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Disturbance decoupling for nonlinear systems by measurement feedback: sensor location

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Abstract: The paper addresses the problem on sensor location regarding the solvability of the disturbance decoupling problem by dynamic measurement feedback. Three methods are given to find a measured output, which guarantees the solution to the problem. It is also shown that the results for linear structured systems on sensor location when applied to infinitesimal linear representation of nonlinear system, do not yield, in general, correct results for nonlinear system.

1. INTRODUCTION

When all state variables are not available for measurement, the solution of the disturbance decoupling problem (DDP) is looked for in the class of measurement feedback. This is much more complex problem than the DDP by state feedback and for nonlinear control systems the full solution is still missing. The DDP via dynamic measurement feedback (DDDPM) has been addressed in Battilotti [1997], Andiarti and Moog [1996], Isidori et al. [1981], Xia and Moog [1999] for continuous-time nonlinear systems and in Kaldmäe et al. [2013], Kaldmäe and Kotta [2012] for discrete-time systems.

However, in this paper, we will focus on the related problem on the location of sensors regarding the solvability of the DDDPM. The sensor location problem aspect has been widely studied in the literature related with different problems like observability Boukhobza and Hamelin [2011] and fault detection Commault and Dion [2007]. In general, it answers the questions how many sensors do we need and where should they be placed so that the problem under consideration is solvable. Sensor location problem is also known as the (measured) output selection problem. However, there are not many papers addressing this problem for nonlinear systems Serpas et al. [2013].

The results of this paper rely on necessary and sufficient solvability conditions for DDDPM for discrete-time nonlinear systems Kaldmäe et al. [2013]. Based on the results of Kaldmäe et al. [2013] we suggest the methodology for choosing the sensor location and will show that though the choice is not always unique, measuring the state variables outside the given subsets is of no additional use.

It is often claimed in the literature, including Commault et al. [2011] that a structured linear system may (often) represent all the linearized models for a non-linear system, pointing that way the possibility to use the linear solution to address the nonlinear problem. Though the claim is correct regarding the representation, it does not necessarily mean that this description is helpful for addressing the nonlinear problem. This approach works in case of some problems, for example in case of accessibility Halas et al. [2009], but not for the other problems like state space realization Belikov et al. [2014]. The bottelneck is in integrability issues. Note that the globally linearized model is given in terms of differential one-forms and though the linear theory yields the solution in terms of one-forms, the solution found that way cannot be always integrated to get back to the equations level. Another goal of this paper is to check whether the results of Commault et al. [2011] for parameter-dependent structural linear systems when applied for linearized system description yield a solution of nonlinear DDDPM or not. Since we address the discrete-time systems, the results of Commault et al. [2011] have to be first adopted for discrete-time case.

2. PRELIMINARIES

2.1 Problem statement

Consider a discrete-time nonlinear control system

$$\begin{aligned} x(k+1) &= f(x(k), u(k), w(k)), \\ y(k) &= h(x(k)), \quad y_*(k) = h_*(x(k)), \end{aligned} \tag{1}$$

where $x(k) \in X \subseteq \mathbb{R}^n$ is the state, $u(k) \in U \subseteq \mathbb{R}^m$ is the control, $w(k) \in W \subseteq \mathbb{R}^\rho$ is the unmeasurable disturbance, $y(k) \in Y \subseteq \mathbb{R}^p$ is the measured output and $y_*(k) \in Y_* \subseteq \mathbb{R}^L$ is the output-to-be-controlled. The disturbance decoupling problem under a dynamic

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measurement feedback (DDDPM) can be stated as follows: find a vector function $z(k) = \alpha(x(k))$, $z(k) \in \mathbb{R}^q$ and a dynamic measurement feedback of the form

$$\begin{aligned} z(k+1) &= F(z(k), y(k), v(k)), \quad z(0) = \alpha(x(0)), \\ u(k) &= G(z(k), y(k), v(k)), \end{aligned} \tag{2}$$

where $v(k) \in V \subseteq \mathbb{R}^m$ and $\text{rank}[\partial G/\partial v] = m$, such that the values of the outputs-to-be-controlled $y_*(k)$, for $k \geq 0$, of the closed-loop system are independent of the disturbances $w(k)$. Note that we call the compensator, described by (2) regular, since it generically defines the (y, z)-dependent one-to-one correspondence between the variables $v(k)$ and $u(k)$. One says that the disturbance decoupling problem is solvable via static output feedback if $u(k) = G(y(k), v(k))$.

Note that the solution to the DDDPM depends on the measured output $y(k)$. In this paper, our goal is to find for a given system

$$\begin{aligned} x(k+1) &= f(x(k), u(k), w(k)) \\ y_*(k) &= h_*(x(k)) \end{aligned} \tag{3}$$

a measured output $y(k) = H(x(k))$, such that the DDDPM is solvable for (3).

2.2 The algebra of functions

Mathematical approach called the algebra of functions Zhirabok and Shumsky [2008] will be used to address the problem. We recall briefly the definitions and concepts to be used in this paper, see also Kotta et al. [2013]. Denote by S_X the set of vector functions with the domain the state space X. The elements of algebra of functions are vector functions on S_X and its main ingredients are: (1) relation of partial preorder, denoted by $\leq$, (2) binary operations, denoted by $\times$ and $\oplus$, (3) binary relation, denoted by Δ, (4) operators $\mathbf{m}$ and $\mathbf{M}$.

Definition 1. (i) Given $\alpha, \beta \in S_X$, one says that $\alpha \leq \beta$ iff there exists a function γ such that $\beta(x) = \gamma(\alpha(x))$ for $x \in X$.
(ii) If $\alpha \leq \beta$ and $\beta \leq \alpha$, then α and β are called strictly equivalent, denoted by $\alpha \cong \beta$.

Note that the relation $\cong$ is reflexive, symmetric and transitive. The equivalence relation divides the set S_X into the *equivalence classes* containing the equivalent functions. If $S_X \backslash \cong$ is the set of all these equivalence classes, then the relation $\leq$ is *partial order* on this set. Recall that a lattice is a set with a partial order where every two elements α and β have a unique supremum (least upper bound) $\sup(\alpha, \beta)$ and an infimum (greatest lower bound) $\inf(\alpha, \beta)$. The equivalent definition of the lattice as an algebraic structure with two binary operations $\times$ and $\oplus$ may be given if for every two elements both operations are commutative and associative and moreover, $\alpha \times (\alpha \oplus \beta) = \alpha$, $\alpha \oplus (\alpha \times \beta) = \alpha$. The equivalence follows from the definition the binary operations $\times$ and $\oplus$ as

$$\alpha \times \beta = \inf(\alpha, \beta), \quad \alpha \oplus \beta = \sup(\alpha, \beta). \tag{4}$$

Therefore, the triple $(S_X \backslash \cong, \times, \oplus)$ is a lattice. In lattice theory it is customary not to operate with $\inf(\alpha, \beta)$ and $\sup(\alpha, \beta)$ but with binary operations $\times$ and $\oplus$, respectively. In the simple cases, (4) may be used to compute $\alpha \oplus \beta$. The rule for operation $\times$ is simple: $(\alpha \times \beta)(x) = [\alpha(x), \beta(x)]^T$. However, the product may contain functionally dependent components that have to be found and removed. Note that there exist two special vector functions $\mathbf{0}$ and $\mathbf{1}$, such that for every vector function $\alpha \in S_X$, $\mathbf{0} \leq \alpha \leq \mathbf{1}$.

Example 2. (Computation of the functions $\alpha \times \beta$ and $\alpha \oplus \beta$). Let $\alpha(x) = [x_1 + x_2, x_3]^T$, $\beta(x) = [x_1 x_3, x_2 x_3]^T$. To compute $\alpha \times \beta$, remove the functionally dependent component $x_2 x_3$ in $[\alpha(x), \beta(x)]^T = [x_1 + x_2, x_3, x_1 x_3, x_2 x_3]^T$ to get $(\alpha \times \beta)(x) \cong [x_1 + x_2, x_3, x_1 x_3]^T$. Clearly, by Definition 1, $\alpha \times \beta \leq \alpha$ and $\alpha \times \beta \leq \beta$ since $\alpha_1 = (\alpha \times \beta)_1$, $\alpha_2 = (\alpha \times \beta)_2$, $\beta_1 = (\alpha \times \beta)_3$, $\beta_2 = (\alpha \times \beta)_1 (\alpha \times \beta)_2 - (\alpha \times \beta)_3$, and therefore both α and β can be expressed via components of $\alpha \times \beta$. Moreover, by Definition 1, $\alpha \leq x_3(x_1 + x_2)$ and $\beta \leq x_3(x_1 + x_2)$ and therefore, $(\alpha \oplus \beta)(x) \cong x_3(x_1 + x_2)$.

Definition 3. (Binary relation Δ) Given $\alpha, \beta \in S_X$, there exists a function f_* such that for all $(x(k), u(k), w(k)) \in X \times U \times W$, $(\alpha, \beta) \in \Delta \iff \beta(f(x(k), u(k), w(k))) = f_*(\alpha(x(k)), u(k), w(k))$. When $(\alpha, \beta) \in \Delta$, it is said that α and β form an ordered pair.

Binary relation Δ is used for definition of the operators $\mathbf{m}$ and $\mathbf{M}$.

Definition 4. Operator $\mathbf{m}(\alpha)$ is a function in S_X that satisfies the following conditions (i) $(\alpha, \mathbf{m}(\alpha)) \in \Delta$, (ii) if $(\alpha, \beta) \in \Delta$, then $\mathbf{m}(\alpha) \leq \beta$.

Definition 5. Operator $\mathbf{M}(\beta)$ is a function in S_X that satisfies the following conditions (i) $(\mathbf{M}(\beta), \beta) \in \Delta$, (ii) if $(\alpha, \beta) \in \Delta$, then $\alpha \leq \mathbf{M}(\beta)$.

From Definitions 4 and 5 it is obvious that given α, $\mathbf{m}(\alpha)$ is the minimal function, forming a pair with α, and given β, $\mathbf{M}(\beta)$ is the maximal function, forming a pair with β.

Computation of the operator $\mathbf{m}$. It has proven in Zhirabok and Shumsky [2008] that the function γ exists that satisfies the condition $(\alpha \times u(k)) \oplus f \cong \gamma(f)$. Define $\mathbf{m}(\alpha) \cong \gamma$.

Computation of the operator $\mathbf{M}$. In the special case when $\beta(f(x(k), u(k)))$ can be represented in the form $\beta(f(x(k), u(k))) = \sum_{i=1}^{d} a_i(x(k)) b_i(u(k))$ where $a_1(x(k))$, $a_2(x(k))$, $\ldots$, $a_d(x(k))$ are arbitrary functions and $b_1(u(k))$, $b_2(u(k))$, $\ldots$, $b_d(u(k))$ are linearly independent, then $\mathbf{M}(\beta) := a_1 \times a_2 \times \cdots \times a_d$.

2.3 The solution of DDDPM

Find first a minimal (containing the maximal number of functionally independent components) vector function $\alpha^0(x)$ such that its forward shift $\alpha^0(f(x, u, w))$ does not depend on the unmeasurable disturbance w. The function $\alpha^0(x)$ plays a key role in Algorithm 1 below, and though it is not unique, all possible choices are equivalent functions. Moreover, applying the operators $\mathbf{m}$ and $\mathbf{M}$ to equivalent functions will yield again equivalent functions. Therefore, the result of Algorithm 1 will be the same for different choices of $\alpha^0(x)$, up to the function equivalence.

Definition 6. The vector function $\alpha(x)$ is said to be (h, f)-invariant if $(\alpha \times h, \alpha) \in \Delta$. In case $h = \mathbf{1}$, function $\alpha(x)$ is said to be f-invariant.

For checking if some function α is (h, f)-invariant (f-invariant), we use the following Lemma.

Lemma 7. (Zhirabok and Shumsky [2008]) Function α is (h, f)-invariant (f-invariant) iff $\alpha \times h \leq \mathbf{M}(\alpha)$ ($\alpha \leq \mathbf{M}(\alpha)$).

Definition 8. The vector function $\alpha(x)$ is said to be a controlled invariant if there exists a regular static state feedback $u(k) = G(x(k), v(k))$ such that function $\alpha(x)$ is f-invariant for the closed-loop system.

Definitions 6 and 8 are generalizations for the systems of the form (1) the concept of conditioned invariant distribution and controlled invariant distribution, respectively; see for example Isidori et al. [1981] and Grizzle [1985].

Theorem 9. Kaldmäe et al. [2013] System (1) is disturbance decoupled iff there exists a f-invariant function ξ such that $\alpha^0 \leq \xi \leq h_*$.

Theorem 10. Kaldmäe et al. [2013] System (1) can be disturbance decoupled by feedback (2) iff there exist a controlled invariant function ξ and a (h, f)-invariant function α such that

$$\alpha^0 \leq \alpha \leq \xi \leq h_*. \tag{5}$$

The algorithm below can be used to compute the minimal (h, f)-invariant vector function α, that satisfies the condition $\alpha^0 \leq \alpha$.

Algorithm 1. Kotta et al. [2013] Given α^0, compute recursively for $i \geq 1$, using the formula $\alpha^{i+1} = \alpha^i \oplus \mathbf{m}(\alpha^i \times h)$, the sequence of non-decreasing functions $\alpha^0 \leq \alpha^1 \leq \alpha^2 \leq \ldots \leq \alpha^i \leq \ldots$. There exists a finite j such that $\alpha^j \ncong \alpha^{j-1}$ but $\alpha^{j+l} \cong \alpha^j$, for all $l \geq 1$. Define $\alpha := \alpha^j$.

3. MAIN RESULTS

Let $h_* = [h_{*1}, \ldots, h_{*L}]^T$ and denote by r_i and d_i the relative degrees of the function $h_{*i}(x)$ with respect to the control input u and disturbance w, respectively. Moreover, we use the notations $y_{*i}(k) = h_{*i}(x(k)) =: h_{*i,1}(x(k)), \ldots, y_{*i}(k + r_i - 1) =: h_{*i,r_i}(x(k))$. We make the following assumption:

Assumption 1. $d_i > r_i$.

From the definition of r_i and Assumption 1, $y_{*i}(k + r_i) = \widehat{f}_i(x(k), u(k))$ for some $\widehat{f}_i$.

Definition 11. (Vector relative degree) [1] Vector $(r_1, \ldots, r_L)$ is called a vector relative degree of output $y_*(k)$ if

$$\text{rank}\left[\frac{\partial(\widehat{f}_1(x(k), u(k)), \ldots, \widehat{f}_L(x(k), u(k)))^T}{\partial u(k)}\right] = L$$

generically, i.e. everywhere except on the set of zero measure.

Assumption 2. Output $y_*(k)$ has a vector relative degree.

Note that Assumption 1 is a standard assumption made in the solution of the disturbance decoupling problem even when the solution is looked for in the form of state feedback. Assumption 2 may be, in principle, replaced by the assumption of right invertibility (regarding the output y_*). However, this type of assumption is often made for simplification reasons; in particular here it allows to find the explicit formula to compute function ξ in Theorem 10. This formula is also an important improvement compared to the results in Kaldmäe et al. [2013], that allows to make the result of Theorem 10 constructive.

[1] This definition is in accordance with Remark 5.1.3. of Isidori [1995].

Consider the set of equations

$$\widehat{f}_i(x(k), u(k)) = v_i(k) \quad i = 1, \ldots, L \tag{6}$$

Under the Assumptions 1 and 2, the set of equations (6) is generically solvable for $u(k)$.

Denote by $\widetilde{f}$ the function f in (1) for the closed-loop system. Note that the definition of operator $\mathbf{M}$ depends on the system equations, i.e. the function f. By $\widetilde{\mathbf{M}}$ we denote the operator $\mathbf{M}$ defined by function $\widetilde{f}$.

Theorem 12. Under the Assumptions 1 and 2, the maximal controlled invariant function ξ that satisfies the inequality $\xi \leq h_*$, may be computed by the formula

$$\xi := \prod_{i=1}^{L}(h_{*i,1} \times \cdots \times h_{*i,r_i}). \tag{7}$$

Proof. Since the equations (6) are solvable for $u(k)$, one can find a static state feedback by solving these equations. We show that the function ξ in (7) is $\widetilde{f}$-invariant. Since $\widetilde{\mathbf{M}}(\alpha \times \beta) \cong \widetilde{\mathbf{M}}(\alpha) \times \widetilde{\mathbf{M}}(\beta)$ for some α, β, see Zhirabok and Shumsky [2008], one obtains

$$\widetilde{\mathbf{M}}(\xi) \cong \prod_{i=1}^{L}(\widetilde{\mathbf{M}}(h_{*i,1}) \times \cdots \times \widetilde{\mathbf{M}}(h_{*i,r_i})),$$

and by the rule of the computation of the operator $\widetilde{\mathbf{M}}$, one has $\widetilde{\mathbf{M}}(h_{*i,j}) = h_{*i,j+1}$, $j = 1, \ldots, r_i - 1$. Since by (6), $h_{*i,r_i}(x(k+1)) = v_i(k)$, $\widetilde{\mathbf{M}}(h_{*i,r_i}) = \mathbf{1}$. Therefore,

$$\begin{aligned}\widetilde{\mathbf{M}}(\xi) &\cong \prod_{i=1}^{L}(h_{*i,2} \times h_{*i,3} \times \cdots \times h_{*i,r_i} \times \mathbf{1}) \\ &\geq \prod_{i=1}^{L}(h_{*i,1} \times h_{*i,2} \times \cdots \times h_{*i,r_i} \times \mathbf{1}) \cong \xi,\end{aligned}$$

i.e. $\xi \leq \widetilde{\mathbf{M}}(\xi)$. By Lemma 7 the function ξ is $\widetilde{f}$-invariant, or controlled invariant function for the original system.

Next, let β be another controlled invariant function such that $\beta \leq h_* = \prod_{i=1}^{L} h_{*i,1}$. Since β is controlled invariant, then $\beta \leq \widetilde{\mathbf{M}}(\beta)$. Then, since $\alpha \leq \beta \Rightarrow \mathbf{M}(\alpha) \leq \mathbf{M}(\beta)$, see Kotta et al. [2013], one obtains

$$\begin{aligned}\beta &\leq \widetilde{\mathbf{M}}(\beta) \leq \widetilde{\mathbf{M}}(h_*) \\ &= \widetilde{\mathbf{M}}(\prod_{i=1}^{L} h_{*i,1}) \cong \prod_{i=1}^{L}\widetilde{\mathbf{M}}(h_{*i,1}) = \prod_{i=1}^{L} h_{*i,2}.\end{aligned}$$

By analogy, $\beta \leq \prod_{i=1}^{L} h_{*i,j}$ for $j = 3, \ldots, r_i$. Then, by the definition of operation $\times$, $\beta \leq \prod_{i=1}^{L}(h_{*i,1} \times h_{*i,2} \times \cdots \times h_{*i,r_i}) = \xi$, meaning that ξ is the maximal $\widetilde{f}$-invariant function satisfying the condition $\xi \leq h_*$. $\square$

Next we describe two methods for finding the unknown measured output function $H(x)$ which makes the DDDPM solvable. By Theorem 10, function H must guarantee the existence of a (H, f)-invariant function α, satisfying $\alpha^0 \leq \alpha \leq \xi$. In Case 1 below, function H is computed based on the function α^0 and in Case 2, function H is computed based on the function ξ.

Case 1. Note that by Algorithm 1, we have $\alpha^1 = \alpha^0 \oplus \mathbf{m}(\alpha^0 \times H)$. If the choice H guarantees that $\mathbf{m}(\alpha^0 \times$

$H) \leq \alpha^0$, then $\alpha^1 \cong \alpha^0$ and $\alpha = \alpha^0$. Therefore $\alpha \leq \xi$ and the problem is solvable for given H. The condition $\mathbf{m}(\alpha^0 \times H) \leq \alpha^0$ is equivalent to another condition $\alpha^0 \times H \leq \mathbf{M}(\alpha^0)$, that is easier to use to find the maximal H, satisfying the last inequality.

Case 2. The function α has to satisfy the conditions $\alpha \leq \xi$ and $\mathbf{m}(\alpha \times H) \leq \alpha$ (or equivalently $\alpha \times H \leq \mathbf{M}(\alpha)$). There exist at least two options to find the function H from the function ξ.

(a) Find the maximal function H such that $\xi \times H \leq \mathbf{M}(\xi)$ is valid. Because of the last inequality, the function ξ is (H, f)-invariant. Since the condition $\alpha^0 \leq \xi$ holds, function ξ can be used as α. Therefore, the condition $\alpha^0 \leq \alpha = \xi \leq h_*$ is valid. The choice $\alpha = \xi$ has the following advantage: the compensator constructed based on this function is of minimal dimension.

(b) Find the function α such that $\alpha \leq \xi$ and $\alpha \ncong \xi$, by choosing the function $\xi' \geq \alpha^0$ such that $\alpha \cong \xi \times \xi'$. Then find the maximal H such that $\alpha \times H \leq \mathbf{M}(\alpha)$ is valid. It is recommended to choose ξ' such that $\mathbf{M}(\xi') \geq \mathbf{M}(\xi)$. In this case $\mathbf{M}(\alpha)$ is equivalent to $\mathbf{M}(\alpha) \times \mathbf{M}(\xi)$ and the solution exists (further analysis is not needed). Otherwise, one has to check if $\mathbf{M}(\alpha) \geq \alpha \times H$.

In all the cases, described above, one needs to find the vector function H that satisfies the condition $\lambda \times H \leq \mu$ for given λ and μ. One may follow the procedure below. Take H such that its elements are

(i) all the functions μ_i that are components of the vector function μ and satisfy the condition $\lambda \nleq \mu_i$

and

(ii) all the variables $x_i(k)$, the vector function μ depends on, but vector function λ does not.

This H is, in general, not maximal, but can be simplified as follows:

(iii) one can replace H by the equivalent, but 'simpler' vector function H';

(iv) some of the elements of H', which can be written in terms of λ and the other elements of H', can be removed.

For example, let $\lambda(x) = [x_1+x_2, x_2+x_3]^T$ and $\mu(x) = [x_1+x_3, x_2+x_4]^T$. Then by the procedure above, one gets $H(x) = [x_1+x_3, x_2+x_4, x_4]^T$. It is easy to see that this vector function is equivalent to $H'(x) = [x_1+x_3, x_2, x_4]^T$. Now, either x_1+x_3 or x_2 can be removed, since they can be written in terms of λ and the other elements of H'. Thus one gets two choices: $H_1 = [x_2, x_4]^T$ and $H_2 = [x_1+x_3, x_4]^T$.

In some cases, there is another heuristic way to improve the solution. Consider the variables x_j such that λ depends on x_j, but μ does not and add these variables to H. For example, let $\lambda = [x_1+x_2, x_2+x_3]^T$ and $\mu = [x_1+x_3, x_4]^T$. Then one gets $H = [x_1+x_3, x_4]^T$ according to (i) and (ii) of the procedure above. This function can not be simplified. However, if one adds x_2 to H, then we get $H' = [x_2, x_4]^T$ as a solution.

4. COMPARISON AND EXAMPLES

In this section, we show that one can not use the method, described in Commault et al. [2011] for linear systems, to solve the DDDPM for nonlinear systems. We first recall the main definitions from Commault et al. [2011]. For simplicity, we consider only the case with single disturbance. A linear structured system Σ_λ with parameterized entries is described by

$$\begin{aligned} x(k+1) &= A_\lambda x(k) + B_\lambda u(k) + E_\lambda w(k) \\ y(k) &= C_\lambda x(k), \qquad y_*(k) = C_{*\lambda} x(k), \end{aligned} \tag{8}$$

where A_λ, B_λ, E_λ, C_λ and $C_{*\lambda}$ are the matrices of appropriate dimensions, depending on a finite number of independent parameters $\lambda_i \in \mathbb{R}$.

With system (8), one can associate a directed graph $G(\Sigma_\lambda) = (V_v, V_a)$, which consists of the vertex set $V_v = X \cup U \cup W \cup Y \cup Y_* = \{x_1(k), \ldots, x_n(k)\} \cup \{u_1(k), \ldots, u_m(k)\} \cup \cdots \cup \{y_{*1}(k), \ldots, y_{*L}(k)\}$ and the arc set V_a, which consists of pairs $(x_i(k), x_j(k))$ such that $(x_i(k), x_j(k)) \in V_a \Leftrightarrow (A_\lambda)_{i,j} \neq 0$ and similarly the pairs $(u_i(k), x_j(k))$, $(w_i(k), x_j(k))$, $(x_i(k), y_j(k))$, $(x_i(k), y_{*j}(k))$.

A path from vertex i_0 to vertex i_μ is a sequence of arcs $(i_t, i_t+1) \in V_a$, $t = 0, \ldots, \mu-1$. The path is called simple if every vertex on the path occurs only once. The length of a path is the number of arcs it consists. Let V_1, V_2 be two nonempty subsets of the set V_v. A simple path P is called a $V_1 - V_2$ path if its initial vertex belongs to V_1 and its final vertex belongs to V_2. $V_1 - V_2$ paths are said to be disjoint if they have no common vertex.

A set of d disjoint and simple $V_1 - V_2$ paths is called a linking from V_1 to V_2 ($V_1 - V_2$ linking) of size d. The maximal number of disjoint $V_1 - V_2$ paths is denoted by $\rho(V_1, V_2)$. A set of $\rho(V_1, V_2)$ disjoint $V_1 - V_2$ paths is called a maximal $V_1 - V_2$ linking. The length of a $V_1 - V_2$ linking is the sum of the lengths of all its paths, and $l(V_1, V_2)$ denotes the minimum length of a maximal $V_1 - V_2$ linking.

The vertex set l^* is defined as follows:

$$\begin{aligned} l^* = \{x_i(k) \in X | \rho(U \bigcup \{x_i(k)\}, Y) = \rho(U, Y), \\ l(U \bigcup \{x_i(k)\}, Y) = l(U, Y)\}. \end{aligned}$$

The frontier F_{l^*} is defined as the set of vertices $F_{l^*} = \{x_i(k) \in l^* | \exists (x_i(k), x_j(k)) \in V_a, x_j(k) \notin l^*\}$.

For a disturbance $w(k)$ such that there exists $x_j(k) \in l^*$ and $(w(k), x_j(k)) \in V_a$, denote by d_w the length of a shortest $\{w(k)\} - F_{l^*}$ path. Define D, the set of vertices $D = \{x_i(k) \in l^* | 0 < l(w(k), x_i(k)) \leq d_w\}$.

Example 1. Consider the system

$$x(k+1) = \begin{bmatrix} w(k)x_3(k) + x_1(k) \\ w(k) \\ x_2(k) \\ x_1(k) + x_3(k) + u(k) \end{bmatrix} \tag{9}$$

$$y_*(k) = x_4(k).$$

Then the globally linearized system equations are

$$\begin{aligned}\mathrm{d}x(k+1) &= A_\lambda \mathrm{d}x(k) + B_\lambda \mathrm{d}u(k) + E_\lambda \mathrm{d}w(k), \qquad (10)\\ \mathrm{d}y_*(k) &= C_{*\lambda}\mathrm{d}x(k)\end{aligned}$$

where

$$A_\lambda = \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & 0\\ 0 & 0 & 0 & 0\\ 0 & \lambda_3 & 0 & 0\\ \lambda_4 & 0 & \lambda_5 & 0\end{pmatrix}, B_\lambda = \begin{pmatrix} 0\\ 0\\ 0\\ \lambda_6\end{pmatrix}, E_\lambda = \begin{pmatrix}\lambda_7\\ \lambda_8\\ 0\\ 0\end{pmatrix}$$

$$C_{*\lambda} = (\,0\ 0\ 0\ \lambda_9\,),$$

$\lambda_2 = w(k)$, $\lambda_7 = x_3(k)$ and the rest of coefficients λ_i are nonzero real numbers. We first try to solve the DDDPM for globally linearized system (10), adopting the method described in Commault et al. [2011] for discrete-time case. The graph, associated with system (10) is given in Figure 1 below.

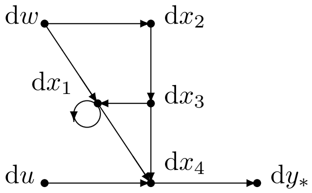

Figure 1. Graph of Example 1

It is easy to see that $l^* = \{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}$, $F_{l^*} = \{\mathrm{d}x_1, \mathrm{d}x_3\}$, $d_w = 1$, $D = \{\mathrm{d}x_1, \mathrm{d}x_2\}$. One can check that for all $(\mathrm{d}w, \mathrm{d}x_i) \in V_a$ one has $\mathrm{d}x_i \in l^*$. Therefore, according to Theorem 11 in Commault et al. [2011], the measurement of $\mathrm{d}x_1$ or $\mathrm{d}x_2$ is sufficient to solve the DDDPM for the linearized system.

Now, let $\mathrm{d}y = \lambda_{10}\mathrm{d}x_2$ and try to find the feedback that solves the DDDPM. Note that the transfer matrix of (10) is Halas and Kotta [2007]

$$\begin{pmatrix}\mathrm{d}y_*\\ \mathrm{d}y\end{pmatrix} = \begin{pmatrix} \dfrac{\lambda_{15}}{z} & \dfrac{\lambda_{14}(\lambda_4 w z^2 + \lambda_{13} z + \lambda_{12} w^{++} - \lambda_{11})}{z^3(z-\lambda_1)}\\ 0 & \dfrac{\lambda_{16}}{z}\end{pmatrix}\begin{pmatrix}\mathrm{d}u\\ \mathrm{d}w\end{pmatrix},$$

where z is the forward shift operator and $\lambda_{11} = \lambda_1\lambda_3\lambda_5$, $\lambda_{12} = \lambda_3\lambda_4$, $\lambda_{13} = \lambda_3\lambda_5$, $\lambda_{14} = \lambda_9\lambda_8$, $\lambda_{15} = \lambda_9\lambda_6$, $\lambda_{16} = \lambda_{10}\lambda_8$. The feedback, that solves the DDDPM is, by (4) in Commault et al. [2011], $\mathrm{d}u = F(z)\mathrm{d}y$, where

$$F(z) = \frac{\lambda_4 w z^3 + \lambda_{13} z^2 + (\lambda_{12} w^{++} - \lambda_{11}) z}{\lambda_6\lambda_{10} z^2 (z-\lambda_1)}$$

or, given in terms of one-forms,

$$\begin{aligned}\lambda_6\lambda_{10}(\mathrm{d}u(k+3) - \mathrm{d}u(k+2)) = \\ \lambda_4 w(k)\mathrm{d}y(k+3) + \lambda_{13}\mathrm{d}y(k+2)\\ +(\lambda_{12} w(k+2) - \lambda_{11})\mathrm{d}y(k+1).\end{aligned}$$

Note that this one-form is not integrable. Therefore, the solution for the linearized system description does not yield the compensator equations.

Consider now the nonlinear system (9). The method described in Commault et al. [2011] suggested that measuring $\mathrm{d}x_1$ or $\mathrm{d}x_2$ is sufficient to solve the DDDPM for the linearized system. Next we show that taking $y = x_1$ or $y = x_2$ is not sufficient to solve the DDDPM for nonlinear system (9). Suppose $y = x_1$, then the function $\alpha^0(x) = [x_3, x_4]^T$, $\alpha^1(x) = x_4$, $\alpha^2(x) = \mathbf{1}$ and the DDDPM is not solvable. Suppose next that $y = x_2$, then $\alpha^1(x) = x_3$, $\alpha^2(x) = \mathbf{1}$ and the DDDPM is again not solvable.

According to Case 1, $\mathbf{M}(\alpha^0)(x) = [x_2, x_1 + x_3]^T$. Clearly, one has to set either $H(x) = [x_1, x_2]^T$ or $H(x) = [x_1 + x_3, x_2]^T$ to satisfy the condition $\alpha^0 \times H \leq \mathbf{M}(\alpha^0)$.

Since $y_* = x_4$, then $\xi = x_4$. According to Case 2 (a), compute of the function $\mathbf{M}(\xi)(x) = x_1 + x_3$. Setting $y = x_1 + x_3$, one can solve the DDDPM.

Example 2. Consider the control system

$$x(k+1) = \begin{bmatrix} x_3(k)x_5(k) + w(k)\\ x_1(k)\\ x_2(k)x_5(k)\\ x_3(k) + u(k)\\ x_3(k)x_4(k)\\ x_5(k)\end{bmatrix} \qquad (11)$$
$$y_*(k) = x_6(k).$$

For this system the matrices of linearized description are

$$A_\lambda = \begin{pmatrix} 0 & 0 & \lambda_1 & 0 & \lambda_2 & 0\\ \lambda_3 & 0 & 0 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0 & \lambda_4 & 0\\ 0 & 0 & \lambda_5 & 0 & 0 & 0\\ 0 & 0 & \lambda_6 & \lambda_2 & 0 & 0\\ 0 & 0 & 0 & 0 & \lambda_7 & 0\end{pmatrix}, B_\lambda = \begin{pmatrix}0\\0\\0\\ \lambda_8\\0\\0\end{pmatrix}, E_\lambda = \begin{pmatrix}\lambda_9\\0\\0\\0\\0\\0\end{pmatrix}$$

$$C_{*\lambda} = (\,0\ 0\ 0\ 0\ 0\ \lambda_{10}\,),$$

where $\lambda_1 = x_5(k)$, $\lambda_2 = x_3(k)$, $\lambda_4 = x_2(k)$, $\lambda_6 = x_4(k)$ and the rest of λ_i-s are nonzero elements of $\mathbb{R}$.

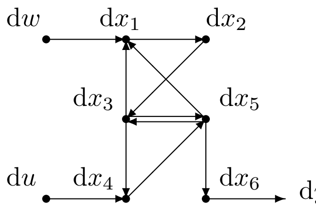

Figure 2. Graph of Example 2

It is easy to see that $l^* = \{\mathrm{d}x_1, \mathrm{d}x_2\}$, $F_{l^*} = \{\mathrm{d}x_2\}$, $d_w = 2$, $D = \{\mathrm{d}x_1, \mathrm{d}x_2\}$. One can check that for all $(\mathrm{d}w, \mathrm{d}x_i) \in V_a$ one has $\mathrm{d}x_i \in l^*$. Therefore according to Theorem 11 of Commault et al. [2011] the measurement of $\mathrm{d}x_1$ or $\mathrm{d}x_2$ is sufficient to solve the DDDPM for the linearized system.

Suppose now that the measurement output is absent, and one has to find it to solve the DDDPM for system (11).

Compute the vector functions $\alpha^0(x) = [x_2, x_3, x_4, x_5, x_6]^T$, $\xi(x) = [x_3x_4, x_6, x_5]^T$ and $\mathbf{M}(\xi)(x) = [x_5, x_4x_3, x_2x_5, x_2x_3x_5]^T$. The inequality $\xi \times H \leq \mathbf{M}(\xi)$ is valid, for example, if $H(x) = [x_2, x_3]^T$. To find a better choice (that needs less sensors) for $H(x)$, set $\xi'(x) = x_3 \geq \alpha^0(x)$ since $\mathbf{M}(x_3) = x_2x_5 \geq \mathbf{M}(\xi)(x)$, then $\alpha(x) = (\xi \times \xi')(x) = [x_3, x_4, x_6, x_5]^T$. One can check that the inequality $\alpha \times H \leq \mathbf{M}(\alpha)$ is valid for $H(x) = x_2$.

Note that in the first case $\alpha(x) = \xi(x) = [x_3x_4, x_5, x_6]^T$ and the dimension of the compensator is 3. The second case with $\alpha(x) = [x_3, x_4, x_5, x_6]^T$ demands the compensator of dimension 4.

Example 3. Consider the control system

$$\begin{aligned}
x_1(k+1) &= \vartheta_1 x_1(k)^2 sign(x_1(k)) + \vartheta_2 x_2(k) \\
&\quad + x_1(k) + \vartheta_6 u_1(k) + \vartheta_6 u_2(k) \\
x_2(k+1) &= \vartheta_3 x_1(k) x_2(k) + x_2(k) + \frac{\vartheta_7}{x_1(k)} u_1(k) \\
&\quad - \frac{\vartheta_7}{x_1(k)} u_2(k) + \frac{\vartheta_8}{x_1(k)} u_3(k) + w(k) \\
x_3(k+1) &= \vartheta_4 x_4(k) + \vartheta_5 x_3(k) sign(x_1(k)) \\
&\quad + x_3(k) + \vartheta_9 u_3(k) \\
x_4(k+1) &= \vartheta_{10} x_3(k) + x_4(k) \\
x_5(k+1) &= \vartheta_{11} x_1(k) x_2(k) + x_5(k) \\
y_*(k) &= x_4(k).
\end{aligned} \tag{12}$$

The equations (12) constitute a simplified sampled-data model of the underwater vehicle moving on a vertical plane, and developed under the assumptions of small x_1 and x_2 values, see Shumsky [2006]. Model variables have the following meaning: x_1 is the velocity, x_2 is the angle of the trajectory, x_4 and x_3 are the trim and its time derivative, respectively, x_5 is the depth. Model coefficients $\vartheta_1 \div \vartheta_{11}$ characterize the masses, inertia and the structural features of the vehicle. The inputs u_1, u_2, and u_3 are the forces of the upper and bottom stern thrusters and the vertical bow thruster, respectively.

Our goal is to find the measurement output in such a manner that allows to solve the DDDPM.

Compute, according to (7), the vector function $\xi(x) = [x_4, \vartheta_{10}x_3 + x_4]^T$, and then $\mathbf{M}(\xi)(x) = [\vartheta_{10}x_3 + x_4, x_3 + \vartheta_4 x_4 + \vartheta_5 x_3 sign(x_1)]^T$. It is obvious that the inequality $\xi \times H \leq \mathbf{M}(\xi)$ is valid for $H(x) = x_1$.

5. CONCLUSION

In this paper, the DDDPM was addressed. A formula was given to find a controlled invariant function ξ, which plays an important role in the solution of the DDDPM. Then, the methods for finding a measured output $H(x)$, which guarantee the solvability of the DDDPM, were suggested. All the given methods require finding H such that $\lambda \times H \leq \mu$ for some λ and μ. Finally, it was shown that one can not solve the DDDPM for nonlinear systems by using the methods from linear theory, and applied to the globally linearized system description.

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