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Stochastic Robust Kalman Filtering for Linear Time-Varying Systems with a Multiplicative Measurement Noise

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Abstract: In this paper, a stochastic robust Kalman filtering problem is investigated for time-varying linear systems with stochastic uncertainties in its measurement matrix. The influence of parametric uncertainties on the nominal Kalman filter estimate is analyzed in the sense of classical weighted least-squares criterion. Stochastic approximation of estimation errors due to uncertainties allows us to obtain a recursive stochastic robust Kalman filter. The procedure of the stochastic error compensation is interpreted as the optimization of an indefinite quadratic cost. Considering the single stage estimation problem, the stochastic robust Kalman filter recursion is derived. As well, its existence condition is recursively checked using the estimation error covariance. It is shown that the weighted estimation error of the suggested filter is zero mean, which is the distinct property compared to the previous robust filters.

1. INTRODUCTION

In practice, the system modeling required to design the Kalman filter may be based on limited information. While the Kalman filter is somewhat insensitive to cost function variations and to the imperfect knowledge of *a priori* noise statistics(Morris [1976]), many researchers have pointed out that inherent uncertainties contained in an available system model might lead to unacceptable estimation performance or even cause divergence of the conventional Kalman filter (Fitzgerald [1971], Toda et al. [1980]). To cope with these problems, the robust Kalman filtering problem has been received much attention during the last decade. In general, the existing robust Kalman filtering algorithms aim to guarantee the upper bound of estimation error variances against the parametric uncertainties (Petersen et al. [1996], Theodor et al. [1996]). Therefore, they can effectively relax the standing assumption on the perfect system model which is necessary for the standard Kalman filter.

In most cases, the deterministic descriptions on the parametric uncertainties such as norm-bounded constraint, polytopic constraint, and sum quadratic constraint(SQC) (Xie et al. [1994], Geromel et al. [1998], Savkin et al. [1998], Ra et al. [2004]) have been adopted to make the robust Kalman filtering problem tractable. The resulting robust filters provided robustness against all available uncertainties. Despite of its validity, the conservatism of robust filters based on the deterministic uncertainty model has blocked the use of the robust Kalman filter in actual applications. In particular, there are many applications whose system model is constructed by the measured information. Therefore, as a natural way to solve the conservatism issue, the *a priori* knowledge on stochastic uncertainties has been taken into account (Wang et al. [2002], Yang et al. [2002]). However, these stochastic robust Kalman filters might be restricted to the systems whose multiplicative measurement noises are correlated with the additive one. Furthermore, they require substantial computations.

Recently, as a substitute of existing robust Kalman filters, we have been investigated the recursive robust least squares (RLS) problem (Ra et al. [2007]). In this work, the single tone frequency estimation from noisy sinusoid was instanced as one of the RLS problems. To solve the RLS problem, the error analysis of the nominal least squares (LS) caused by the multiplicative measurement noises were carried out. It has been shown that the stochastic approximation of the LS estimation errors could be successfully applied to the robust filtering problem. It was concluded that the weighted error of RLS estimate become zero-mean, hence it is not conservative. This stochastic property distinguishes the RLS estimator from the previous robust Kalman filters. However, since it cannot reflect the additive noise variances on the filtering equations, it may not be applicable for the non-stationary additive noise cases.

This flaw motivates us to develop a novel stochastic robust Kalman filter which evolves the RLS estimation scheme in Ra et al. [2007]. To enjoy the benefit of computational efficiency and structural degree of freedom, the robust Kalman filtering problem for the system with multiplicative measurement noises are reformulated within the framework of weighted LS (WLS) estimation. Then, the robust WLS (RWLS) estimation problem could be reduced to the error compensation problem of the nominal WLS estimator in the presence of multiplicative noises. It is also shown that the proposed RWLS estimation scheme could be reinterpreted as the minimization of a certain indefinite quadratic cost function. In consequence, by solving the single state stochastic optimization problem, the recursive stochastic robust filter recursion and its existence condition are derived.

2. LEAST SQUARES ESTIMATION FOR UNCERTAIN LINEAR SYSTEMS

The Kalman filter has been referred as a optimal estimator in the sense of weighted least squares (WLS) estimation. Hence, the basic concept of the stochastic robust Kalman filtering could be readily explained by addressing and solving the robust weighted least squares (RWLS) estimation problem.

2.1 Weighted least squares criterion

Consider the following vectorial measurement equation.

$$y = \mathcal{H}x + v = [\tilde{\mathcal{H}} - \Delta\mathcal{H}]x + v \tag{1}$$

where $x \in \mathcal{R}^n$ is the vector should be estimated, $y \in \mathcal{R}^m$ is the measurement vector and $v \in \mathcal{R}^m$ is the zero-mean white additive measurement noise. The available measurement matrix $\tilde{\mathcal{H}} \in \mathcal{R}^{m\times n}$ can be represented as the sum of the unknown noise-free measurement matrix $\mathcal{H}$ and the stochastic uncertainty $\Delta\mathcal{H}$ consisting of several zero-mean white noise sources, namely $\tilde{\mathcal{H}} = \mathcal{H} + \Delta\mathcal{H}$. It is also assumed that the following *a priori* statistics is given for state estimation.

$$\begin{aligned} &E\{v\} = 0,\ E\{vv^T\} = \mathcal{R},\ E\{\Delta\mathcal{H}\} = 0, \\ &E\{\Delta\mathcal{H}^T\mathcal{R}^{-1}\Delta\mathcal{H}\} = W,\ E\{\Delta\mathcal{H}^T\mathcal{R}^{-1}v\} = V \end{aligned} \tag{2}$$

Remark 2.1. At this point, it should be noted that, in our problem formulation, the measurement vector y is generated by the noise-free measurement matrix $\mathcal{H}$ and is contaminated by the additive measurement noise v. This is same as the setting of standard Kalman filtering problem.

However, if the noise corrupted measurement matrix $\tilde{\mathcal{H}}$ is only available for state estimation, the measurement equation can be understood in a different aspect. That is, the given measurement equation is rewritten as

$$y = [\tilde{\mathcal{H}} - \Delta\mathcal{H}]x + v \tag{3}$$

Thus, in (3), $\Delta\mathcal{H}$ is regarded as a stochastic parametric uncertainty or multiplicative measurement noise. Moreover, it is obvious that *the available noisy measurement matrix $\tilde{\mathcal{H}}$ is not deterministic and correlated with $\Delta\mathcal{H}$.*

On the other hand, in the previous robust Kalman filtering problem, the measurement vector y is made by not the noise-free measurement matrix $\mathcal{H}$ but the noisy measurement matrix $\tilde{\mathcal{H}}$ containing parametric uncertainty $\Delta\mathcal{H}$.

$$y = [\mathcal{H} + \Delta\mathcal{H}]x + v \tag{4}$$

In the above equation, *the given measurement matrix $\mathcal{H}$ is deterministic and is uncorrelated with $\Delta\mathcal{H}$.* Therefore, the robust Kalman filtering problem associated with (3) is totally different from that related to (4).

Problem 2.1. (*Optimal WLS Criterion*) If the vector $\bar{x} = E\{x\}$, the noise free measurement matrix $\mathcal{H}$ and the weighting matrices $\mathcal{Q} > 0, \mathcal{R} > 0$ are given, the optimal weighted least squares(OWLS) estimation problem is defined as the minimization of the following regularized quadratic cost.

$$J_{OWLS} = J^a_{OWLS} + J^b_{OWLS} \tag{5}$$

where

$$\begin{aligned} J^a_{OWLS} &\triangleq \frac{1}{2}(x-\bar{x})^T\mathcal{Q}^{-1}(x-\bar{x}) \\ J^b_{OWLS} &\triangleq \frac{1}{2}(y-\mathcal{H}x)^T\mathcal{R}^{-1}(y-\mathcal{H}x) \end{aligned} \tag{6}$$

■

From the above regularized cost function, one can consider an equivalent WLS estimation problem for the augmented measurement equation of the form.

$$\bar{y} = \bar{\mathcal{H}}x + \bar{v} = \left[\tilde{\bar{\mathcal{H}}} - \Delta\bar{\mathcal{H}}\right]x + \bar{v} \tag{7}$$

where

$$\bar{y} \triangleq \begin{bmatrix}\bar{x}\\ y\end{bmatrix},\ \bar{\mathcal{H}} \triangleq \begin{bmatrix}I\\ \mathcal{H}\end{bmatrix},\ \tilde{\bar{\mathcal{H}}} \triangleq \begin{bmatrix}I\\ \tilde{\mathcal{H}}\end{bmatrix},\ \Delta\bar{\mathcal{H}} \triangleq \begin{bmatrix}0\\ \Delta\mathcal{H}\end{bmatrix}$$

$$\bar{v} \triangleq \begin{bmatrix}\bar{x}-x\\ v\end{bmatrix},\ E\{\bar{v}\} = 0,\ \bar{\mathcal{R}} \triangleq cov\langle\bar{v},\bar{v}\rangle = \begin{bmatrix}\mathcal{Q} & 0\\ 0 & \mathcal{R}\end{bmatrix}$$

For notational convenience, the cost function J_{OWLS} can be rewritten as

$$J_{OWLS} = \frac{1}{2}(\bar{y}-\bar{\mathcal{H}}x)^T\bar{\mathcal{R}}^{-1}(\bar{y}-\bar{\mathcal{H}}x) \tag{8}$$

If the noise-free measurement matrix $\mathcal{H}$ is given, it is straightforward to derive the optimal solution satisfying the above mentioned OWLS criterion.

Lemma 2.1. (*Optimal WLS Solution*) From the stationary condition derived by the first differentiation of (5) with respected to x

$$\frac{\partial J_{OWLS}}{\partial x} = -\bar{\mathcal{H}}^T\bar{\mathcal{R}}^{-1}(\bar{y}-\bar{\mathcal{H}}x) = 0, \tag{9}$$

one can obtain the OWLS estimate as follows:

$$\begin{aligned} \hat{x}_{OWLS} &= \left(\bar{\mathcal{H}}^T\bar{\mathcal{R}}^{-1}\bar{\mathcal{H}}\right)^{-1}\bar{\mathcal{H}}^T\bar{\mathcal{R}}^{-1}\bar{y} \\ &= \left(\mathcal{Q}^{-1}+\mathcal{H}^T\mathcal{R}^{-1}\mathcal{H}\right)^{-1}\left(\mathcal{Q}^{-1}\bar{x}+\mathcal{H}^T\mathcal{R}^{-1}y\right) \\ &= \bar{x} + \left(\mathcal{Q}^{-1}+\mathcal{H}^T\mathcal{R}^{-1}\mathcal{H}\right)^{-1}\mathcal{H}^T\mathcal{R}^{-1}\left(y-\mathcal{H}^T\bar{x}\right). \end{aligned} \tag{10}$$

The OWLS estimate (10) exists if and only if

$$\frac{\partial^2 J_{OWLS}}{\partial x^2} = \bar{\mathcal{H}}^T\bar{\mathcal{R}}^{-1}\bar{\mathcal{H}} = \mathcal{Q}^{-1}+\mathcal{H}^T\mathcal{R}^{-1}\mathcal{H} > 0. \tag{11}$$

■

The noise-free measurement matrix $\mathcal{H}$ required to define the OWLS solution would not be available in many actual applications. In those cases, the estimate is inevitably constructed by given matrix $\tilde{\mathcal{H}}$ and it is commonly called the *nominal* WLS estimate.

Lemma 2.2. (Nominal WLS Solution) Replacing $\mathcal{H}$ with $\tilde{\mathcal{H}}$ in (10) gives us the nominal WLS estimate.

$$\begin{aligned} \hat{x}_{WLS} &= \left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}\right)^{-1}\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\bar{y} \\ &= \left(\mathcal{Q}^{-1}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\tilde{\mathcal{H}}\right)^{-1}\left(\mathcal{Q}^{-1}\bar{x}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}y\right) \\ &= \bar{x} + \left(\mathcal{Q}^{-1}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\tilde{\mathcal{H}}\right)^{-1}\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\left(y-\tilde{\mathcal{H}}^T\bar{x}\right) \end{aligned} \tag{12}$$

The existence condition of the nominal WLS estimate is given by

$$\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}} = \mathcal{Q}^{-1}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\tilde{\mathcal{H}} > 0 \tag{13}$$

■

It is obvious that the above nominal WLS estimate provides erroneous result in the presence of the multiplicative measurement noise $\Delta\mathcal{H}$. The quantitative analysis will unveil the characteristics of these estimation errors.

Lemma 2.3. (*Error Properties of Nominal WLS Solution*) Using the measurement vector (1), the WLS estimate (12) can be rewritten by

$$\begin{aligned}\hat{x}_{WLS} &= \left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}\right)^{-1}\tilde{\bar{\mathcal{H}}}^T\tilde{\bar{\mathcal{R}}}^{-1}\left([\tilde{\bar{\mathcal{H}}}-\Delta\bar{\mathcal{H}}]x+\bar{v}\right) \\ &\triangleq (I-\alpha)\,x+\beta\end{aligned} \tag{14}$$

From (14), it becomes clear that the nominal WLS estimate contains the scale-factor error α and the bias error β.

$$\begin{aligned}\alpha &\triangleq \left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}\right)^{-1}\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\Delta\bar{\mathcal{H}} \qquad (15)\\ &= \left(\mathcal{Q}^{-1}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\tilde{\mathcal{H}}\right)^{-1}\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\Delta\mathcal{H}\\ \beta &\triangleq \left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}\right)^{-1}\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\bar{v} \qquad (16)\\ &= \left(\mathcal{Q}^{-1}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\tilde{\mathcal{H}}\right)^{-1}\left(\mathcal{Q}^{-1}(\bar{x}-x)+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\Delta\mathcal{H}\right)\end{aligned}$$

■

As shown in Lemma 2.3, the scale-factor error in the nominal WLS solution occurs due to $\Delta\mathcal{H}$ itself. Similarly, the correlation between the multiplicative measurement noise $\Delta\mathcal{H}$ and the additive measurement noise v generates the bias error.

2.2 Robust weighted least squares estimation based on stochastic approximation

The RWLS estimation strategy to be suggested is strongly motivated from the fact that the nominal WLS solution could be sensitive to the multiplicative measurement noise $\Delta\mathcal{H}$ which frequently appears in actual applications.

Proposition 2.1. (*Approximation of Nominal WLS Estimation Errors*) Considering a large ensemble average of $\Delta\mathcal{H}$ will make sense the following approximation.

$$\left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}\right)\alpha \approx E\left\{\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\Delta\bar{\mathcal{H}}\right\} = W \tag{17}$$

$$\left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}\right)\beta \approx E\left\{\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\bar{v}\right\} = V \tag{18}$$

Then the scale-factor and bias errors of the nominal WLS solution can be successfully approximated without knowing the multiplicative measurement noise $\Delta\mathcal{H}$.

$$\hat{\alpha} \triangleq \left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}\right)^{-1}W = \left(\mathcal{Q}^{-1}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\tilde{\mathcal{H}}\right)^{-1}W \tag{19}$$

$$\hat{\beta} \triangleq \left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}\right)^{-1}V = \left(\mathcal{Q}^{-1}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\tilde{\mathcal{H}}\right)^{-1}V \tag{20}$$

■

Combining the nominal WLS estimate and error compensating terms (19) and (20), one can derive a RWLS solution.

Proposition 2.2. (*RWLS Solution*) Under the assumption that the nominal WLS solution always exists, from the form of WLS estimation errors (14), one gets the RWLS solution.

$$\begin{aligned}\hat{x}_{RWLS} &\triangleq (I-\hat{\alpha})^{-1}\left(\hat{x}_{WLS}-\hat{\beta}\right)\\ &= \left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}-W\right)^{-1}\left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\bar{y}-V\right)\\ &= \mathcal{P}\left(\mathcal{Q}^{-1}\bar{x}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}y-V\right)\\ &= (I+\mathcal{P}W)\,\bar{x}+\mathcal{P}\tilde{\mathcal{H}}^T\mathcal{R}^{-1}(y-\tilde{\mathcal{H}}\bar{x})-\mathcal{P}V\end{aligned} \tag{21}$$

where the Gramian matrix $\mathcal{P}$ is

$$\mathcal{P} \triangleq \left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}-W\right)^{-1} = \left(\mathcal{Q}^{-1}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\tilde{\mathcal{H}}-W\right)^{-1} \tag{22}$$

■

Eq. (22) implies that the Gramian matrix $\mathcal{P}$ must be invertible to ensure the existence of the RWLS solution (21).

Lemma 2.4. (*Existence of RWLS Solution*) The RWLS solution exists if $\mathcal{P} > 0$. □

Proof. Recalling the fact that the proposed RWLS solution is derived from the nominal WLS solution, the existence condition of the nominal WLS solution should be checked together with the nonsingularity of the Gramian matrix $\mathcal{P}$. Since one can assume that $W \geq 0$ without loss of generality, if the Gramian matrix $\mathcal{P}$ is positive definite, the following result can be obtained.

$$\mathcal{P}^{-1} > 0 \;\rightarrow\; \mathcal{Q}^{-1}+\tilde{\mathcal{H}}^T\mathcal{R}^{-1}\tilde{\mathcal{H}} > W \geq 0 \tag{23}$$

Therefore, the positive definiteness of $\mathcal{P}$ is a sufficient condition for the existence of the nominal WLS solution as well as the proposed RWLS solution.

■

Different from the existing robust filters, the proposed RWLS estimation scheme provides unique stochastic property.

Lemma 2.5. (*Unbiasedness of Weighted RWLS Estimate*) The weighted RWLS estimation error is zero mean.

$$E\left\{\mathcal{P}^{-1}\left(\hat{x}_{RWLS}-x\right)\right\} = 0 \tag{24}$$

□

Proof. It is straightforward to prove the unbiasedness of weighted RWLS estimate. Substituting (1) into (21) results in

$$\mathcal{P}^{-1}\left(\hat{x}_{RWLS}-x\right) = -\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\Delta\bar{\mathcal{H}}x+\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\bar{v}-V+Wx.$$

Taking expectation for both sides, one can obtain the desired result.

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3. STOCHASTIC ROBUST KALMAN FILTERING

3.1 Reformulation of RWLS estimation problem

Now, an alternative viewpoint to the proposed RWLS solution is introduced. The RWLS solution derived by using the stochastic error compensation could be regarded as the unique minimizing solution of a certain indefinite quadratic cost function.

Recall the cost function (8) of the optimal WLS problem in Lemma 2.1 and decompose it as follows:

$$J_{OWLS} = J^{a}_{OWLS} + J^{b1}_{OWLS} + J^{b2}_{OWLS} \tag{25}$$

where

$$J^{b1}_{OWLS} \triangleq \frac{1}{2}(\bar{y}-\tilde{\bar{\mathcal{H}}}x)^T\bar{\mathcal{R}}^{-1}(\bar{y}-\tilde{\bar{\mathcal{H}}}x)$$

$$J^{b2}_{OWLS} \triangleq \frac{1}{2}(\Delta\bar{\mathcal{H}}x)^T\bar{\mathcal{R}}^{-1}(\bar{y}-\tilde{\bar{\mathcal{H}}}x)+\frac{1}{2}(\bar{y}-\tilde{\bar{\mathcal{H}}}x)^T\bar{\mathcal{R}}^{-1}(\Delta\bar{\mathcal{H}}x) + \frac{1}{2}(\Delta\bar{\mathcal{H}}x)^T\bar{\mathcal{R}}^{-1}(\Delta\bar{\mathcal{H}}x)$$

Taking expectation for the second term gives us the approximated cost function.

$$\begin{aligned}\hat{J}_{OWLS} &= J^{a}_{OWLS} + J^{b1}_{OWLS} + E\{J^{b2}_{OWLS}\} \\ &= \frac{1}{2}\left(\begin{bmatrix}\bar{y}\\0\end{bmatrix}-\begin{bmatrix}\tilde{\bar{\mathcal{H}}}\\-I\end{bmatrix}x\right)^T\begin{bmatrix}\bar{\mathcal{R}}^{-1} & 0\\ 0 & -W\end{bmatrix}\left(\begin{bmatrix}\bar{y}\\0\end{bmatrix}-\begin{bmatrix}\tilde{\bar{\mathcal{H}}}\\-I\end{bmatrix}x\right) \\ &+\frac{1}{2}\left(x^TV+V^Tx\right)\end{aligned} \tag{26}$$

It is noteworthy that the resultant quadratic cost function is not positive or negative definite but indefinite. Hence, the problem is to find the saddle point of the given indefinite cost. Differentiating the above indefinite cost yields the stationarizing condition.

$$\frac{\partial \hat{J}_{OWLS}}{\partial x} = -\left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}(\bar{y}-\tilde{\bar{\mathcal{H}}}x)+Wx\right)+V=0 \tag{27}$$

From the above stationarizing condition, one gets the same estimate with the RWLS solution (21). If the second differentiation of $\hat{J}_{OWLS}$ at the stationarizing point is positive definite, it becomes the unique minimum of the indefinite cost function $\hat{J}_{OWLS}$.

$$\frac{\partial^2 \hat{J}_{OWLS}}{\partial x^2} = \left(\tilde{\bar{\mathcal{H}}}^T\bar{\mathcal{R}}^{-1}\tilde{\bar{\mathcal{H}}}-W\right) = \mathcal{P}^{-1} > 0 \tag{28}$$

This minimum condition exactly coincides with the existence condition in Lemma 2.4 required to derive the RWLS solution using stochastic approximation. Accordingly, the cost function J_{RWLS} of the proposed RWLS estimation scheme is equivalent to the approximated optimal WLS criterion $\hat{J}_{OWLS}$.

Problem 3.1. (*RWLS Criterion*) The RWLS estimate is the minimizing solution of the indefinite quadratic cost (26). That is,

$$J_{RWLS} \triangleq \hat{J}_{OWLS} \tag{29}$$

3.2 Stochastic robust Kalman filter recursion

Let's consider the linear time-varying uncertain system which contains the multiplicative noise ΔH_k.

$$\begin{cases} x_{k+1} = F_k x_k + G_k u_k \\ y_k \;\;= H_k x_k + v_k = [\tilde{H}_k - \Delta H_k]x_k + v_k\end{cases} \tag{30}$$

In the above state-space realization, with the initial guess $\hat{x}_{0|-1}$ and its estimation error $\tilde{x}_{0|-1} = x_0 - \hat{x}_{0|-1}$, the zero-mean white noises u_k and v_k satisfies

$$cov\langle \begin{bmatrix}\tilde{x}_{0|-1}\\u_k\\v_k\end{bmatrix}, \begin{bmatrix}\tilde{x}_{0|-1}\\u_j\\v_j\end{bmatrix}\rangle = \begin{bmatrix}P_{0|-1} & 0\\ 0 & \begin{bmatrix}Q_k & 0\\0 & R_k\end{bmatrix}\delta_{kj}\end{bmatrix}. \tag{31}$$

where δ_{kj} means the Dirac-delta function.

In addition, it is assumed that the noisy measurement matrix $\tilde{H}_k$ and the statistics on the unknown variable ΔH_k are given for the Kalman filtering.

$$\begin{aligned} E\{\Delta H_k^T R_k^{-1}\Delta H_k\} = W_k, \\ E\{\Delta H_k^T u_k\} = 0,\ E\{\Delta H_k^T v_k\} = V_k\end{aligned} \tag{32}$$

If the *a posteriori* estimate $\hat{x}_{k|k}$ is given at k, , it is well-known that, in the context of the stochastic optimization in Bryson et al. [1975], the standard Kalman filtering problem is defined as a single-stage minimization problem of

$$\begin{aligned} J_{KF} &= \|x_k - \hat{x}_{k|k}\|^2_{P^{-1}_{k|k}} + \|u_k\|^2_{Q_k^{-1}} \\ &+\|y_{k+1} - [\tilde{H}_{k+1} - \Delta H_{k+1}]x_{k+1}\|^2_{R^{-1}_{k+1}} \\ &= J^a_{KF} + J^{b1}_{KF} + J^{b2}_{KF}\end{aligned} \tag{33}$$

where

$$J^a_{KF} = \|x_k - \hat{x}_{k|k}\|^2_{P^{-1}_{k|k}} + \|u_k\|^2_{Q_k^{-1}}$$

$$J^{b1}_{KF} = \|y_{k+1}-\tilde{H}_{k+1}x_{k+1}\|^2_{R^{-1}_{k+1}}$$

$$J^{b2}_{KF} = \|y_{k+1}-[\tilde{H}_{k+1}-\Delta H_{k+1}]x_{k+1}\|^2_{R^{-1}_{k+1}} - J^{b1}_{KF}$$

According to the observation in Problem 3.1, we are able to set the cost function of a stochastic robust Kalman filtering.

$$\begin{aligned} J_{SRKF} &= \hat{J}_{KF} = J^a_{KF} + J^{b1}_{KF} + E\{J^{b2}_{KF}\} \\ &= \left\|\begin{bmatrix}x_k - \hat{x}_{k|k}\\ u_k\end{bmatrix}\right\|^2_{P^{-1}_{k|k}\oplus Q_k^{-1}} \\ &+ \left\|\begin{bmatrix}y_{k+1}\\0\end{bmatrix} - \begin{bmatrix}\tilde{H}_{k+1}\\I\end{bmatrix}x_{k+1}\right\|^2_{R^{-1}_{k+1}\oplus -W_{k+1}} \\ &+ x^T_{k+1}V_{k+1} + V^T_{k+1}x_{k+1}\end{aligned} \tag{34}$$

Theorem 3.1. (*Stochastic Robust Kalman Filter Recursion*) For linear time-varying systems (30) with multiplicative measurement noise ΔH_k and the additive noises u_k and v_k, the stationarizing solution of the indefinite quadratic cost function J_{SRKF} is recursively computed by the following formulas.

(*measurement update*):

$$\begin{aligned} \hat{x}_{k|k} &= \left(I + P_{k|k}W_k\right)\hat{x}_{k|k-1} \\ &+ P_{k|k}\tilde{H}_k^T R_k^{-1}\left(y_k - \tilde{H}_k\hat{x}_{k|k-1}\right) - P_{k|k}V_k \\ P^{-1}_{k|k} &= P^{-1}_{k|k-1} + \tilde{H}_k^T R_k^{-1}\tilde{H}_k - W_k\end{aligned} \tag{35}$$

(*time update*):

$$\begin{aligned} \hat{x}_{k+1|k} &= F_k\hat{x}_{k|k} \\ P_{k+1|k} &= F_kP_{k|k}F_k^T + G_kQ_kG_k^T\end{aligned} \tag{36}$$

□

Proof. Using the state-space equation (30) describing the given system, J_{SRKF} can be rewritten as

$$J_{SRKF}=\left\|\begin{bmatrix} x_k - \hat{x}_{k|k} \\ u_k \end{bmatrix}\right\|^2_{P^{-1}_{k|k}\oplus Q^{-1}_k} \tag{37}$$
$$+\left\|\begin{bmatrix} y_{k+1} \\ 0 \end{bmatrix}-\begin{bmatrix}\tilde{H}_{k+1}\left[F_k\ G_k\right] \\ I \end{bmatrix}\begin{bmatrix} x_k \\ u_k \end{bmatrix}\right\|^2_{R^{-1}_{k+1}\oplus -W'_{k+1}}$$
$$+\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \left[F_k\ G_k\right]^T V_{k+1}+V^T_{k+1}\left[F_k\ G_k\right]\begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

where

$$W'_{k+1} \triangleq \left[F_k\ G_k\right]^T W_{k+1}\left[F_k\ G_k\right]$$

Comparing (37) with (26) yields the following correspondences.

$$x \mapsto \begin{bmatrix} x_k \\ u_k \end{bmatrix},\ \bar{x} \mapsto \begin{bmatrix} \hat{x}_{k|k} \\ 0 \end{bmatrix},\ y \mapsto y_{k+1}$$
$$\tilde{H} \mapsto \tilde{H}_{k+1}\left[F_k\ G_k\right], \mathcal{Q} \mapsto \begin{bmatrix} P_{k|k} & 0 \\ 0 & Q_k \end{bmatrix}, \mathcal{R} \mapsto R_{k+1}, \tag{38}$$
$$W \mapsto W'_{k+1},\ V \mapsto \left[F_k\ G_k\right]^T V_{k+1}$$

Substituting the above relation for the RWLS solution (21) results in

$$\mathcal{P}^{-1}\begin{bmatrix} \hat{x}_{k|k+1} \\ \hat{u}_{k|k+1} \end{bmatrix} \tag{39}$$
$$=\begin{bmatrix} P^{-1}_{k|k} & 0 \\ 0 & Q^{-1}_k \end{bmatrix}\begin{bmatrix} \hat{x}_{k|k} \\ 0 \end{bmatrix}+\begin{bmatrix} F^T_k \\ G^T_k \end{bmatrix}\tilde{H}^T_{k+1}R^{-1}_{k+1}y_{k+1}-\begin{bmatrix} F^T_k \\ G^T_k \end{bmatrix}V_{k+1}$$

where, by definitions of (2) and (22),

$$\mathcal{P}^{-1}=\begin{bmatrix} P^{-1}_{k|k} & 0 \\ 0 & Q^{-1}_k \end{bmatrix}+\begin{bmatrix} F^T_k \\ G^T_k \end{bmatrix}\left(\tilde{H}^T_{k+1}R^{-1}_{k+1}\tilde{H}_{k+1}-W_{k+1}\right)\left[F_k\ G_k\right] \tag{40}$$

At this point, it is helpful for further argument to define the *a posteriori estimate* at $k+1$ using interim variables $\hat{x}_{k|k+1}$ and $\hat{u}_{k|k+1}$.

$$\hat{x}_{k+1|k+1} \triangleq F_k\hat{x}_{k|k+1} + G_k\hat{u}_{k|k+1} \tag{41}$$

Then (39) can be simplified as follows:

$$\begin{bmatrix} \hat{x}_{k|k+1}-\hat{x}_{k|k} \\ \hat{u}_{k|k+1} \end{bmatrix}=\begin{bmatrix} P_{k|k} & 0 \\ 0 & Q_k \end{bmatrix}\begin{bmatrix} F^T_k \\ G^T_k \end{bmatrix}\times \tag{42}$$
$$\left(\left[\tilde{H}^T_{k+1}R^{-1}_{k-1}\ \ W_{k+1}\right]\begin{bmatrix} y_{k+1}-\tilde{H}_{k+1}\hat{x}_{k+1|k+1} \\ \hat{x}_{k+1|k+1} \end{bmatrix} - V_{k+1}\right)$$

Inserting $\hat{x}_{k|k+1}$ and $\hat{u}_{k|k+1}$ in (42) into (41) yields

$$\begin{aligned}\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} \\ &+ P_{k+1|k}\tilde{H}^T_{k+1}R^{-1}_{k+1}(y_{k+1} - \tilde{H}_{k+1}\hat{x}_{k+1|k+1}) \\ &+ P_{k+1|k}W_{k+1}\hat{x}_{k+1|k+1} - P_{k+1|k}V_{k+1} \\ &= \left(I + P_{k+1|k}(\tilde{H}^T_{k+1}R^{-1}_{k+1}\tilde{H}_{k+1} - W_{k+1})\right)^{-1} \times \\ &\left(\hat{x}_{k+1|k} + P_{k+1|k}\tilde{H}^T_{k+1}R^{-1}_{k+1}y_{k+1} - P_{k+1|k}V_{k+1}\right)\end{aligned} \tag{43}$$

where the *a priori* estimate $\hat{x}_{k+1|k}$ and its error covariance $P_{k+1|k}$ are

$$\hat{x}_{k+1|k} = F_k\hat{x}_{k|k}, \tag{44}$$
$$P_{k+1|k} = F_kP_{k|k}F^T_k + G_kQ_kG^T_k. \tag{45}$$

As a result, these equations constitute time-update equations of the proposed filter.

Defining the *a posteriori* estimation error covariance

$$P^{-1}_{k+1|k+1} = P^{-1}_{k+1|k} + \tilde{H}^T_{k+1}R^{-1}_{k+1}\tilde{H}_{k+1} - W_{k+1}, \tag{46}$$

after simple matrix manipulations, one gets the measurement update equation .

$$\begin{aligned}\hat{x}_{k+1|k+1} &= \left(I + P_{k+1|k+1}W_{k+1}\right)\hat{x}_{k+1|k} \\ &+ P_{k+1|k+1}\tilde{H}^T_{k+1}R^{-1}_{k+1}\left(y_{k+1} - \tilde{H}_{k+1}\hat{x}_{k+1|k}\right) \\ &- P_{k+1|k+1}V_{k+1}\end{aligned} \tag{47}$$

This is the end of proof. ■

Theorem 3.2. (*Existence Condition of the Stochastic Robust Kalman Filter*) If $P_{0|-1} > 0,\ Q_k > 0,\ R_k$ is invertible and $[F_k\ G_k]$ has full rank, then the stochastic robust Kalman filter estimates minimizes the indefinite quadratic cost function if and only if

$$P^{-1}_{k|k} = P^{-1}_{k|k-1} + \tilde{H}^T_kR^{-1}_k\tilde{H}_k - W_k > 0. \tag{48}$$

□

Proof. If the symmetric matrix W_{k+1} can be decomposed by $W_{k+1} = E^T_{k+1}E_{k+1}$, from (28) and (40), the unique minimum of the indefinite quadratic cost J_{RWS} is exists if and only if

$$\mathcal{P}^{-1} = R_z - R_{zy}R^{-1}_yR_{yz} > 0 \tag{49}$$

where

$$R_z = \begin{bmatrix} P^{-1}_{k|k} & 0 \\ 0 & Q^{-1}_k \end{bmatrix},\ R_y = -\begin{bmatrix} R_{k+1} & 0 \\ 0 & -I \end{bmatrix},$$
$$R_{yz} = R^T_{zy} = \begin{bmatrix} H_{k+1} \\ E_{k+1} \end{bmatrix}\left[F_k\ G_k\right]$$

According to the Sylvester's law, the above inequality condition is equivalent to the following inertia conditions.

$$\begin{aligned}(49) &\Leftrightarrow \mathcal{I}_-\left\{R_y\right\} = \mathcal{I}_{-1}\left\{R_z\right\} + \mathcal{I}_-\left\{R_y - R_{yz}R^{-1}_zR_{zy}\right\} \\ &\Leftrightarrow R_z > 0,\ \mathcal{I}_-\left\{R_y\right\} = \mathcal{I}_-\left\{R_y - R_{yz}R^{-1}_zR_{zy}\right\}\end{aligned} \tag{50}$$

where $\mathcal{I}_-\{\bullet\}$ means the number of negative eigenvalues of the given matrix.

Let's assume that $R_z > 0$, and $[F_k\ G_k]$ has full rank. Then, from the Riccati equation (36), it is obvious that $P_{k+1|k} > 0$. Therefore, to prove the existence condition in the above lemma, we should check the inertia condition (50).

Using the congruent transform, we have

$$\mathcal{I}_-\left\{\begin{bmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & \mathcal{C} \end{bmatrix}\right\} = \mathcal{I}_-\left\{\begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{C} - \mathcal{B}^T\mathcal{A}^{-1}\mathcal{B} \end{bmatrix}\right\} \tag{51}$$
$$= \mathcal{I}_-\left\{\begin{bmatrix} \mathcal{A} - \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^T & 0 \\ 0 & \mathcal{C} \end{bmatrix}\right\} \tag{52}$$

where it has been defined that

$$\mathcal{A} = \begin{bmatrix} P^{-1}_{k+1|k} & 0 \\ 0 & Q^{-1}_{k+1} \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} H^T_{k+1} & E^T_{k+1} \\ 0 & 0 \end{bmatrix},\ \mathcal{C} = R_y,$$
$$\mathcal{A} - \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^T = \begin{bmatrix} P^{-1}_{k+1|k+1} & 0 \\ 0 & Q^{-1}_{k+1} \end{bmatrix},$$
$$\mathcal{C} - \mathcal{B}^T\mathcal{A}^{-1}\mathcal{B} = R_y - R_{yz}R^{-1}_zR_{zy}.$$

Since $\mathcal{A} > 0$ when $P_{k|k} > 0,\ Q_k > 0$ and $[F_k\ G_k]$ has full rank, we can analogize the following results.

$$\mathcal{A} - \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^T > 0,\ \mathcal{I}_- \left\{\mathcal{C} - \mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}\right\} = \mathcal{I}_- \left\{\mathcal{C}\right\} \quad (53)$$

That is, the inertia condition (50) is satisfied.

$$P_{k+1|k+1}^{-1} > 0,\ \mathcal{I}_- \left\{R_y\right\} = \mathcal{I}_- \left\{R_y - R_{yz}R_z^{-1}R_{zy}\right\} \quad (54)$$

Therefore, the existence of the stochastic robust Kalman filtering solution can be recursively checked by $P_{k|k} > 0$ under the assumption that $P_{0|-1} > 0$, $Q_k > 0$, and $[F_k\ G_k]$ has full rank for all k. This is the end of proof. ■

4. CONCLUSION

A new approach has been taken to the problem of robust Kalman filtering for linear time-varying uncertain systems with a noisy measurement matrix and a multiplicative measurement noise. Aside from the classical approaches, the problem was treated in view of classical weighted least squares estimation. The estimation errors of the nominal Kalman filter caused by multiplicative noise were characterized and approximated by the stochastic approximation method. The error compensating methodology has been used to solve the stochastic robust Kalman filtering problem without conservatism. It was shown that the stochastic robust Kalman filtering problem is equivalent to the minimization of a certain indefinite quadratic cost. The proposed robust Kalman filter and its existence condition have been derived in terms of just single discrete Riccati recursion. Moreover, the resulting filter had the structural flexibility of the RLS estimator, hence it could account for the cross-correlation between the multiplicative and additive measurement noises as well as the auto-correlation of the state-dependent noise itself.

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