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Characteristic equation for linear periodic systems with distributed delay ⋆

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Abstract: The paper presents a method for constructing the characteristic equation of SISO linear periodic dynamical systems with distributed delay. The construction bases on a description in frequency domain, and a relation between the characteristic function of the periodic system and the Fredholm denominator of an associated Fredholm integral equation of the second kind. Using the Fredholm calculus, polynomial approximations of the characteristic function are derived. Together with estimation formulae, sufficient conditions for the location of the characteristic roots with respect to the unit circle are provided, i.e. ordinary criteria can be applied for stability investigations. The method is illustrated by an example.

Keywords: Linear systems, Periodic structures, Time delay, Distributed feedback, Integral equations, Characteristic roots, Stability criteria

1. INTRODUCTION

Methods for the calculation of eigenvalues play an important role in the theory of linear systems with delay. These numbers determine the character of the transient processes and the asymptotic behavior of the system for $t \to \infty$. A comprehensive description of the general methods and a careful analysis of the existing literature in this direction is given in Michiels and Niculescu (2007); Gil' (2013).

In particular, in the theory of linear periodic systems (LPS) with delay, an important role plays the problem of constructing the characteristic equation, where its roots are the eigenvalues of the investigated system. The majority of contributions in this direction are connected with the study of the properties of the monodromy operator Halanay (1961); Shimanov (1963) and the construction of the associated characteristic matrix Gasimov (1972); Zverkin (1988); Dolgii and Nikolaev (1999); Dolgii (2006); Kaashoek and Verduyn Lunel (1992); Siebel and Szalai (2011).

However, when using existing metods for real systems, the constuction of the characteristic matrix bases on the solution of a special boundary value problem for ordinary differential equations, and this procedure is connected with honest technical difficulties.

An alternative approach for the solution of the stability problem for single loop LPS with delay is presented in Rosenwasser (1969, 1964). This approach bases on the detected relation between the characteristic function and the Fredholm denominator of an especially constructed Fredholm integral equation of the second kind, where the kernel depends on a parameter. Hereby the use of the calculus of Fredholm's theory opens constructive possibilities for building the characteristic equation directly from the original equations of the LPS with delay, and needs no degression over the solution of a boundary value problem. Further on, this method is denoted as IFE.

Further development of the IFE method has been done in the works Lampe and Rosenwasser (2010, 2011, 2013a,b). In particular, the papers Lampe and Rosenwasser (2010, 2011) consider the IFE method on the example of a single loop system with one concentrated delay. The papers Lampe and Rosenwasser (2013a,b) investigate multidimensional systems with one and several concentrated delays, respectively.

On basis of the IFE method, the present paper provides a method for the construction of the characteristic equation for a single loop LPS with one distributed delay. Then, sufficient stability conditions are obtained by studying the root locations of a certain polynomial with respect to the unit circle.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

The paper considers the single loop system $\mathcal{S}_\tau$, described by the equations

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= Ax(t) + \mu B a(t) \int_0^h y(t-\tau) m(\tau)\,\mathrm{d}\tau, \\ y(t) &= Cx(t). \end{aligned} \tag{1}$$

Here $x(t)$ is an $\chi \times 1$ vector, A is a constant $\chi \times \chi$ matrix, and B, C are constant vectors of size $\chi \times 1$ and $1 \times \chi$, respectively. Assume the pair (A, B) to be completely controllable, and the pair (A, C) completely observable.

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With respect to the functions $a(t) = a(t+T)$ and $m(t)$ we assume that they are of finite variation. Hereby, the function $a(t)$ is continuous, and the function $m(t)$ is piecewise continuous. These conditions are assumed for simplifying the proofs, but they could be weakened. Moreover, in (1) the quantity h is a positive constant, and μ an auxiliary parameter, which is introduced for calculatory reasons, but it is not small.

Introduce the transfer function

$$W(p) = C(pI_\chi - A)^{-1}B. \tag{2}$$

Then system (1) can be written as operator equation

$$\begin{aligned} y(t) &= \mu W(p)L[y(t)], \\ L[y(t)] &= a(t)\int_0^h y(t-\tau)m(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{3}$$

where $p = \frac{\mathrm{d}}{\mathrm{d}t}$ is the differential operator. Under the above mentioned suppositions, the function $W(p)$ can be presented in the form

$$W(p) = \frac{n(p)}{d(p)}, \tag{4}$$

where $n(p)$, $d(p)$ are coprime polynomials with $\deg n(p) < \deg d(p) \le \chi$. Hereby, we assume the product

$$d(p) = (p-p_1)^{\mu_1}\cdots(p-p_\rho)^{\mu_\rho}, \tag{5}$$

where $p_1, ..., p_\rho$ are all different eigenvalues of the matrix A. Below the numbers

$$p_{ik} = p_i + k\mathrm{j}\omega, \quad (i = 1, ..., \rho,\ k = 0, \pm 1, ...), \tag{6}$$

where $\mathrm{j} = \sqrt{-1}$, $\omega = 2\pi/T$, are called eigenindices of the system $\mathcal{S}_\tau$, and the numbers

$$\zeta_i^0 = \mathrm{e}^{-p_i T} = \mathrm{e}^{-p_{ik}T} \tag{7}$$

are the inverse eigenmultipliers. Further on, the sets of numbers p_{ik} and ζ_i^0 are denoted by $\mathcal{M}_p$ and $\mathcal{M}_\zeta^0$, respectively.

Let $f(\zeta)$ be a rational function. Then the function

$$\tilde{f}(s) = f(\zeta)|_{\zeta=\mathrm{e}^{-sT}} \tag{8}$$

is called rational-periodic (RP). The functions $f(\zeta)$ and $\tilde{f}(s)$, related by (8), are called associated.

As is known Hale (1971), equation (1) has a set of solutions of the form

$$y(t) = \mathrm{e}^{\lambda t} z(t), \quad z(t) = z(t+T), \tag{9}$$

where λ is a constant, in general complex. Those solutions are usually called Floquet solutions. The corresponding numbers λ are called indices of the system $\mathcal{S}_\tau$. Obviously, when λ_i is an index, then all numbers

$$\lambda_{ik} = \lambda_i + k\mathrm{j}\omega, \quad (k = 0, \pm 1, ...) \tag{10}$$

are indices too. Below, the set of all indices λ_{ik} is denoted as $\mathcal{M}_\lambda$. The numbers

$$\zeta_i = \mathrm{e}^{-\lambda_i T} = \mathrm{e}^{-\lambda_{ik}T} \tag{11}$$

are called the inverse multipliers of the system $\mathcal{S}_\tau$, and the set of all ζ_i by $\mathcal{M}_\zeta$. It is known that the sets $\mathcal{M}_\lambda$ and $\mathcal{M}_\zeta$ are countable.

The system $\mathcal{S}_\tau$ is called (asymptotically) stable, when all its solutions for arbitrary initial conditions tend to zero for $t \to \infty$. As is known Halanay (1961), for the stability of the system $\mathcal{S}_\tau$ the condition

$$\operatorname{Re}\lambda_{ik} < 0, \quad \forall \lambda_{ik} \in \mathcal{M}_\lambda, \tag{12}$$

is necessary and sufficient, which is equivalent to

$$|\zeta_i| > 1, \quad \forall \zeta_i \in \mathcal{M}_\zeta. \tag{13}$$

Using the mathematical apparatus of Fredholm integral equations of the second kind, the present paper constructs an integral function $L(\zeta, \mu)$ of the arguments ζ and μ such that for fixed μ the set of roots of the function $L(\zeta, \mu)$ coincides with the set of inverse multipliers $\mathcal{M}_\zeta$. It is shown that applying the function $L(\zeta, \mu)$ allows to find sufficient conditions for stability, by studying the location of the roots of certain polynomials in ζ in relation to the unit circle.

3. PRELIMINARIES

In this section we prove some theorems establishing the basis for the suggested methods to investigate the stability of the system $\mathcal{S}_\tau$.

Theorem 1. For the fact that for fixed μ the number $\lambda \notin \mathcal{M}_p$ becomes an index of the system $\mathcal{S}_\tau$, it is necessary and sufficient that the homogeneous integral equation

$$z(t) = \mu \int_0^T K(\lambda, t, u) z(u)\,\mathrm{d}u \tag{14}$$

possesses a nonzero solution. The in (14) configured function $K(\lambda, t, u)$ is determined by the relation

$$K(\lambda, t, u) = \int_0^h \phi_W(T, \lambda, t-u-\tau) a(u+\tau) m(\tau)\,\mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau. \tag{15}$$

Here $\phi_W(T, \lambda, t)$ is defined by the series

$$\phi_W(T, \lambda, t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} W_k(\lambda)\,\mathrm{e}^{k\mathrm{j}\omega t}, \tag{16}$$

where

$$W_k(\lambda) = W(\lambda + k\mathrm{j}\omega). \tag{17}$$

Proof. Assume in (3)

$$y(t) = \mathrm{e}^{\lambda t} z(t), \tag{18}$$

where λ is a constant. Then we obtain an operator equation for the function $z(t)$

$$\begin{aligned} z(t) &= \mu W(p+\lambda) L_\lambda[z(t)], \\ L_\lambda[z(t)] &= a(t)\int_0^h z(t-\tau)m(\tau)\,\mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau. \end{aligned} \tag{19}$$

As it follows from Rosenwasser (1970); Rosenwasser and Lampe (2006), for $\lambda \notin \mathcal{M}_p$ equation (19) has a periodic solution $z_T(t) = z_T(t+T)$ if and only if

$$z_T(t) = \mu \int_0^T \phi_W(T, \lambda, t-\nu) L_\lambda[z_T(\nu)]\,\mathrm{d}\nu. \tag{20}$$

Applying (19), the last equation can be written in the form

$$\begin{aligned} z_T(t) &= \\ \mu &\int_0^T \int_0^h \phi_W(T, \lambda, t-\nu) a(\nu) z_T(\nu-\tau) m(\tau)\,\mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau\,\mathrm{d}\nu. \end{aligned} \tag{21}$$

Substituting here $\nu - \tau = u$, we find

$$\begin{aligned} z_T(t) &= \mu \int_{-\tau}^{T-\tau} \int_0^h \\ &\phi_W(T, \lambda, t-u-\tau) a(u+\tau) z_T(u) m(\tau)\,\mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau\,\mathrm{d}u. \end{aligned} \tag{22}$$

Note that the integrand in (22) is periodic regarding u with period T. Hence, this relation can be presented in the form

$$z_T(t) = \mu \int_0^T \int_0^h \phi_W(T, \lambda, t-u-\tau) a(u+\tau) m(\tau)\, \mathrm{e}^{-\lambda\tau}\, \mathrm{d}\tau \cdot z_T(u)\, \mathrm{d}u, \tag{23}$$

which is equivalent to (14)-(15). ■

Here and further on the symbol ■ indicates the end of the proof.

Below, function (15) is called the kernel of the system $\mathcal{S}_\tau$.

Theorem 2. The following statements hold:

i) For $0 < t, u < T$ and $\lambda \notin \mathcal{M}_p$ equation (20) is an homogenous Fredholm integral equation of the second kind with kernel (15).
ii) For fixed μ, the number $\lambda \notin \mathcal{M}_p$ is an index of the system $\mathcal{S}_\tau$, if and only if

$$\tilde{D}_K(\lambda, \mu) = 0, \tag{24}$$

where $\tilde{D}_K(\lambda, \mu)$ is the Fredholm denominator of the kernel $K(\lambda, t, u)$.

Proof. Inserting (16) into (15), after term-wise integration, we obtain

$$K(\lambda, t, u) = \frac{1}{T} \sum_{k=-\infty}^{\infty} W_k(\lambda) a_k(\lambda, u)\, \mathrm{e}^{k\mathrm{j}\omega(t-u)}, \tag{25}$$

where

$$a_k(\lambda, u) = \int_0^h a(u+\tau) m(\tau)\, \mathrm{e}^{-(\lambda+k\mathrm{j}\omega)\tau}\, \mathrm{d}\tau. \tag{26}$$

Under the taken suppositions, the function $a_k(\lambda, u)$ depends continuously on u, and for fixed λ, with respect to u and in $0 \le u \le T$ the estimate

$$a_k(\lambda, u) < \frac{d}{|k|}, \quad d = \mathrm{const.} \tag{27}$$

holds. Since the terms of series (25) are continuous regarding t, u, and decrease as k^{-2}, also the sum of series (25) depends continuously on t, u for $0 \le t, u \le T$.

Statement ii) follows from i) and the general Fredholm theory, Goursat (1927); Tricomi (1957). ■

4. CONSTRUCTION OF THE FREDHOLM DENOMINATOR

As it follows from the Fredholm theory, the Fredholm denominator for the kernel $K(\lambda, t, u)$ when $\lambda \notin \mathcal{M}_p$ is determined by the formula Goursat (1927); Tricomi (1957)

$$\tilde{D}_K(\lambda, \mu) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \mu^m}{m!} \tilde{D}_m(\lambda), \tag{28}$$

where

$$\tilde{D}_m(\lambda) = \int_0^T \ldots \int_0^T \Delta_K(\lambda, t_1, \ldots, t_m)\, \mathrm{d}t_1 \ldots \mathrm{d}t_m \tag{29}$$

and

$$\Delta_K(\lambda, t_1, \ldots, t_m) = \det \begin{bmatrix} K(\lambda, t_1, t_1) & K(\lambda, t_1, t_2) & \ldots & K(\lambda.t_1, t_m) \\ K(\lambda, t_2, t_1) & K(\lambda, t_2, t_2) & \ldots & K(\lambda, t_2, t_m) \\ \ldots & \ldots & \ldots & \ldots \\ K(\lambda, t_m, t_1) & K(\lambda, t_m, t_2) & \ldots & K(\lambda, t_m, t_m) \end{bmatrix}. \tag{30}$$

In this formula, in consensus with (25), the function $K(\lambda, t_\alpha, t_\beta)$ is defined by the sum of the series

$$K(\lambda, t_\alpha, t_\beta) = \frac{1}{T} \sum_{k=-\infty}^{\infty} W_k(\lambda) a_k(\lambda, t_\beta)\, \mathrm{e}^{k\mathrm{j}\omega(t_\alpha - t_\beta)}. \tag{31}$$

Substituting (31) in (30), and applying the addition theorem for determinants, we obtain

$$\Delta_K(\lambda, t_1, \ldots, t_m) = \frac{1}{T^m} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_m=-\infty}^{\infty} a_{k_1}(\lambda, t_1) \cdots a_{k_m}(\lambda, t_m) \Delta_{k_1 \ldots k_m}(t_1, \ldots, t_m), \tag{32}$$

where

$$\Delta_{k_1 \ldots k_m}(t_1, \ldots, t_m) = \det \begin{bmatrix} \mathrm{e}^{k_1\mathrm{j}\omega(t_1-t_1)} & \mathrm{e}^{k_2\mathrm{j}\omega(t_1-t_2)} & \ldots & \mathrm{e}^{k_m\mathrm{j}\omega(t_1-t_m)} \\ \mathrm{e}^{k_1\mathrm{j}\omega(t_2-t_1)} & \mathrm{e}^{k_2\mathrm{j}\omega(t_2-t_2)} & \ldots & \mathrm{e}^{k_m\mathrm{j}\omega(t_2-t_m)} \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{e}^{k_1\mathrm{j}\omega(t_m-t_1)} & \mathrm{e}^{k_2\mathrm{j}\omega(t_m-t_2)} & \ldots & \mathrm{e}^{k_m\mathrm{j}\omega(t_m-t_m)} \end{bmatrix}. \tag{33}$$

It is easily seen that in the case when some of the numbers k_i, $(i = 1, ..., m)$ are equal, then determinant (33) is equal to zero. Therefore, formula (32) can be written in the form

$$\Delta_K(\lambda, t_1, \ldots, t_m) = \frac{1}{T^m} \sum_{k_1=-\infty}^{\infty}{}^{*} \cdots \sum_{k_m=-\infty}^{\infty}{}^{*} a_{k_1}(\lambda, t_1) \cdots a_{k_m}(\lambda, t_m) \Delta_{k_1 \ldots k_m}(t_1, \ldots, t_m), \tag{34}$$

where the symbol $*$ means that under the numbers $k_1, \ldots, k_m$ are no equal ones. Inserting (34) in (29), we find

$$\tilde{D}_m(\lambda) = \frac{1}{T^m} \sum_{k_1=-\infty}^{\infty}{}^{*} \cdots \sum_{k_m=-\infty}^{\infty}{}^{*} W_{k_1}(\lambda) \cdots W_{k_m}(\lambda) \times \int_0^T \cdots \int_0^T a_{k_1}(\lambda, t_1) \cdots a_{k_m}(\lambda, t_m) \Delta_{k_1 \ldots k_m}(t_1, \ldots, t_m)\, \mathrm{d}t_1 \cdots \mathrm{d}t_m. \tag{35}$$

On basis of the above relations, we will state a number of properties of the function $\tilde{D}_m(\lambda)$, needed for the further considerations.

Theorem 3. The following statements hold:

i) For all m, the function $\tilde{D}_m(\lambda)$ is rational periodic.
ii) Let the roots p_i of polynomial (5) satisfy the conditions

$$\mathrm{e}^{p_i T} \ne \mathrm{e}^{p_k T}, \quad (i \ne k;\ i, k = 1, ..., \rho). \tag{36}$$

Then for arbitrary m, the associated functions

$$D_m(\zeta) = \tilde{D}_m(\lambda)|_{\mathrm{e}^{-\lambda T} = \zeta} \tag{37}$$

permit the representations

$$D_m(\zeta) = \frac{b_m(\zeta)}{b_0(\zeta)}, \tag{38}$$

where the $b_m(\zeta)$ are polynomials and

$$b_0(\zeta) = (1 - \zeta\, e^{p_1 T})^{\mu_1} \cdots (1 - \zeta\, e^{p_\rho T})^{\mu_\rho}. \tag{39}$$

Proof. i) Notice that the function

$$\tilde{R}(\lambda, t, u) = e^{\lambda(t-u)}\, K(\lambda, t, u) \tag{40}$$

is RP. Indeed, as it follows from (15),

$$\tilde{R}(\lambda, t, u) = \int_0^h D_W(T, \lambda, t-u-\tau) a(u+\tau) m(\tau)\, d\tau, \tag{41}$$

where $\tilde{D}_W(T, \lambda, t-u-\tau)$ is the sum of the series

$$\tilde{D}_W(T, \lambda, t-u-\tau) = \frac{1}{T} \sum_{k=-\infty}^{\infty} W_k(\lambda)\, e^{(\lambda + k j\omega)(t-u-\tau)}. \tag{42}$$

As it was shown in Rosenwasser and Lampe (2006), the function $\tilde{D}_W(T, \lambda, t-u-\tau)$ is determined interval wise by the formula

$$\begin{aligned} \tilde{D}_W(T, \lambda, t-u-\tau) = \\ C(I_\chi - e^{-\lambda T} e^{AT})^{-1} e^{A(t-u-\tau-kT)}\, B\, e^{k\lambda T}, \\ kT < t-u-\tau < (k+1)T. \end{aligned} \tag{43}$$

Inserting (43) into (41) emerges, that for t, u the function (41) is RP.

ii) It is easily seen that

$$\Delta_K(\lambda, t_1, \ldots, t_m) = \Delta_{\tilde{R}}(\lambda, t_1, ..., t_m), \tag{44}$$

where the function $\Delta_{\tilde{R}}(\lambda, t_1, \ldots, t_m)$ is determined by formula (30), wherein the kernel $K(\lambda, t, u)$ is substituted by the RP function $\tilde{R}(\lambda, t, u)$. Then, substituting in (29) the kernel $K(\lambda, t, u)$ by the function $\tilde{R}(\lambda, t, u)$, we find that the function $\tilde{D}_m(\lambda)$ is RP.

Since the $a_n(\lambda, u)$ are integral functions of the argument λ, it follows from (35) that the set of poles of the function $\tilde{D}_m(\lambda)$ lies in the set $\mathcal{M}_p$. Hence from (36) emerge, that the multiplicity of the poles $p_i + kj\omega$ of the function $\tilde{D}_m(\lambda)$ is equal to μ_i. Therefore, from the property of RP functions Whittaker and Watson (1927); Rosenwasser and Lampe (2000), we find

$$\tilde{D}_m(\lambda) = \frac{\tilde{b}_m(\lambda)}{\tilde{b}_0(\lambda)}, \tag{45}$$

where $\tilde{b}_m(\lambda)$ is a polynomial in the variable $e^{-\lambda T}$ and

$$\tilde{b}_0(\lambda) = (1 - e^{-\lambda T} e^{p_1 T})^{\mu_1} \cdots (1 - e^{-\lambda T} e^{p_\rho T})^{\mu_\rho}. \tag{46}$$

Substituting in (45) $e^{-\lambda T}$ by ζ, representation (38) is achieved. ■

5. CHARACTERISTIC FUNCTION AND STABILITY INVESTIGATION

Substituting in (28) ζ by $e^{-\lambda T}$, we obtain

$$D_K(\zeta, \mu) = \tilde{D}_K(\lambda, \mu)|_{e^{-\lambda T} = \zeta} = \frac{1}{T} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{m!} D_m(\zeta), \tag{47}$$

where the series on the right side converges for all μ and all $\zeta \notin \mathcal{M}_\zeta^0$. Using (38), from (47) we achieve

$$D_K(\zeta, \mu) = \frac{L(\zeta, \mu)}{b_0(\zeta)}, \tag{48}$$

where

$$L(\zeta, \mu) = b_0(\zeta) + \sum_{m=1}^{\infty} \frac{(-1)^m \mu^m}{m!} b_m(\zeta), \tag{49}$$

and the series on the right side converges for all ζ and μ. Below, the function $L(\zeta, \mu)$ is called characteristic function of the system $\mathcal{S}_\tau$, and the equation

$$L(\zeta, \mu) = 0 \tag{50}$$

its characteristic equation.

Theorem 4. Under assumption (36) for the stability of the system $\mathcal{S}_\tau$, it is necessary and sufficient that the roots of equation (50) for a fixed μ are located outside the unit circle.

The proof of Theorem 4 is not provided, because it is identical with the proof of Theorem 6.2 in Lampe and Rosenwasser (2011).

The direct application of Theorem 4 for stability investigations is problematic, because expressions for the sum of series (49) obviously do not exist. However, on basis of this theorem we are able to derive strict sufficient stability conditions. For this reason, we introduce the polynomial

$$L_N(\zeta) = b_0(\zeta) + \sum_{m=1}^{N} \frac{(-1)^m \mu^m}{m!} b_m(\zeta) \tag{51}$$

and denote

$$b_{\max} = \max_{|\zeta|=1} |b_0(\zeta)|. \tag{52}$$

Theorem 5. Assume (36), and there are none of the numbers p_i, $(i = 1, ..., \rho)$ on the imaginary axis. Then the following statements hold:

i) For $0 \le t, u \le T$ and any real ν

$$|K(j\nu, t, u)| < M = \text{const}. \tag{53}$$

ii) Introduce for $n > 0$

$$q(n) = \frac{2TMe|\mu|}{\sqrt{n}}. \tag{54}$$

If the estimate

$$\min_{|\zeta|=1} |L_N(\zeta, \mu)| > \frac{q^{N+1}(N+1)}{1 - q(N+1)} b_{\max} \tag{55}$$

is valid, then characteristic equation (50) and the approximated equation

$$L_N(\zeta, \mu) = 0 \tag{56}$$

do not possess roots on the unit circle, and they have the same number of roots inside $|\zeta| = 1$. Hereby, when in particular the polynomial $L_N(\zeta, \mu)$ do not possess roots inside the circle $|\zeta| = 1$, then the system $\mathcal{S}_\tau$ is stable.

The proof is not provided, because it deviates only marginally from the proof of Theorem 6.4 in Lampe and Rosenwasser (2011).

6. EXAMPLE

Build the polynomial $L_2(\zeta, \mu)$ for the system

$$\frac{dy(t)}{dt} = ay(t) + \mu \cos(\omega t) \int_0^T y(t-\tau)\, d\tau, \tag{57}$$

where $a \neq 0$, $\omega = 2\pi/T$. The system coincides with a system of form (3), where $\chi = 1$, $a(t) = \cos\omega t$, $m(t) = 1$, $h = T$. Moreover,

$$W(p) = \frac{1}{p-a}\,. \tag{58}$$

For the construction of the kernel $K(\lambda, t, u)$ we apply formulae (25), (26). Actually, from (26) we obtain

$$a_k(\lambda, u) = \mathrm{e}^{\mathrm{j}\omega u}\, P_k(\lambda)^{+} + \mathrm{e}^{-\mathrm{j}\omega u}\, P_k(\lambda)^{-}, \tag{59}$$

where

$$P_k(\lambda)^{\pm} = \frac{1}{2}\frac{1-\mathrm{e}^{-\lambda T}}{\lambda + k\mathrm{j}\omega \mp \mathrm{j}\omega}\,. \tag{60}$$

Inserting (59), (60) into (25), we find

$$\begin{aligned} K(\lambda, t, u) = {} & \frac{\mathrm{e}^{\mathrm{j}\omega u}(1-\mathrm{e}^{-\lambda T})}{2} \times \\ & \frac{1}{T}\sum_{k=-\infty}^{\infty} \frac{\mathrm{e}^{k\mathrm{j}\omega(t-u)}}{(\lambda - a + k\mathrm{j}\omega)(\lambda + k\mathrm{j}\omega - \mathrm{j}\omega)} \\ & + \frac{\mathrm{e}^{-\mathrm{j}\omega u}(1-\mathrm{e}^{-\lambda T})}{2} \times \\ & \frac{1}{T}\sum_{k=-\infty}^{\infty} \frac{\mathrm{e}^{k\mathrm{j}\omega(t-u)}}{(\lambda - a + k\mathrm{j}\omega)(\lambda + k\mathrm{j}\omega + \mathrm{j}\omega)}\,. \end{aligned} \tag{61}$$

Formulae (29), (30) yield

$$\tilde{D}_1(\lambda) = \int_0^T K(\lambda, t, t)\,\mathrm{d}t. \tag{62}$$

Inserting here (61), for $t = u$ we obtain directly

$$\tilde{D}_1(\lambda) = 0. \tag{63}$$

For the calculation of $\tilde{D}_2(\lambda)$, we apply formula (35), which for $m = 2$, $k_1 = k$, $k_2 = n$ can be written in the form

$$\begin{aligned} \tilde{D}_2(\lambda) = {} & \frac{1}{T^2}\sum_{k=-\infty}^{\infty}\sum_{\substack{n=-\infty \\ n\neq k}}^{\infty} W_k(\lambda)W_n(\lambda) \times \\ & \int_0^T\int_0^T a_k(\lambda_1 t_1)a_n(\lambda_1 t_2)\Delta_{kn}(t_1, t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2, \end{aligned} \tag{64}$$

where, due to (33)

$$\Delta_{kn}(t_1, t_2) = 1 - \mathrm{e}^{n(n-k)\mathrm{j}\omega t_1}\,\mathrm{e}^{(k-n)\mathrm{j}\omega t_2}\,. \tag{65}$$

From (59) and (65), we achieve

$$\begin{aligned} \tilde{D}_2(\lambda) = {} & -\frac{1}{T^2}\sum_{k=-\infty}^{\infty}\sum_{\substack{n=-\infty \\ n\neq k}}^{\infty} W_k(\lambda)W_n(\lambda) \times \\ & \int_0^T a_k(\lambda, t_1)\,\mathrm{e}^{(n-k)\mathrm{j}\omega t_1}\,\mathrm{d}t_1 \int_0^T a_n(\lambda, t_2)\,\mathrm{e}^{(k-n)\mathrm{j}\omega t_2}\,\mathrm{d}t_2. \end{aligned} \tag{66}$$

Substituting here $n = k + l$, we obtain

$$\begin{aligned} \tilde{D}_2(\lambda) = {} & -\frac{1}{T^2}\sum_{k=-\infty}^{\infty}\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} W_k(\lambda)W_{k+l}(\lambda) \times \\ & \int_0^T a_k(\lambda, t_1)\,\mathrm{e}^{l\mathrm{j}\omega t_1}\,\mathrm{d}t_1 \int_0^T a_{k+l}(\lambda, t_2)\,\mathrm{e}^{-l\mathrm{j}\omega t_2}\,\mathrm{d}t_2. \end{aligned} \tag{67}$$

It follows from (59) that the integrals on the right side vanish for all l, except for $l = \pm 1$. Therefore, formula (67) essentially simplifies and takes the form

$$\begin{aligned} \tilde{D}_2(\lambda) = {} & -\frac{1}{T^2}\sum_{k=-\infty}^{\infty} W_k(\lambda)W_{k+1}(\lambda) \times \\ & \int_0^T a_k(\lambda, t_1)\,\mathrm{e}^{l\mathrm{j}\omega t_1}\,\mathrm{d}t_1 \int_0^T a_{k+l}(\lambda, t_2)\,\mathrm{e}^{-\mathrm{j}\omega t_2}\,\mathrm{d}t_2 \\ & -\frac{1}{T^2}\sum_{k=-\infty}^{\infty} W_k(\lambda)W_{k-1}(\lambda) \times \\ & \int_0^T a_k(\lambda, t_1)\,\mathrm{e}^{-\mathrm{j}\omega t_1}\,\mathrm{d}t_1 \int_0^T a_{k-l}(\lambda, t_2)\,\mathrm{e}^{\mathrm{j}\omega t_2}\,\mathrm{d}t_2. \end{aligned} \tag{68}$$

Moreover, notice that due to (59), (60)

$$\begin{aligned} \int_0^T a_k(\lambda, t_1)\,\mathrm{e}^{\pm\mathrm{j}\omega t_1}\,\mathrm{d}t_1 &= \frac{T}{2}\frac{1-\mathrm{e}^{\lambda T}}{\lambda + k\mathrm{j}\omega \pm \mathrm{j}\omega}\,, \\ \int_0^T a_{k+1}(\lambda, t_2)\,\mathrm{e}^{-\mathrm{j}\omega t_2}\,\mathrm{d}t_2 &= \frac{T}{2}\frac{1-\mathrm{e}^{\lambda T}}{\lambda + k\mathrm{j}\omega}\,, \\ \int_0^T a_{k-1}(\lambda, t_2)\,\mathrm{e}^{\mathrm{j}\omega t_2}\,\mathrm{d}t_2 &= \frac{T}{2}\frac{1-\mathrm{e}^{\lambda T}}{\lambda + k\mathrm{j}\omega}\,. \end{aligned} \tag{69}$$

With the help of (69) and (57), from (68) we find

$$\begin{aligned} \tilde{D}_2(\lambda) = {} & -\frac{(1-\mathrm{e}^{-\lambda T})^2}{4} \times \\ & \Bigg[\sum_{k=-\infty}^{\infty} \frac{1}{\lambda - a + k\mathrm{j}\omega}\frac{1}{\lambda - a + k\mathrm{j}\omega + \mathrm{j}\omega}\,\cdot \\ & \frac{1}{\lambda + k\mathrm{j}\omega}\frac{1}{\lambda + k\mathrm{j}\omega + \mathrm{j}\omega} \\ & -\sum_{k=-\infty}^{\infty} \frac{1}{\lambda - a + k\mathrm{j}\omega}\frac{1}{\lambda - a + k\mathrm{j}\omega - \mathrm{j}\omega}\,\cdot \\ & \frac{1}{\lambda + k\mathrm{j}\omega}\frac{1}{\lambda + k\mathrm{j}\omega - \mathrm{j}\omega}\Bigg]. \end{aligned} \tag{70}$$

Introduce the notations

$$\begin{aligned} W(\lambda)^{o} &= \frac{W(\lambda)}{\lambda} = \frac{1}{(\lambda - a)\lambda}\,, \\ W_k(\lambda)^{o} &= W^{o}(\lambda + k\mathrm{j}\omega). \end{aligned} \tag{71}$$

Then formula (70) yields

$$\begin{aligned} & \tilde{D}_2(\lambda) = \\ & -\frac{(1-\mathrm{e}^{-\lambda T})^2}{4}\sum_{k=-\infty}^{\infty} W_k(\lambda)^{o}\left[W_{k+1}(\lambda)^{o} + W_{k-1}(\lambda)^{o}\right], \end{aligned} \tag{72}$$

where

$$W_{k\pm 1}(\lambda)^{o} = \frac{1}{(\lambda - a + k\mathrm{j}\omega \pm \mathrm{j}\omega)(\lambda + k\mathrm{j}\omega \pm \mathrm{j}\omega)}\,. \tag{73}$$

Applying (73), formula (72) can be represented in the form

$$\tilde{D}_2(\lambda) = -\frac{(1-\mathrm{e}^{-\lambda T})^2}{2}\sum_{k=-\infty}^{\infty} W^{(1)}(\lambda + k\mathrm{j}\omega), \tag{74}$$

where

$$W^{(1)}(\lambda) = \frac{\lambda(\lambda - a) - \omega^2}{(\lambda - a)\left[(\lambda - a)^2 + \omega^2\right]\lambda(\lambda^2 + \omega^2)} \tag{75}$$

is a real rational function. The sum of series in (74) can be calculated with the help of formulae given in Rosenwasser and Lampe (2000). Here, rather extensive transformations show that

$$\sum_{k=-\infty}^{\infty} W^{(1)}(\lambda + k\mathrm{j}\omega) = \frac{2T}{a(a^2+\omega^2)}\, \frac{\mathrm{e}^{-\lambda T}(1-\mathrm{e}^{-aT})}{(1-\mathrm{e}^{-\lambda T})(1-\mathrm{e}^{-\lambda T}\,\mathrm{e}^{aT})}\,. \tag{76}$$

Hence, (74) and (76) yield

$$\tilde{D}_2(\lambda) = \frac{2T(\mathrm{e}^{aT}-1)}{a(a^2+\omega^2)}\, \frac{\mathrm{e}^{-\lambda T}(1-\mathrm{e}^{-\lambda T})}{1-\mathrm{e}^{-\lambda T}\,\mathrm{e}^{aT}}\,, \tag{77}$$

and finally

$$D_2(\zeta) = \frac{2T(\mathrm{e}^{aT}-1)}{a(a^2+\omega^2)}\, \frac{\zeta(1-\zeta)}{(1-\zeta\,\mathrm{e}^{aT})}\,. \tag{78}$$

Using the above results, we achieve the demanded function $L_2(\zeta,\mu)$ as

$$L_2(\zeta,\mu) = (1-\zeta\,\mathrm{e}^{aT}) + \frac{T\mu^2(\mathrm{e}^{aT}-1)}{a(a^2+\omega^2)}\,\zeta(1-\zeta). \tag{79}$$

If $a < 0$, this expression together with Theorem 5, allows to specify a region for the value $|\mu|$, for which system (57) is guaranteed to be stable.

7. CONCLUSIONS

The application of Fredholm's theory on integral equations has allowed an exact description of linear periodic systems with distributed delay. On this basis, the characteristic equation for the closed system was found directly from the given system components without solving a boundary value problem in state space. Together with the results of the Fredholm theory on integral equations of the second kind, a method was developed, that yields to state sufficient stability conditions for the closed loop. This conditions are formulated as location of the roots of certain polynomials with respect to the unit circle. Therefore ordinary stability criteria could be used. Hereby, as it was shown by an example, beside numerical solutions, also closed expressions in the original parameters could be achieved.

Since the concepts are not restricted to the SISO case, future work will be directed to the extension on MIMO systems. The authors are preparing a toolbox in MATLAB® to support various steps in analysis and design of such systems.

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