σ if

$$A_i \mathcal{V}_R \subseteq \mathcal{V}_R + Im\ B_i \text{ for all } i = 1, \ldots, N.$$

If Σ is the family of the modes of a switching linear system Σ_σ of the form (1), any robust controlled invariant subspace $\mathcal{V}_R$ for Σ is said to be a *controlled invariant subspace* for Σ_σ.

Proposition 2. Given a family $\Sigma = \{\Sigma_i\}_{i \in I}$ of linear systems of the form (2), a subspace $\mathcal{V}_R \subseteq \mathcal{X}$ is a robust controlled invariant for Σ if and only if there exists an indexed family $F = \{F_i\}_{i \in I}$ of feedbacks $F_i : \mathcal{X} \to \mathcal{U}$, with $i \in I$, such that

$$(A_i + B_i F_i)\mathcal{V}_R \subseteq \mathcal{V}_R \text{ for all } i = 1, \ldots, N.$$

Any family F of that kind is called a *family of friends* of $\mathcal{V}$.

For any subspace $\mathcal{K} \subseteq \mathcal{X}$, the set $V_R(\mathcal{K})$ of all robust controlled invariant subspaces contained in $\mathcal{K}$ forms a semilattice with respect to inclusion and sum of subspaces, therefore $V_R(\mathcal{K})$ has a maximum element, denoted by $\mathcal{V}_R^*(\mathcal{K})$, or simply $\mathcal{V}_R^*$ if no confusion arises.

An algorithm to compute $\mathcal{V}_R^*(\mathcal{K})$, which works, under suitable hypotheses, also in the case of infinite families of systems, was given in [3] and it was recently applied to the framework of switching systems in [15].

Remark 3. Let us remark that, for all $i \in I$, $\mathcal{V}_R^*(\mathcal{K})$ is contained in $\mathcal{V}_i^*(\mathcal{K})$, that is in the maximum controlled invariant subspace of the i-th mode contained in $\mathcal{K}$, and that, in general, it may be smaller than the $\bigcap_{i \in I} \mathcal{V}_i^*(\mathcal{K})$.

Standard techniques of the geometric approach can be employed to prove Proposition 2 and the above statements in a straightforward way (see [2] for proofs and related notions).

3. PROBLEM FORMULATION

Let P be a switching system of the form (1), called the Plant, defined by the equations

$$P_\sigma \equiv \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}w(t) \\ y(t) = C_{\sigma(t)}x(t) \end{cases} \quad (3)$$

with state $x \in \mathcal{X} = \mathbb{R}^{n'}$; input $w \in \mathcal{W} = \mathbb{R}^{m'}$; output $y \in \mathcal{Y} = \mathbb{R}^p$.
Let M_σ be a switching system of the form (1), called the Model, defined by the equations

$$M_\sigma \equiv \begin{cases} \dot{x}_M(t) = A_{M\sigma(t)}x_M(t) + B_{M\sigma(t)}u(t) \\ y_M(t) = C_{M\sigma(t)}x_M(t) \end{cases} \quad (4)$$

with state $x_M \in \mathcal{X}_M = \mathbb{R}^n$; input $u \in \mathcal{U} = \mathbb{R}^m$; output $y_M \in \mathcal{Y} = \mathbb{R}^p$.
Without loss of generality, we assume that the matrices B_{Mi} and B_i are full-column rank for all $i \in I$.
Note that, while the input and state spaces of the two systems above differ and may have different dimensions, their output space is assumed to be the same.

Remark 4. Note that in considering two or more switching systems, like the Plant and the Model, on the same time interval, say $[0, +\infty)$, we can assume without loss of generality that they are governed by a single, common switching law σ. In fact, letting the plant P and the model M be governed by two different switching laws, respectively, $\sigma_P : \mathbb{R}^+ \to I_P$ and $\sigma_M : \mathbb{R}^+ \to I_M$, we can define a common switching law $\sigma : \mathbb{R}^+ \to I$, where $I = I_P \times I_M$, by $\sigma(t) = (\sigma_P(t), \sigma_M(t))$ for all $t \in [0, +\infty)$.

By remarking that the forced response and free response of the plant P and of the model M depend, in particular, on the specific switching law, we can consider the problem

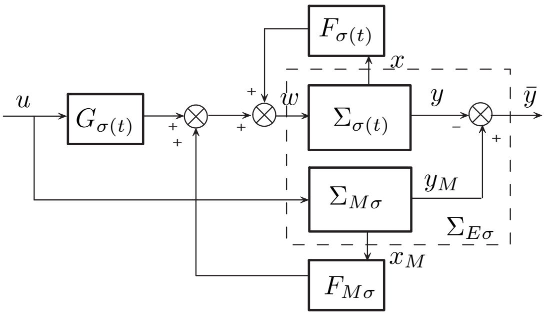

Fig. 1. Block diagram for model matching with static state feedback

of compensating the plant in such a way that its forced response matches, for any switching law σ, that of the model.

A stronger request is that, in addition to the matching of the forced responses, the plant is compensated in such a way that its free response asymptotically matches that of the model, at least for all the switching laws in a sufficiently large class.

It is reasonable to assume, in the latter case, that the model to match is asymptotically stable, at least for all the switching laws in a sufficiently large class. Hence, the stronger requirement implies, as we will see, the same kind of stability for the compensated system.

It is well known that matching problems of the above kinds are equivalent to disturbance decoupling problems for a suitable system, that essentially compares the output of the plant and that of the model (see [8]).

This fact has been used by many authors to reduce matching problems to disturbance decoupling ones and to investigate them by means of geometric tools and methods.

In order to follow the same approach and to state formally the above problems, let us introduce the output-difference switching system $\Sigma_{E\sigma}$ and the disturbed output-difference switching system $\Sigma_{D\sigma}$, defined respectively by the equations

$$\Sigma_{E\sigma} \equiv \begin{cases} \dot{x}_M(t) = A_{M\sigma(t)} x_M(t) \\ \dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} w(t) \\ \bar{y}(t) = C_{M\sigma(t)} x_M(t) - C_{\sigma(t)} x(t) \end{cases} \tag{5}$$

and

$$\Sigma_{D\sigma} \equiv \begin{cases} \dot{x}_M(t) = A_{M\sigma(t)} x_M(t) + B_{M\sigma(t)} u(t) \\ \dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} w(t) \\ \bar{y}(t) = C_{M\sigma(t)} x_M(t) - C_{\sigma(t)} x(t) \end{cases} \tag{6}$$

Model Matching Problem with Static Feedback

Given the plant (3) and the model (4) and assuming that the state of the model is measurable, the Model Matching Problem with Static Feedback (MMPSF) consists in finding a static, switching feedback law

$$w(t) = F_{M\sigma} x_M(t) + F_{\sigma(t)} x(t) + G_{\sigma(t)} u(t) \tag{7}$$

for the disturbed system (6), such that the forced response of compensated system

$$\Sigma_{C\sigma} \equiv \begin{cases} \dot{x}_M(t) = A_{M\sigma} x(t) + B_{M\sigma} u(t) \\ \dot{x}(t) = (A_\sigma + B_\sigma F_\sigma) x(t) + \\ \qquad +B_\sigma F_{M\sigma} x_M(t) + B_\sigma G_\sigma u(t) \\ \bar{y}(t) = C_{M\sigma} x_M(t) - C_{\sigma(t)} x(t) \end{cases} \tag{8}$$

is null for every switching law σ.

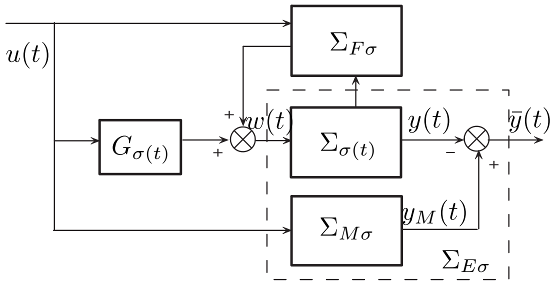

Fig. 2. Block diagram for model matching with dynamic feedback

Model Matching Problem with Dynamic Feedback

Given the plant (3) and the model (4), the Model Matching Problem with Dynamic Feedback (MMPDF) consists in finding an integer q and a dynamic, switching feedback law

$$\dot{x}_a(t) = H_{1\sigma} x(t) + H_{2\sigma} x_a(t) + K_\sigma u(t) \tag{9}$$

$$w(t) = F_{1\sigma} x(t) + F_{2\sigma} x_a(t) + G_{\sigma(t)} u(t) \tag{10}$$

with $x_a \in \mathcal{X}_a = \mathbb{R}^q$ such that the forced response of the extended compensated system

$$\Sigma_{exC\sigma} \equiv \begin{cases} \dot{x}_M(t) = A_{M\sigma} x(t) + B_{M\sigma} u(t) \\ \dot{x}(t) = \left(A_{\sigma(t)} + B_{\sigma(t)} F_{1\sigma}\right) x(t) + \\ \qquad B_{\sigma(t)} F_{2\sigma} x_a(t) + B_{\sigma(t)} G_{\sigma(t)} u(t) \\ \dot{x}_a(t) = H_{1\sigma} x(t) + H_{2\sigma} x_a(t) + \\ \qquad K_\sigma u(t) \\ \bar{y}(t) = C_{M\sigma} x_M(t) - C_{\sigma(t)} x(t) \end{cases} \tag{11}$$

is null for every switching law σ.

Note that in stating the above problems we have implicitly assumed that the switching signal is simultaneously available to the output-difference switching system and to the switching feedback regulator. The same assumption holds also for the matching problem with stability defined in the sequel.

The above problems actually consist in decoupling the input u, viewed as a measurable disturbance, from the output of the output-difference switching system $\Sigma_{D\sigma}$ given by (6), in the case in which the entire state or, respectively, only its component in $\mathcal{X}$ is measurable.

Since the feedback modifies only the dynamics of the plant, achieving decoupling, that is annihilating the forced response of the compensated system, causes the output of the compensated plant to match that of the model.

In order to consider a realistic stability requirement for the class of switching system we are dealing with, let us recall that, given a switching law σ, a positive constant τ_σ is called the dwell time of σ if the time interval between any two consecutive switchings is no smaller than τ_σ.

Stability of switching systems can be conveniently dealt with if we restrict the set of admissible switching laws by considering only those whose dwell time verifies $\tau_\sigma \geq \alpha > 0$ for a given value α.

Strong Model Matching Problems

Given the plant (3) and the model (4), the Strong Model Matching Problems (SMMP) consists in finding a solution of the form (7) of the MMPSF (respectively, an integer q and a solution of the form (9-10) of the MMPDF) for which there exists $\alpha \in \mathbb{R}^+$ such that the compensated system

(8) (respectively, the extended compensated system (11)) is asymptotically stable for every switching law σ with $\tau_\sigma \geq \alpha$.

4. PROBLEM SOLUTION

We give now necessary and sufficient conditions for the solution of the MMPSF and of the MMPDF in geometric terms.

For this, let us consider the output-difference switching system $\Sigma_{E\sigma}$ given in (5) and let us denote by $\mathcal{K}$ the subspace of $\mathcal{X}_M \oplus \mathcal{X}$ defined by

$$\mathcal{K} = \cap_{i\in N} Ker \left[C_{Mi} \quad -C_i\right].$$

We will denote by $\mathcal{V}_R^*$ the maximum controlled invariant subspace $\mathcal{V}_R^*(\mathcal{K})$ for $\Sigma_{E\sigma}$ contained in $\mathcal{K}$.

Proposition 5. Given the plant (3) and the model (4), the MMPSF (respectively, the MMPDF) is solvable if and only if the condition

$$Im \begin{bmatrix} B_{Mi} \\ 0 \end{bmatrix} \subseteq \mathcal{V}_R^* + Im \begin{bmatrix} 0 \\ B_i \end{bmatrix} \tag{12}$$

holds for all $i \in I$.

Proof. As remarked above, the MMPSF consists in decoupling a measurable disturbance and, therefore, necessity and sufficiency of (12) can be shown using geometric methods as in [15] in the case of measurable disturbance. For the MMPDF, assume that (12) holds and without loss of generality let $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & 0 \end{bmatrix}$ be a $(n+n') \times (q+q')$ matrix whose columns are a basis of $\mathcal{V}_R^*$ (note that in particular V_3 is left invertible). This implies, in particular, the equalities

$$\begin{bmatrix} B_{Mi} \\ 0 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \\ V_3 & 0 \end{bmatrix} \begin{bmatrix} D_{1i} \\ D_{2i} \end{bmatrix} + \begin{bmatrix} 0 \\ B_i \end{bmatrix} G_i$$

and

$$\begin{bmatrix} A_{Mi} & 0 \\ 0 & A_i \end{bmatrix} \begin{bmatrix} V_1 & V_2 \\ V_3 & 0 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \\ V_3 & 0 \end{bmatrix} \begin{bmatrix} L_{1i} & L_{2i} \\ L_{3i} & L_{4i} \end{bmatrix} + \begin{bmatrix} 0 \\ B_i \end{bmatrix} \left[\, M_{1i} \;\; M_{2i} \,\right]$$

for suitable matrices $D_{1i}, D_{2i}, G_i, L_{1i}, L_{2i}, L_{3i}, L_{4i}, M_{1i}, M_{2i}$ for all $i \in I$. Then, we have that the subspace $\mathcal{V}_{ex}$ spanned in $\mathcal{X}_M \times \mathcal{X} \times \mathcal{X}_a$ by the columns of the matrix $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & 0 \\ 0 & I_q \end{bmatrix}$ is a controlled invariant subspace for the switched system that one obtains by extending the output-difference system $\Sigma_{E\sigma}$ by the switching dynamics (9) and also that $\mathcal{V}_{ex}$ contains $Im \begin{bmatrix} B_{Mi} \\ 0 \\ K_{1i} \end{bmatrix}$ for $K_{1i} = D_{2i}$. Taking the matrices H_{1i}, H_{2i}, K_{1i} as $H_{1i} = L_{3i}W$ where W is a left inverse of V_3 (i.e. $WV_3 = I$), $H_{2i} = L_{4i}$ and $K_{1i} = D_{2i}$ for all $i \in I$, it is possible to choose a friend $F = \{(F_{Mi}, F_{1i}, F_{2i})\}_{i\in I}$ of $\mathcal{V}_{ex}$ letting $F_{Mi} = 0$, $F_{1i}V_3 = M_{1i}$ and $F_{2i} = M_{2i}$ for all $i \in I$ and to construct a compensator of the form (9), (10) that achieves decoupling of u from $\bar{y}$. □

Remark 6. Note that the feedback F constructed in the proof of Proposition 5 has an arbitrary component. The equalities that F_{2i} must verify, in fact, assign its value only on the subspace of $\mathcal{X}$ spanned by V_3. The remaining degree of freedom will be exploited to solve, when possible, the SMMP.

In order to give conditions for the solution of the SMMP, we restrict our attention to the case in which the condition

$$Im \begin{bmatrix} 0 \\ B_i \end{bmatrix} \cap \mathcal{V}_R^* = 0 \tag{13}$$

holds for all $i \in I$. Note that, B_i being full-column rank for all $i \in I$, the previous condition is equivalent to left invertibility of every mode of the plant (3).

Proposition 7. Assume that condition (13) holds for all $i \in I$. Then, the set of all controlled invariant subspaces $\mathcal{V}$ of $\Sigma_{E\sigma}$ such that $\mathcal{V} \subseteq \mathcal{K}$ and $Im \begin{bmatrix} B_{iM} \\ 0 \end{bmatrix} \subseteq \mathcal{V} + \begin{bmatrix} 0 \\ B_i \end{bmatrix}$ for all $i \in I$ has a minimum element.

Proof. It is enough to show that the intersection of two controlled invariant subspaces of $\Sigma_{E\sigma}$ is a controlled invariant subspace of Σ_e.

Denoting by $\mathcal{V}_{*R}$ the minimum controlled invariant subspace of $\Sigma_{E\sigma}$ contained in $\mathcal{K}$ such that

$$Im \begin{bmatrix} B_{Mi} \\ 0 \end{bmatrix} \subseteq \mathcal{V}_{*R} + Im \begin{bmatrix} 0 \\ B_i \end{bmatrix}, \text{ for all } i \in I,$$

we have the following Proposition.

Proposition 8. Assume that condition (13) holds for all $i \in I$ and let $\mathcal{V}$ be a controlled invariant subspace of $\Sigma_{E\sigma}$ that contain $\mathcal{V}_{*R}$ and is contained in $\mathcal{K}$. Then, for any family $F' = \{F'_i\}_{i\in I}$ of friends of $\mathcal{V}$ and any family $F = \{F_i\}_{i\in I}$ of friends of $\mathcal{V}_{*R}$ we have $F'_i(v) = F_i(v)$ for all $v \in \mathcal{V}_{*R}$ and all $i \in I$.

Proof. For all $v \in \mathcal{V}_{*R}$ and all $i \in I$ we have $\begin{bmatrix} A_{Mi} & 0 \\ 0 & A_i \end{bmatrix} v = v' + \begin{bmatrix} 0 \\ B_i \end{bmatrix} F'_i(v) = v" + \begin{bmatrix} 0 \\ B_i \end{bmatrix} F_i(v)$ for some $v' \in \mathcal{V}$ and $v" \in \mathcal{V}_{*R}$ and the conclusion follows using condition (13) and the assumption that B_i is full-column rank for all $i \in I$. □

Remark 9. Let V be any matrix whose columns are a basis of $\mathcal{V}_{*R}$. Then, we have from the above Propositions that, for all $i \in I$, there exist a unique square matrix L_i and a unique matrix M_i of suitable dimension such that

$$\begin{bmatrix} A_{Mi} & 0 \\ o & A_i \end{bmatrix} V = VL_i + \begin{bmatrix} 0 \\ B_i \end{bmatrix} M_i \tag{14}$$

for all $i \in I$.

Proposition 10. Let B_i be full-column rank and assume that condition (13) holds for all $i \in I$. Also assume that all the modes of the model M are stable and that all the modes of the plant P are stabilizable. Then, the SMMP is solvable by a static feedback of the form (7) or, respectively, by a dynamic feedback of the form (9-10), if and only if the matrix L_i defined, for any choice of a basis V of V_{*R}, by (14) is Hurwitz.

Sketch of Proof *Static feedback case*

Let V be a matrix whose columns are a basis of V_{*R} and let $[V' \; V]$ be a matrix whose columns are a basis of $\mathcal{X}_M \oplus \mathcal{X}$. By Proposition 8 and Remark 9, we have that, for any solution $v(t) = F_{M\sigma}x_M(t) + F_{\sigma(t)}x(t) + G_{\sigma(t)}u(t)$ of the form (7) of the MMPSF, the dynamic matrices of the i-th mode of the compensated system $\Sigma_{C\sigma}$ given by (8), in the

basis $[V' \; V]$, takes the form $\begin{bmatrix} * & * \\ 0 & L_i \end{bmatrix}$, where $*$ denotes a component of no interest for the present discussion. Since any static feedback solution of the SMMP is a solution of the MMPSF and since the condition of asymptotic stability for any switching law σ with $\tau_\sigma \geq \alpha$ implies asymptotic stability of any mode, necessity of the above condition follows.
Sufficiency of the above condition for the existence of a static feedback solution of the SMMP follows by remarking, first, that it guarantees (using the same argument as in the proof of Theorem 4.2-2 in [2]) the existence of a feedback $v(t) = F_{Mi}x_M(t)+F_ix(t)+G_iu(t)$ that decouples with stability the input u from the output $\bar{y}$ in each mode of the disturbed system $\Sigma_{D\sigma}$. Then, all the modes of $\Sigma_{C\sigma}$ being asymptotically stable, the existence of α such that $\Sigma_{C\sigma}$ is asymptotically stable for any switching law σ with $\tau_\sigma \geq \alpha$ follows from [14] Lemma 2 (a procedure to find α is also given in that paper).

Dynamic feedback case Let us consider, now the SMMP with dynamic feedback. Existence of a solution of the form (9-10) implies the existence of a subspace $\mathcal{V}_e$ of $\mathcal{X}_M \oplus \mathcal{X} \oplus \mathcal{X}_a$ which is invariant for the dynamics of the extended compensated Σ_{exC} given in (11), which contains $Im \begin{bmatrix} B_{Mi} \\ B_iG_i \\ K_i \end{bmatrix}$ for all $i \in I$ and which is contained in $Ker \,[\, C_{Mi} \; -C_i \; 0 \,]$ for all $i \in I$. Using the assumption that the reachable subspace and the kernel of the output map of each mode of the model intersect only at the origin and assuming, without loss of generality, that the dimension q of $\mathcal{X}_a$ is the minimal for which a solution of the above kind can be found, we can show that the subspace $\mathcal{V}_e$ intersects $\mathcal{X}_M$ only at the origin and that, in a suitable basis of $\mathcal{X}_M \oplus \mathcal{X} \oplus \mathcal{X}_a$, it is generated by the columns of a matrix V_e of the form $V_e = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \\ 0 & I_q \end{bmatrix}$, where V_3 is full-column rank. This means that there exists a set of matrices $\{L_i\}_{i \in I}$ such that

$$\begin{bmatrix} A_{Mi} & 0 & 0 \\ 0 & (A_i + B_iF_{1i}) & B_iF_{2i} \\ 0 & H_{1i} & H_{2i} \end{bmatrix} V_e = V_e L_i \qquad (15)$$

for all $i \in I$. Moreover, since asymptotic stability for any switching law σ with $\delta_\sigma \geq \alpha$ implies asymptotic stability of any mode, L_i is Hurwitz for all $i \in I$. From these facts, we can conclude that the subspace $\mathcal{V}$ spanned in $\mathcal{X}_M \oplus \mathcal{X}$ by the columns of the matrix $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & 0 \end{bmatrix}$ is a controlled invariant subspace for the system $\Sigma_{E\sigma}$, given by (5), that contains $Im \begin{bmatrix} B_{Mi} \\ 0 \end{bmatrix}$ for all $i \in I$ and such that the equality

$$\begin{bmatrix} A_{Mi} & 0 \\ 0 & A_i \end{bmatrix} V = VL_i + \begin{bmatrix} 0 \\ B_i \end{bmatrix} [\, F_{1i} \; F_{2i} \,] \qquad (16)$$

holds for all $i \in I$. Necessity of the condition in the statement, then, follows from Proposition 8.
To show sufficiency, let us construct a solution of the MMPDF exactly in the same way as in the proof of Proposition 5 by using $\mathcal{V}_{*R}$ in the place of $\mathcal{V}_R^*$, that is letting the column of $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & 0 \end{bmatrix}$ denote a basis of $\mathcal{V}_{*R}$ instead of a basis of $\mathcal{V}_R^*$. Note, that in such case, V is the same matrix appearing in (14). If V' is a matrix such that the columns of $[\, V' \; V_3 \,]$ form a basis of $\mathcal{X}$, the dynamics of the modes of the extended compensated system $\Sigma_{exC\sigma}$, in the basis given by the columns of $\begin{bmatrix} I_n & 0 & V_1 & V_2 \\ 0 & V' & V_3 & 0 \\ 0 & 0 & 0 & I_q \end{bmatrix}$, takes the form $\begin{bmatrix} A_{Mi} & * & 0 & 0 \\ 0 & S_i & 0 & 0 \\ 0 & * & L_{1i} & L_{2i} \\ 0 & * & L_{3i} & L_{4i} \end{bmatrix}$ for all $i \in I$, where $L_i = \begin{bmatrix} L_{1i} & L_{2i} \\ L_{3i} & L_{4i} \end{bmatrix}$ is the matrix appearing in (14). Note that A_{Mi} and L_i are Hurwitz for all $i \in I$ and, using [2] Property 4.1-16, it can be shown that the remaining degree of freedom in F_{1i} (see Remark 6) can be used to stabilize the dynamics described by S_i. Hence, all the modes of $\Sigma_{exC\sigma}$ being asymptotically stable, the existence of α such that $\Sigma_{C\sigma}$ is asymptotically stable for any switching law σ with $\tau_\sigma \geq \alpha$ follows again from [14] Lemma 2.

□

Remark 11. By introducing the concept of average dwell time $\tau_{a\sigma}$ of a switching law σ, the SMMP can be stated in a slightly different way by asking for the existence of a positive α such that asymptotic stability is guaranteed for all switching law σ whose average dwell time $\tau_{a\sigma} \geq \alpha$. Then, the statement of Proposition 10 remains valid thank to the results of [9] (see also [11] Theorem 12).

5. EXAMPLE

Let us consider the switching system $\Sigma_{\sigma(t)}$, of the form (1), where $\sigma : \mathbb{R}^+ \to \{1, \; 2\}$, whose modes are

$$A_1 = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C_1 = [\, -1 \; 0 \,]$$
$$A_2 = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C_2 = [\, -1 \; 0 \,].$$

Let the Model be the switching system $M_{\sigma(t)}$, defined by equations of the form (4) where

$$A_{M1} = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix}, \; B_{M1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_{M1} = [\, -1 \; -1 \,]$$
$$A_{M2} = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}, \; B_{M2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C_{M2} = [\, 1 \; -1 \,].$$

In the disturbed output-difference switching system $\Sigma_{D\sigma}$ defined by equations (6), the subspace $\mathcal{V}_R^*(\mathcal{K})$ is spanned by the columns of the matrix

$$V = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and conditions (12) and (13) hold. In this case $\mathcal{V}_R^*(\mathcal{K})$ coincides with $\mathcal{V}_{*R}$. Following the construction outlined in the proof of Proposition 5, we get the dynamic feedback of the form (9) and (10), where

$$\begin{aligned} H_{11} &= [\, -1 \; 0 \,], \; H_{21} = [\, 0 \,], K_1 = [\, 0 \,] \\ F_{11} &= [\, -5 \; 2 \,], \; F_{21} = [\, 3 \,], G_1 = [\, -1 \,], \end{aligned}$$

and

$$H_{12} = [\,1\ 0\,],\ H_{22} = [\,-2\,], K_2 = [\,1\,]$$
$$F_{12} = [\,-3\ 0\,],\ F_{22} = [\,0\,], G_2 = [\,0\,]$$

that provides the matching. The modes of the switched extended compensated system (11), in a suitable basis, are described by the matrices

$$\begin{bmatrix} A_{M1} & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & H_{11} & H_{21} \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 3 \\ 0 & 0 & -2 & -1 & 3 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} B_{M1} \\ B_1 G_1 \\ K_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \quad [\,C_{M1}\ -C_1\ 0\,] = [\,1\ -1\ 1\ 0\ 0\,]$$

$$\begin{bmatrix} A_{M2} & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & H_{12} & H_{22} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} B_{M2} \\ B_2 G_2 \\ K_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\,C_{M2}\ -C_2\ 0\,] = [\,1\ -1\ 1\ 0\ 0\,]$$

which are easily seen to be Hurwitz stable. Hence, there exists $\alpha > 0$ such that the switched compensated extended system is asymptotically stable for every switching rule σ with $\tau_\sigma \geq \alpha$ by [14] Lemma 2.

6. CONCLUSION

Geometric methods have proved to be applicable and useful in characterizing solvability conditions of model matching problems for switching systems and in constructing solutions. Stability has been considered for families of switching laws with bounded dwell time and results can be extended to families with bounded average dwell time. Model matching problems with output feedback will be the object of future investigations.

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