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Performance and Stability for Combined Economic and Regulatory Control in MPC

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Abstract: We propose a dual-objective MPC formulation in which the dual objective is the convex combination of an economic- and regulatory stage cost, using a specially formulated state- and input-dependent dynamic weight function. The purpose of the dynamic weight function is to promote increased economic performance while ensuring asymptotic stability for the economically optimal steady-state setpoint. First, sufficient conditions are derived for which the dual-objective MPC value function is a Lyapunov candidate function. Next, we propose a weight function which satisfies these conditions. We implement the combined economic and regulatory MPC, with proposed weight function, in an isothermal CSTR numerical case study, which illustrates how economics are emphasized during process transients, while retaining stability by emphasizing the regulatory cost close to the setpoint.

Keywords: Multiple-criterion optimisation, Predictive control, Asymptotic stability, Lyapunov functions, Optimality, Adaptive weighting

1. INTRODUCTION

Model Predictive Control (MPC) is a well-established method for the optimal control of linear and nonlinear systems (Rawlings and Mayne, 2009). The practical utilization thereof has seen increased preference in industry, ranging from petro-chemical and food processing up to aerospace applications (Qin and Badgwell, 2003). Results pertaining stability, optimality and robustness of the underlying MPC formulations are now well documented (Rawlings and Mayne, 2009; Mayne et al., 2000; Grüne and Pannek, 2012). However, most of these results often apply only to the special case of *standard* MPC, also called regulatory MPC. Regulatory MPC often optimize a simplistic convex and quadratic cost function which embeds steady-state economic criteria in a setpoint tracking formulation. However, the true economic cost involved for operating a process may be far from the cost as measured by a convex and quadratic cost function. Therefore, the resulting control law for regulatory MPC may often prove suboptimal for minimizing actual economic costs over a receding horizon (Angeli et al., 2012).

Recently, Economic MPC (EMPC) attempts to reconcile the control design with process economics by replacing the regulatory stage cost with the actual economic stage cost (Rawlings and Amrit, 2009). The latter has shown to outperform economic performance of regulatory MPC in the transients, as well as on time average, compared to steady-state operation (Angeli et al., 2012; Amrit et al., 2011). Results on stability analysis of EMPC are still at an early stage. Results to date for stability either depends on restrictive assumptions made on convexity, linearity or strict dissipativity of the underlying process models and objectives (Diehl et al., 2011; Angeli et al., 2012). Even though dissipativity theory is an elegant theoretical concept for analyzing stability in EMPC, finding a dissipative function that satisfies strict dissipativity often proves to be hard (Angeli et al., 2012; Amrit et al., 2011).

It has been observed, via simulation (Angeli et al., 2011; Amrit et al., 2011), stability, or at least convergence, can be obtained for EMPC if the convexity of the underlying stage cost can be tuned up sufficiently. In the context of dissipativity theory (Angeli et al., 2012), one can modify the convexity of the underlying cost function by adding an appropriately chosen convex term, and consequently enforce strict dissipativity. The latter, however, still requires the solution of a nontrivial off-line optimization problem. Tuning the convexity of an EMPC cost function bears close resemblance to dual-objective MPC (Maree and Imsland, 2011) in which one considers the weighted economic and regulatory cost objective. The technical challenge is how to tune the weight in such a way that convergence (strict dissipativity (Angeli et al., 2012), or sufficient convexity (Angeli et al., 2011; Amrit et al., 2011)) is guaranteed, while simultaneously promoting economic performance during process transients, by optimizing a favourably weighted economic stage cost over a receding horizon. To date, it seems (Rawlings et al., 2012), no intuitive, and practical viable method has yet been proposed in how to tune a weighted dual-objective MPC formulation, that guarantees asymptotic stability, while also promoting economic benefits during process transients in an optimal way.

1.1 Contribution

This work aligns with the ideas stipulated in Angeli et al. (2012) in which the convexity of the underlying stage cost

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of interest for EMPC is tuned to achieve asymptotic stability, while promoting economic objectives during transients. In particular, focus is placed on the weighted dual-objective (economic-regulatory) formulation (Maree and Imsland, 2011), where the aim is to formulate conditions on the weighted dual-objective function such that good control performance (asymptotic tracking of optimal economic steady-state setpoints) and economic performance (minimize actual economic costs over a receding horizon) are achieved in an optimal way.

Contributions in this work are the following: (i) sufficient conditions are stipulated, such that for a weighted dual-objective MPC formulation, the resulting optimal MPC value function admits being Lyapunov, where asymptotic stability of the origin follows; (ii) optimality of a general weight function, which favourably weights the economic stage cost, is analysed by inspecting Karush-Kuhn-Tucker (KKT) conditions of optimality. An explicit weight function is subsequently proposed by embedding these KKT conditions in the proposed weight function; (iii) practical suggestions are given in how the proposed weight function can be implemented for a dual-objective MPC formulation. The theory is applied to an isothermal CSTR case study.

NOTATION

The symbols $\mathbb{I}$ and $\mathbb{R}$ define integer and real numbers respectively. A vector z consists of components z_i, $i \in \mathbb{I}_{1:n_z}$ such that $z \in \mathbb{R}^{n_z}$. A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is a class-$\mathcal{K}$ function if $\gamma(0) = 0$, continuous, and strictly increasing.

2. PRELIMINARIES

We consider the nonlinear, discrete-time system model

$$x^{+} = f(x, u) \tag{1}$$

with state, $x \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$, and control input $u \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$ respectively. We define the mixed constraint set $(x(k), u(k)) \in \mathbb{Z}$, $\forall k \in \mathbb{I}_{\geqslant 0}$ for a compact set $\mathbb{Z} = \mathbb{X} \times \mathbb{U}$.

2.1 Dual-objective MPC

Consider the economic cost function, $l_e : \mathbb{Z} \to \mathbb{R}$. The solution to the economic steady-state optimization problem

$$(x_s, u_s) := \arg \min_{(x,u)\in\mathbb{Z}} \{l_e(x, u) \,|x = f(x, u)\} \tag{2}$$

defines the optimal economic steady-state, (x_s, u_s). In regulatory MPC, (x_s, u_s) is usually embedded in a regulatory stage cost function, $l_r : \mathbb{Z} \to \mathbb{R}$, which measures the tracking distance from (x_s, u_s). We will adopt a quadratic regulatory stage cost function

$$l_r(x, u) := \tfrac{1}{2} \|x - x_s\|_Q^2 + \tfrac{1}{2} \|u - u_s\|_R^2 \tag{3}$$

in which Q and R are positive (semi-)definite matrices used for tuning. Since the true economic cost, l_e, involved for operating the process (1), can be different from the tracking cost measured by a regulatory stage cost function, l_r; it may prove beneficial (Maree and Imsland, 2011), from an economical point of view, to combine the economic and regulatory cost objectives in a weighted, dual-objective function. For such purposes, consider the general continuous weight function $\mu : \mathbb{Z} \to [0, 1]$. Then, we can define the weighted, dual-objective stage cost functional, $l_\mu : \mathbb{Z} \to \mathbb{R}$,

$$l_\mu(x, u) := \mu(x, u)\, l_{\bar{e}}(x, u) + (1 - \mu(x, u))\, l_r(x, u) \tag{4}$$

in which $l_{\bar{e}}(x, u) := l_e(x, u) - l_e(x_s, u_s)$. This dual-objective stage cost function (in addition to the economic steady-state, (x_s, u_s), embedded in l_r) promotes increased economic performance during dynamic operation, by favourably tuning μ close to one. A technical concern, however, is that no intuitive method has yet been presented in how to determine the optimal choice of μ (for increased economic performance), while still retaining desirable stability characteristic as guaranteed for regulatory MPC (Rawlings et al., 2012). In regulatory MPC, one often considers a terminal cost function in the MPC formulation to promote recursive feasibility, and stable regulatory MPC optimization problems (Mayne et al., 2000). We define such a terminal cost function, $V_f(x) \geqslant 0$, on a compact local neighbourhood $\mathcal{X}_f \subseteq \mathbb{X}$ of x_s where $V_f(x_s) = 0$ holds (see also Rawlings and Mayne (2009)). Next, we define the dual-objective MPC value function for a N-step horizon

$$V_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} l_\mu(x(k), u(k)) + V_f(x(N)) \tag{5}$$

in which $\mathbf{u} := [u(0), u(1), \cdots, u(N-1)]$, and $x = x(0)$ being the control sequence and initial state, respectively. The dual-objective MPC optimal control problem is then defined

$$\min_{\mathbf{u}} V_N(x, \mathbf{u}) \tag{6a}$$
$$\text{s.t. } x(k+1) = f(x(k), u(k)),\ \forall k \in \mathbb{I}_{0:N-1} \tag{6b}$$
$$(x(k), u(k)) \in \mathbb{Z},\ \forall k \in \mathbb{I}_{0:N-1} \tag{6c}$$
$$x(N) \in \mathcal{X}_f \tag{6d}$$

The optimal MPC value function and control sequence are defined $V_N^0(x)$ and $\mathbf{u}^0(x)$ respectively. The MPC receding-horizon control law is the first optimal control input, $\kappa_N(x) := u^0(0; x)$, such that the system (1) evolves according to this control law with closed-loop trajectories

$$x^{+} = f(x, \kappa_N(x)) \tag{7}$$

The admissible set for the dual-objective MPC problem (6) is the set of state-control sequence pairs in which the initial state x can be steered to $\mathcal{X}_f$, with an admissible input sequence $\mathbf{u}$, while satisfying the state constraints

$$\mathcal{Z}_N := \{(x, \mathbf{u}) \,|(\phi(k; x, \mathbf{u}), u(k)) \in \mathbb{Z},\ \forall k \in \mathbb{I}_{0:N-1},\ \phi(N; x, \mathbf{u}) \in \mathcal{X}_f\}$$

in which $\phi(k; x, \mathbf{u})$ defines the solution of (1) at sample time $k \in \mathbb{I}_{0:N}$ for initial state x and control sequence $\mathbf{u}$. The feasible set of initial states is the projection of $\mathcal{Z}_N$ onto $\mathbb{X}$

$$\mathcal{X}_N := \left\{x \in \mathbb{X} \,\middle|\, \exists \mathbf{u} \in \mathbb{U}^N \text{s.t.}\, (x, \mathbf{u}) \in \mathcal{Z}_N \right\}$$

The feasible set $\mathcal{X}_N$ is not empty since it contains x_s.

Assumption 1. There exists an admissible solution to the steady-state optimization problem (2), being unique in some local region of interest.

Assumption 2. (Continuity of cost and system).
The regulatory cost, $l_r(\cdot)$, economic cost, $l_e(\cdot)$, and system, $f(\cdot)$, are continuously differentiable.

Assumption 3. (Basic Stability). There exists an admissible control input, $u \in \mathbb{U}$, such that

$$V_f(f(x, u)) - V_f(x) \leqslant -(1 + \varepsilon) l_r(x, u), \quad \forall x \in \mathcal{X}_f \tag{8}$$

holds in which $\varepsilon \in \mathbb{R}_{>0}$ is some small scalar constant. Define the terminal control law, $\kappa_f : \mathcal{X}_f \to \mathbb{U}$, as an admissible control input for which (8) holds.

3. STABILITY FOR DUAL-OBJECTIVE MPC

We proceed in presenting sufficient conditions for the dual-objective MPC formulation (6), such that the optimal steady-state point, (x_s, u_s), is asymptotically stable with respect to closed-loop system trajectories (7).

Assumption 4. [Bounds on l_μ]. There exists a class-$\mathcal{K}$ function $\gamma_\mu(\cdot)$, an admissible control for all $u \in \mathbb{U}$, and a unique function $\mu(x,u) \in \mathbb{Z} \to [0,1]$, not identically zero, such that

$$\gamma_\mu(|x - x_s|) \leqslant l_\mu(x,u) \leqslant (1+\varepsilon) l_r(x,u) \tag{9}$$

in which $\varepsilon \in \mathbb{R}_{>0}$ being some small scalar constant.

Note 1. It is clear that Assumption 4 is satisfied for a weighting function of $\mu(x,u) = 0$ for all $(x,u) \in \mathbb{Z}$, since $l_\mu(x,u)$ evaluates $l_r(x,u)$ for $\mu(x,u) = 0$. We emphasize that it is desirable to have $\mu(x,u)$ not identically zero such that we can directly optimize process economics. For the design of $\mu(x,u)$ that evaluates not identically zero, but dependent on the current state of process operation, we refer the reader to Section 4.

The inclusion of a weighted, economic cost function $\mu(x,u) l_{\bar{e}}(x,u)$ in the dual-objective cost formulation (4) promotes increased economic performance by optimizing actual economic costs over a receding-horizon. It is possible, however, that the inclusion of $\mu(x,u) l_{\bar{e}}(x,u)$ may imply that (4) becomes non-convex, or indefinite, and it may occur that $l_\mu(x,u) < l_\mu(x_s,u_s)$ (Angeli et al., 2012). As consequence, one cannot analyse for asymptotic stability using the general framework presented by Mayne et al. (2000). Instead, one needs to consider an alternative stability analysis framework (Diehl et al., 2011; Angeli et al., 2012) which require additional assumptions either on convexity, or dissipativty. In contrast to the contributions of Diehl et al. (2011); Angeli et al. (2012), this work entails finding a weight function $\mu(x,u)$, which explicitly satisfies Assumption (4). To be shown is how the latter can be used to analyse asymptotic stability of the dual-objective MPC formulation.

Note 2. Scaling of stage costs $l_r(x,u)$ and $l_{\bar{e}}(x,u)$, with respect to each other, plays an important role in how a dynamic weight (4) will adapt to satisfy the bounds (9). For this work we assume that $l_r(x,u)$ and $l_{\bar{e}}(x,u)$ are appropriately scaled on the defined set $\mathbb{Z}$.

Lemma 3.1. (Terminal cost decrease). Suppose Assumptions 3-4 hold. Then, there exists a control input $u \in \mathbb{U}$ such that

$$V_f(f(x,u)) - V_f(x) + l_\mu(x,u) \leqslant 0, \ \forall x \in \mathcal{X}_f \tag{10}$$

Proof. From Assumption 4, manipulation on the upper bound of (9) reveals that

$$\mu(x,u)\left[l_{\bar{e}}(x,u) - l_r(x,u)\right] - \varepsilon l_r(x,u) \leqslant 0 \tag{11}$$

holds for all $x \in \mathcal{X}_N \supseteq \mathcal{X}_f$. Next, expand relation (10)

$$V_f(f(x,u)) - V_f(x) + l_\mu(x,u) \tag{12a}$$
$$= V_f(f(x,u)) - V_f(x) + l_\mu(x,u) \tag{12b}$$
$$+ \varepsilon l_r(x,u) - \varepsilon l_r(x,u) \tag{12c}$$
$$= V_f(f(x,u)) - V_f(x) + (1+\varepsilon) l_r(x,u) \tag{12d}$$
$$+ \mu(x,u)\left[l_{\bar{e}}(x,u) - l_r(x,u)\right] - \varepsilon l_r(x,u) \tag{12e}$$

Assumption 3 implies the existence of an admissible control input $u = \kappa_f(x)$ such that (12d) is negative semi-definite for all $x \in \mathcal{X}_f$. The latter, in conjunction with (11), concludes the proof.

Lemma 3.2. (Bounds on V_N^0). Let Assumptions 1 and 4 hold. Then, $V_N^0(x_s) = 0$, and there exist class-$\mathcal{K}$ functions $\gamma_1(\cdot), \gamma_2(\cdot)$ such that

$$\gamma_1(|x - x_s|) \leqslant V_N^0(x) \leqslant \gamma_2(|x - x_s|), \ \forall x \in \mathcal{X}_N \tag{13}$$

Proof. The lower bound in (13) is clear from Assumption 4. The upper bound in (13) follows since the horizon length N is finite; $\mathbb{U}$ is bounded; and, $V_f(x)$ is defined on the closed set $\mathcal{X}_f$. We evaluate the stage- and terminal-cost, given the optimal steady-state (x_s, u_s), as $l_\mu(x_s, u_s) = 0$, and $V_f(x_s) = 0$, respectively. The latter in conjunction with (5) subsequently implies $V_N^0(x_s) = 0$.

Lemma 3.3. (Descent of V_N^0). Suppose that Assumption 4 holds. Then, for all $x \in \mathcal{X}_N$ there exists a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that

$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leqslant -\gamma(|x - x_s|), \ \forall x \in \mathcal{X}_N \tag{14}$$

Proof. Consider the optimal control sequence, $\mathbf{u}^0(x)$, which evaluates $V_N^0(x)$

$$\mathbf{u}^0(x) := \left[u^0(0;x), u^0(1;x), \cdots, u^0(N-1;x)\right]$$

The corresponding system trajectory of (1), which evolves with $\mathbf{u}^0(x)$ is

$$\mathbf{x}^0(x) := \left[x^0(0;x), x^0(1;x), \cdots, x^0(N;x)\right]$$

in which $x^0(0;x) = x$. Next, define an admissible control sequence

$$\tilde{\mathbf{u}}(x) := \left[u^0(1;x), \cdots, u^0(N-1;x), \kappa_f\left(x^0(N;x)\right)\right]$$

and corresponding evolving admissible state sequence

$$\tilde{\mathbf{x}}(x) := \left[x^0(1;x), \cdots, f\left(x^0(N;x), \kappa_f\left(x^0(N;x)\right)\right)\right]$$

Express the MPC value function for the next evolved state, given the previously defined admissible sequences,

$$\begin{aligned} V_N(f(x,\kappa_N(x)), \tilde{\mathbf{u}}(x)) &= V_N^0(x) - l_\mu\left(x, u^0(0;x)\right) \\ &+ V_f\left(f\left(x^0(N;x), \kappa_f\left(x^0(N;x)\right)\right)\right) - V_f\left(x^0(N;x)\right) \\ &+ l_\mu\left(x^0(N;x), \kappa_f\left(x^0(N;x)\right)\right) \end{aligned}$$

From the optimality of (6) we have $V_N^0(f(x,\kappa_N(x))) \leqslant V_N(f(x,\kappa_N(x)), \tilde{\mathbf{u}}(x))$. From Assumption 4, and applying Lemma 3.1, we conclude for all $x \in \mathcal{X}_N$ the closed-loop trajectories (7) satisfy

$$V_N^0(f(x,\kappa_N(x))) - V_N^0(x) \leqslant -l_\mu(x, \kappa_N(x)) \leqslant -\gamma_\mu(|x - x_s|)$$

Theorem 3.1. (Asymptotic stability). Let Assumptions 1-4 hold. Then, the steady-state solution x_s is an asymptotically stable equilibrium point for the closed-loop trajectories (7) with region of attraction $\mathcal{X}_N$.

Proof. Lemmata 3.2-3.3 are sufficient conditions for the optimal MPC value to be a Lyapunov function (Mayne et al., 2000).

4. ON OPTIMALITY OF A WEIGHT FUNCTION

As discussed earlier, a benefit of choosing[1] $\mu > 0$ is that the resulting receding horizon control law promotes increased economic performance. To optimize for increased economic performance, it is desirable to have μ as large as possible (close to one). Remark 1, and the discussion thereafter, emphasize the technical challenge of selecting a weight function which promotes increased economic performance while *simultaneously* satisfying some stabilizing criteria. In this work, we derive an economic optimal weight function which strictly satisfies Assumption 4, required for asymptotic stability. It is sufficient to define such an optimal weight function, $\mu^0 : \mathbb{Z} \to \mathbb{R}$, as the optimal solution the parametric optimization problem

$$\mu^0 := arg \min_{\mu} \; -\mu \tag{16a}$$

$$\text{s.t.} \quad c_1(\mu) := \; (1+\varepsilon) \, l_r - \mu l_{\bar{e}} - (1-\mu) \, l_r \geqslant 0 \tag{16b}$$

$$c_2(\mu) := \; \mu l_{\bar{e}} + (1-\mu) \, l_r - \gamma_\mu \geqslant 0 \tag{16c}$$

$$c_3(\mu) := \; (1-\mu) \geqslant 0 \tag{16d}$$

$$c_4(\mu) := \; \mu \geqslant 0 \tag{16e}$$

The function μ^0 defines the optimal choice for μ that gives maximum economic performance, while satisfying the stability conditions (9), for the current point of operation (x, u). μ^0 is therefore a natural choice for the adaptive weight function.

Note 3. Parametric optimization problem (16) is always feasible for $\mu = 0$.

Proposition 1. The optimal weight function, $\mu^0 : \mathbb{Z} \to \mathbb{R}$, satisfies Assumption 4.

Proof. By inspection, feasibility of the constraints (16b-16c) implies fulfilment of Assumption 4.

Proposition 2. (Explicit Optimal Weight).
Suppose Assumption 2 holds. The optimal weight function μ^0 for (16) is explicitly defined for all $(x, u) \in \mathbb{Z}$ as

$$\mu^0 := \begin{cases} \mathbf{min}\{1, \varepsilon l_r/(l_{\bar{e}} - l_r)\}, & l_{\bar{e}} > l_r \\ \mathbf{min}\{1, (\gamma_\mu - l_r)/(l_{\bar{e}} - l_r)\}, & l_{\bar{e}} < l_r \\ c, & l_{\bar{e}} = l_r \end{cases} \tag{17}$$

for any $c \in [0, 1]$.

Proof. Write inequalities (16b-16e) in the compact form $a\mu \geqslant b$. Define the vectors a and b respectively as

$$a := \left[-\left(l_{\bar{e}} - l_r\right), \; \left(l_{\bar{e}} - l_r\right), \; -1, \; 1\right]^T, \tag{18a}$$

$$b := \left[-\varepsilon l_r, \; \left(\gamma_\mu - l_r\right), \; -1, \; 0\right]^T \tag{18b}$$

Next, define the Lagrangian function

$$\mathcal{L}(\mu, \lambda) := -\mu - \lambda^T (a\mu - b) \tag{19}$$

in which λ is a vector of Lagrange multipliers with components $\lambda_i, i \in \mathbb{I}_{1:4}$. Let μ^0 be an optimal solution to (16). Then, there exists a Lagrange multiplier vector λ^0 with components $\lambda_i^0, i \in \mathbb{I}_{1:4}$ such that the following conditions are satisfied at (μ^0, λ^0) (Nocedal and Wright, 2006)

$$\nabla_\mu \mathcal{L}\left(\mu^0, \lambda^0\right) := -1 - \lambda^{0T} a = 0 \tag{20a}$$

$$a\mu^0 - b \geqslant 0 \tag{20b}$$

$$\lambda_i^0 c_i\left(\mu^0\right) = 0, \; \forall i \in \mathbb{I}_{1:4} \tag{20c}$$

$$\lambda_i^0 \geqslant 0, \; \forall i \in \mathbb{I}_{1:4} \tag{20d}$$

These conditions, known as the KKT conditions, are necessary and sufficient for the optimality of the linear (parametric) program (16). We proceed by arguing that (17) implies that the KKT conditions hold. Suppose $l_{\bar{e}} = l_r$. Then, the optimization problem (16) becomes degenerate, meaning, any $\mu^0 \in [0, 1]$ gives equivalent performance. For the general case in which $l_{\bar{e}} \neq l_r$, inspection of the equality constraints (16b)-(16c) reveals that if $c_1(\mu)$ is active then $c_2(\mu)$ must be inactive. The reverse also holds. Now consider the case when constraints $c_i(\mu)$ for any $i = \{1, 2\}$ are active. Since the constraints are linear, we can directly evaluate $\mu^0 = \frac{b_i}{a_i}$ and $\lambda_i^0 = \frac{-1}{a_i}$. From property (20d), $\lambda_i^0 \geqslant 0$, we can conclude that a_i must be negative since b_i for any $i = \{1, 2\}$ is negative. It follows that if $a_1 = -\left(l_{\bar{e}} - l_r\right) < 0$ then either $c_1(\mu)$ or $c_3(\mu)$ will be active. Hence, for $l_{\bar{e}} > l_r$ it follows that μ^0 is upper bounded by $\min\{1, b_1/a_1\}$. We apply similar reasoning in the case when $a_1 = -\left(l_{\bar{e}} - l_r\right) > 0$ or equivalently $a_2 < 0$. It follows that either $c_2(\mu)$ or $c_3(\mu)$ will be active. Hence, for $l_{\bar{e}} < l_r$ it follows that μ^0 is upper bounded by $\min\{1, b_2/a_2\}$.

5. IMPLEMENTATION CONSIDERATIONS FOR WEIGHT

Section 4 derived the KKT conditions (20) of optimality for the dual-objective weight function μ which were used in defining an explicit weight function (17). The explicit weight function (17), however, is not suitable for immediate numerical implementation when a sufficiently smooth and differential dual-objective MPC value function (5) is required (desirable for most optimization environments). We subsequently proceed in presenting a candidate weight function which approximates the optimal weight function μ^0 and is differentiable and sufficiently smooth for implementation. First, we define two intermediate weight functions.

Definition 1. (Intermediate weights). We define the intermediate weight functions $\bar{\mu}_{lb} : \mathbb{Z} \to \mathbb{R}$ and $\bar{\mu}_{ub} : \mathbb{Z} \to \mathbb{R}$

$$\bar{\mu}_{lb}(x, u) := \frac{(\gamma_\mu - l_r)(l_{\bar{e}} - l_r)}{(l_{\bar{e}} - l_r)^2 + \kappa^2}; \; \bar{\mu}_{ub}(x, u) := \frac{\varepsilon l_r (l_{\bar{e}} - l_r)}{(l_{\bar{e}} - l_r)^2 + \kappa^2} \tag{21}$$

where κ is chosen sufficiently small.

For the remainder of this section we will adopt the functional approximations for the *minimum* and *maximum* functional operators,

$$\mathbf{min}\{x, y\} \approx \underline{\mathbf{min}}\{x, y\} \quad := \frac{(x+y) - \sqrt{(x-y)^2 + \rho^2}}{2} \tag{22a}$$

$$\mathbf{max}\{x, y\} \approx \underline{\mathbf{max}}\{x, y\} \quad := \frac{(x+y) + \sqrt{(x-y)^2 + \rho^2}}{2} \tag{22b}$$

where ρ is some sufficiently small constant. Using the intermediate weight functions (21), and functional approximations (22), we can proceed in formulating a candidate weight function, $\mu : \mathbb{Z} \to \mathbb{R}$.

Definition 2. (Candidate weight function). We define the candidate weight function, $\mu : \mathbb{Z} \to \mathbb{R}$, for all $(x, u) \in \mathbb{Z}$ as

$$\mu(x, u) := \underline{\mathbf{max}}\{\underline{\mathbf{min}}\{1, \bar{\mu}_{lb}\}, \underline{\mathbf{min}}\{1, \bar{\mu}_{ub}\}\} \tag{23}$$

[1] To present the work in a compact manner, the following short-hand notation will be adopted, for present and future definitions, throughout Sections 4-5: $l_r := l_r(x, u)$, $l_{\bar{e}} := l_{\bar{e}}(x, u)$, $l_\mu := l_\mu(x, u)$, $\mu := \mu(x, u)$, $\bar{\mu} := \bar{\mu}(x, u)$, $\mu^0 := \mu^0(x, u)$, and $\gamma_\mu := \gamma_\mu(|x - x_s|)$.

in which κ and ρ are sufficiently small scalars defined for (21) and (22), respectively.

Theorem 5.1. (Optimality of candidate weight). Suppose Assumption 4 holds. Given the dual-objective MPC formulation (6), the candidate weight function (23) approximates the optimal weight function (17) in the limits $\kappa \to 0$ and $\rho \to 0$, for all $(x, u) \in \mathbb{Z}$.

Proof. From inspection, it is clear that in the limit $\kappa \to 0$

$$\underline{\mathbf{min}}\{1, \bar{\mu}_{lb}\} \to \underline{\mathbf{min}}\{1, (\gamma_\mu - l_r) / (l_{\bar{e}} - l_r)\} \tag{24a}$$

$$\underline{\mathbf{min}}\{1, \bar{\mu}_{ub}\} \to \underline{\mathbf{min}}\{1, \varepsilon l_r / (l_{\bar{e}} - l_r)\} \tag{24b}$$

Next, suppose $l_{\bar{e}} = l_r$. Then, $l_\mu = l_r \geqslant \gamma_\mu$ by definition of l_μ, and Assumption 4. Also $\varepsilon l_r \geqslant 0$ by definition. It follows that when $l_{\bar{e}} = l_r$ we have $\bar{\mu}_{lb} = -\infty$ which implies $\underline{\mathbf{min}}\{1, \bar{\mu}_{lb}\} \to -\infty$ in the limit $\rho \to 0$. Similarly, $\bar{\mu}_{ub} \to \infty$ which results in $\underline{\mathbf{min}}\{1, \bar{\mu}_{ub}\} \to 1$. Subsequently, from Definition 2 it follows that $\mu \to 1$ which approximates $\mu^0(x, u)$ for $l_{\bar{e}} = l_r$. Next, for $l_{\bar{e}} > l_r$ we have $\underline{\mathbf{min}}\{1, \bar{\mu}_{lb}\} < 0$, and in conjunction with (23) implies μ will evaluate as (24b). Similarly, for the case in which $l_{\bar{e}} < l_r$ it follows that $\underline{\mathbf{min}}\{1, \bar{\mu}_{ub}\} < 0$, and from (23) we conclude that μ evaluate as (24a) which in turn corresponds with the optimal weight $\mu_0(x, u)$.

6. NUMERICAL EXAMPLE

For a numerical example we will consider the continuous-flow, stirred-tank reactor (CSTR) with heat flux control. The CSTR case exhibits the following stoichiometry (consecutive-competitive) reactions

$$R \to P_1; \quad R \to P_2$$

with R being the reaction concentration, P_1 the desired product, and P_2 the waste product. The dimensionless form of the conservation equations of the CSTR is expressed (Bailey et al., 1971)

$$\begin{aligned} \dot{x}_1 &= 1 - a_1 e^{-1/x_3} x_1^\alpha - a_2 e^{-\delta/x_3} x_1 - x_1 \\ \dot{x}_2 &= a_1 e^{-1/x_3} x_1^\alpha - x_2 \\ \dot{x}_3 &= u - x_3 \end{aligned} \tag{25}$$

The dimensionless control input u is the heat-flux. States x_1, x_2 and x_3 are the dimensionless concentration R, desired product P_1 and waste product P_2 respectively. The following parameter values $\alpha = 2$, $\delta = 0.55$, $a_1 = 10^4$, $a_2 = 400$ are assumed.

6.1 Performance criteria

The economic performance criteria of process operation is to maximize the amount of desired product P_1 being produced, i.e., minimize the economic objective $l_e(x, u) = -x_2$. The optimal economic steady-state, (x_s, u_s), is obtained by solving the steady-state optimization problem (2), which evaluates $x_s = [0.083,\ 0.085, 0.149]^T$ and $u_s = 0.149$. This economic steady-state is embedded in a regulatory quadratic cost objective, $l_r(x, u)$, where the choice of penalty weight matrices (Q, R) are stipulated according to Table 1. A prediction-horizon of $N = 30$ time steps, sampling time of $\Delta t = 1/6$ seconds, and optimization horizon of $t_f = 5$ seconds were implemented. Assumption 3 is satisfied for a terminal constraint MPC formulation (Mayne et al., 2000), in which we choose $\mathcal{X}_f = \{x_s\}$. The interested reader is referred to Mayne et al. (2000); Rawlings and Mayne (2009) for a comprehensive coverage on formulating a MPC optimal control problem in which $\mathcal{X}_f$ is chosen as some sub-level set of a Lyapunov function, being control invariant for some admissible control law $\kappa_f : \mathbb{X} \to \mathbb{U}$. The latter consequently allows for a larger admissible region of operation, however, this is not in the scope of this work.

6.2 Implementation details

The MPC formulation (6) was numerically solved in a Modelica-based open source platform called JModelica.org (Åkesson et al., 2010) (ver. 1.11). For integration, and evaluation of sensitivities, we incorporated the CasADi package. A Legendre-Gauss-Radau collocation scheme was used for discretization, where the subsequent NLP problem was solved using IPOPT. The constants $\kappa = 10^{-2}$ and $\rho = 10^{-4}$ were considered for the candidate weight function (23). For Assumption 4, we chose $\varepsilon = 10^{-3}$ and $\gamma_\mu(|x - x_s|) = 10^{-3}\|x - x_s\|_2$.

6.3 Results

The CSTR case under investigation is known to have improved economic process performance during nonsteady, cyclical process operation (Lee and Bailey, 1980). For a pure economic objective (select $\mu(x, u) = 1$ in (4)) we observe from Figure 1 how the resulting receding horizon control law exploits nonsteady process operation for increased economic performance, on average. Such nonsteady operation, however, is not always desirable for industrial process operation. Instead, a control philosophy in which one optimize process economics during process transients (when nonsteady process operation is unavoidable), and then within a finite period of time start tracking some admissible steady-state, is more desirable. The candidate weight function (23), incorporated in (5), adopts these operational philosophies simultaneously: (i) economic performance is increased (see Proposition 2) during process transients by optimizing a favourably weighted

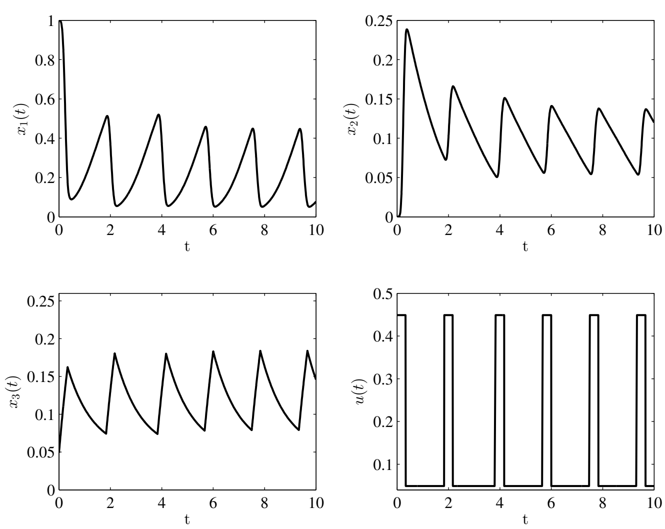

Fig. 1. Economic MPC with a receding horizon control law which exploits nonsteady process operation for increased economic performance.

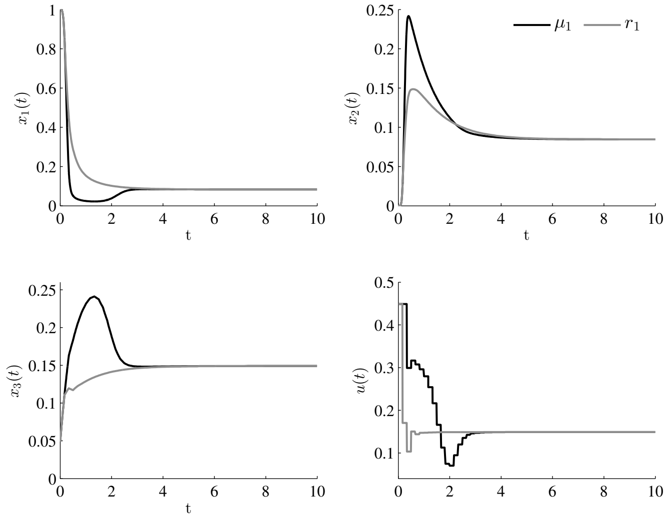

Fig. 2. Closed-loop state and control trajectories for: (i) a regulatory MPC formulation, r_1, and (ii) a weighted dual-objective MPC formulation, μ_1; $((Q, R) = ([1.0, 10, 1], 0.1)$ for regulatory weights used for (i-ii)).

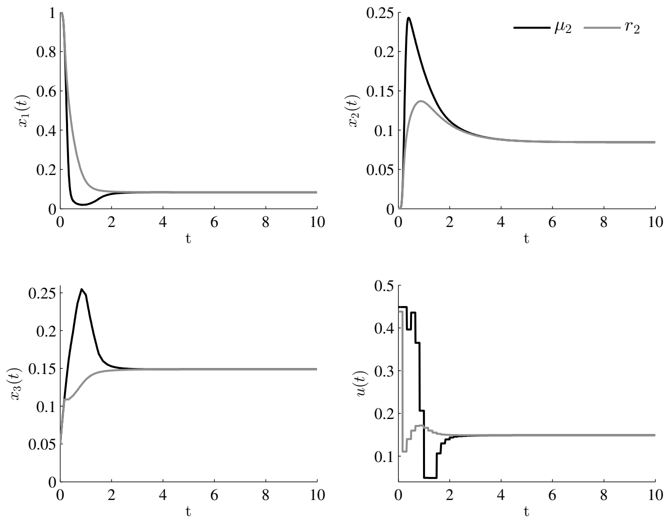

Fig. 3. Closed-loop state and control trajectories for: (i) a regulatory MPC formulation, r_2, and (ii) a weighted dual-objective MPC formulation, μ_2; $((Q, R) = ([0.1, 5, 10], 0.5)$ for regulatory weights used for (i-ii)).

economic cost objective over a receding horizon. Figures 2-3 illustrates closed-loop profiles. Table 1 tabulates realized cumulative economic gains, for two experiments $i \in \{1, 2\}$, where either a dual-objective stage cost l_{μ_i}, or, regulatory stage cost, l_{r_i} is optimized over a receding horizon; (ii) regulatory control objectives are promoted by admissibly start tracking steady-state economic setpoints after initial process transients. Figure 4 illustrates how the candidate weight function (23) adapts for different control philosophies, i.e., economic performance during process transients (μ close to one) and regulatory (control) performance thereafter (μ converge to zero). Figure 4 also illustrates that for the chosen candidate weight function (23), one retains asymptotic stability of (x_s, u_s) by observing strict monotonicity of the optimal MPC value function $V_N^0(x)$. The latter validates Theorem 3.1 by simulation.

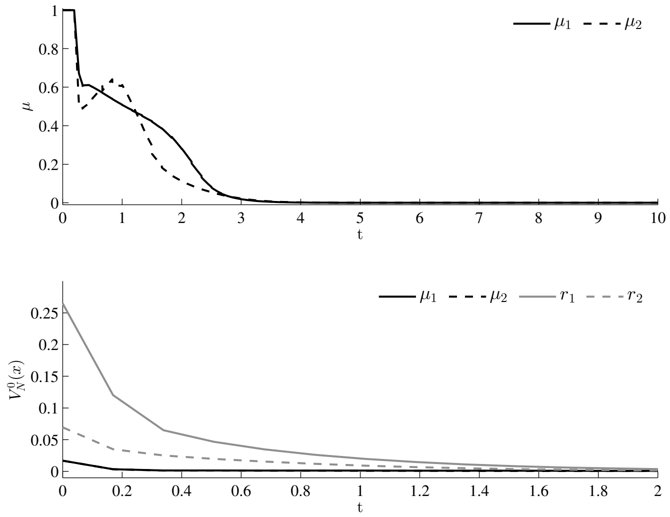

Fig. 4. Top figure illustrates adaptive on-line evolution of the candidate weight function (23). Bottom figure illustrates the optimal MPC value function for (6). Note that the value function for μ_1 is indistinguishable from the one for μ_2.

Table 1. Economic Performance $[\times 10^{-2}]$

i	(Q,R)	Profit (r_i)	Profit (μ_i)	Gain $\left(\frac{\mu_i - r_i}{r_i}\right)$
1	$([1.0, 10, 1], 0.1)$	9.36	10.01	7.01
2	$([0.1, 5, 10], 0.5)$	9.18	10.05	9.49

7. CONCLUSION

In this work a dynamically weighted dual-objective MPC formulation was proposed. First, sufficient conditions were stipulated under which any general dynamic weight function will result in the weighted dual-objective MPC value being a candidate Lyapunov function. Secondly, KKT conditions under which the dynamic weight will promote optimal economic performance were derived. These KKT conditions were subsequently embedded in an explicit dynamic weight function used for the convex combination of regulatory and economic objectives. For this explicit dynamic weight function, is was observed that increased economic performance is achieved by optimizing over a favourably weighted economic objective during process transients. Lyapunov properties of the MPC value still admits desirable steady-state tracking once nonsteady process behaviour, during process transients, has been exploited.

Future work will consider the effect of ill-scaled economic and regulatory cost objectives (see Remark 2), and how the latter relates to the dynamic behaviour of the proposed weight function. Also, parallels between the particular weight function and dissipativity theory needs to be investigated. Lastly, improvements on the explicit weight function (17), which relax the restrictive upper bound of Assumption 4, is envisioned for future work.

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