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On the regularity for singular linear systems with Markov jump parameters ⋆

Amanda L. P. Manfrim * **Marco H. Terra** *
Eduardo F. Costa ** **João Y. Ishihara** ***

* *University of São Paulo at São Carlos, C.P.359, São Carlos, SP, 13566-590, Brazil (e-mail: amanfrim,terra@sel.eesc.usp.br).*
** *Depto. de Matemática Aplicada e Estatística, University of São Paulo at São Carlos, C.P.668, São Carlos, SP, 13560-970, Brazil, currently visiting the Electrical and Electronic Engineering Dept., Imperial College London SW72AZ, London, UK (e-mail: efcosta@icmc.usp.br)*
*** *Department of Electrical Engineering, University of Brasília, CEP 70.919-970, Brasília/DF, Brazil, (e-mail: ishihara@ene.unb.br)*

Abstract
In this paper we consider how to extend the regularity notion of usual singular (descriptor) systems to singular systems with Markovian jumping parameters. Three regularity definitions are introduced: the first one is based on a collection of matrices which defines the transitions of the continuous state and the other ones take into account the stochastic nature of the system, by using information regarding conditional first and second moments. Numerical examples illustrate the difference between these three notions.

Keywords: Singular systems; Markov jump parameters; Regularity; Stochastic systems; Descriptor systems.

1. INTRODUCTION

A problem of prime concern in many practical situations is related to analysis of dynamical systems which are subject to abrupt changes in their parameters. In this paper, we consider particularly that class of systems which can be modeled as a discrete-time linear descriptor (singular) system with abrupt changes that can be described by Markovian chain with finite state-space. The associated literature has increased steadily, with particular attention to continuous-time singular systems with Markov jumping (see Ibrir and Boukas [2003], Yan-Ming et al. [2005], Xu and Lam [2006]).

The study of singular linear system with Markov jump parameter (SLSMJP) is motivated, for example, by the fact that systems in singular formulation frequently arise naturally in the process of modeling (see Xu and Lam [2006], and references in therein). During the past ten years much attention has been devoted to investigate such a class of stochastic systems (Costa and do Val [2002], Costa et al. [2005], de Souza et al. [2006] and the references therein), which is of both practical and theoretical importance.

For standard singular systems, the most primary condition to be verified is the regularity property (Yip and Sincovec [1981], Lewis [1986], Dai [1989]). This is due to the fact that regularity assures that the system has solutions and each solution is unique for each admissible initial condition. We observe that the attempts to extend this notion to singular systems with Markov jumping found in the literature usually consider the regularity of each subsystem. They do not consider the stochastic nature of the system. These attempts can be considered as a particular case of our first notion of regularity which is based on the analysis of a collection of matrices. This first definition does not fully take into account the stochastic nature of the SLSMJP. It also disregards the available information on the transition probabilities of the jump variable (despite it considers the matrices that characterizes the continuous state transitions), see Xu and Lam [2006].

Therefore, two more notions are introduced related to the first and the second moment of a certain system variable, respectively. It turns out that the three definitions are not equivalent. However we can show that second moment regularity implies first moment regularity. More insight about the difference among the three notions of regularity is provided by some numerical examples.

The paper is organized as follows. In Section 2 we present notation and preliminary concepts. The concepts of regularity for SLSMJP are described in Section 3. In Section 4 we present numerical examples.

2. NOTATION AND PRELIMINARY CONCEPTS

Let $\mathbb{R}^n$ be the Euclidean linear space formed by n-vectors. Let $\mathcal{R}^{r,n}$ (respectively, $\mathcal{R}^r$) represent the normed linear space formed by all $r \times n$ real matrices (respectively, $r \times r$) and $\mathcal{R}^{r0}$ ($\mathcal{R}^{r+}$) the closed convex cone $\{U \in \mathcal{R}^r : U =$

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$U' \geq 0\}$, (the open cone $\{U \in \mathcal{R}^r : U = U' > 0\}$), where U' denotes the transpose of U; $U \geq V$ ($U > V$) signifies that $U - V \in \mathcal{R}^{r0}$ ($U - V \in \mathcal{R}^{r+}$). Let $\mathcal{M}^{r,n}$ denote the linear space formed by a number N of matrices such that $\mathcal{M}^{r,n} = \{U = (U_1, \ldots, U_N) : U_i \in \mathcal{R}^{r,n}, i = 1, \ldots, N\}$; also, $\mathcal{M}^r \equiv \mathcal{M}^{r,r}$. We denote by $\mathcal{M}^{r0}$ ($\mathcal{M}^{r+}$) the set $\mathcal{M}^r$ when it is made up of $U_i \in \mathcal{R}^{r0}$ ($U_i \in \mathcal{R}^{r+}$) for all $i = 1, \ldots N$. We denote the expectation $\mathcal{E}\{\cdot|x_0, \theta_0\}$ symply by $\mathcal{E}\{\cdot\}$.

Define the operators $\varphi : \mathcal{R}^{m\times n} \to \mathbb{R}^{mn}$ and $\hat{\varphi} : \mathcal{M}^{n0} \to \mathcal{R}^{n^2N}$ as follows. For $V \in \mathcal{R}^n$, let v_i, $0 \leq i \leq n$, be such that $V = \left[v_1 \vdots v_2 \vdots \cdots \vdots v_n \right]$ and let

$$\varphi(V) := \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

For $U = (U_1, \ldots, U_N)$, we introduce the linear and invertible operator Costa and Fragoso [1993] $\hat{\varphi}(U) : \mathcal{M}^{n0} \to \mathcal{R}^{n^2N}$ as:

$$\hat{\varphi}(U) := \begin{bmatrix} \varphi(U_1) \\ \vdots \\ \varphi(U_N) \end{bmatrix}.$$

The operators $\varphi(\cdot)$ and $\hat{\varphi}(\cdot)$ are employed throughout this paper in order to facilitate the handling of various technical details.

For the Kronecker product $L \otimes K \in \mathcal{R}^{n^2}$ defined in the usual way, we have the next proposition, presented in Brewer [1978].

Proposition 1. For any L, $K \in \mathcal{R}^n$

(i) $(L \otimes K)' = L' \otimes K'$
(ii) for $H \in \mathcal{R}^n$, $Y = LKH' \Leftrightarrow \varphi(Y) = (H \otimes L)\varphi(K)$

Consider the standard linear singular system (SS)

$$Sx(k+1) = Fx(k), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, F is real constant matrix of appropriate dimension, and the matrix S may be singular, with $\text{rank}(S) = n_S \leq n$. The next definition is standard in the literature of SS (see Dai [1989], Yip and Sincovec [1981], and references therein).

Definition 2.1. The pencil (S, F) is called regular if there exist a constant scalar $\lambda \in \mathbb{C}$ such that $|\lambda S + F| \neq 0$, i.e., $\det(\lambda S - F) \neq 0$ except a finite number of $\lambda \in \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers.

The following characterization for regularity, presented in Dai [1989], Yip and Sincovec [1981], is based on the analysis of matrix pencils Gantmacher [1974] and on the analysis of discrete descriptor systems Luenberger [1978].

Theorem 1. The following statements are equivalent.

(i) (S, F) is regular;
(ii) If $X(0)$ is the null space of F (denoted by $Ker(F)$) and $X(k) = \{x|Fx \in SX(k-1)\}$, then $Ker(S) \cap X(k) = 0$, for $k = 0, 1, \ldots$;
(iii) If $Y(0) = Ker(F^T)$, $Y(k) = \{x|F'x \in S'Y(k-1)\}$ then $Ker(S') \cap Y(k) = 0$, for $k = 0, 1, \ldots$;
(iv) Let

$$G(t) = \begin{bmatrix} S & & & & \\ F & S & & & \\ & F & \ddots & & \\ & & \ddots & S & \\ & & & F & \end{bmatrix} \in \mathbb{R}^{(t+1)n \times nt}.$$

Then $\text{rank}(G(t)) = nt$, $t = 1, 2, \ldots$;
(v) Let

$$H(t) = \begin{bmatrix} S & F & & & \\ & S & F & & \\ & & \ddots & \ddots & \\ & & & S & F \end{bmatrix} \in \mathbb{R}^{nt \times n(t+1)}.$$

Then $\text{rank}(H(t)) = nt$, $t = 1, 2, \ldots$;
(vi) rank $(G(n)) = n^2$;
(vii) rank $(H(n)) = n^2$.

We consider the discrete-time SLSMJP, defined in a fundamental probability space $(\Omega, \bar{\mathcal{F}}, \mathbb{P})$, as

$$\Phi : \begin{cases} S_{\theta(k+1)}x(k+1) = F_{\theta(k)}x(k), \ k = 0, 1, \ldots \\ \qquad x(0) = x_0, \quad \theta(0) = \theta_0 \end{cases} \tag{2}$$

where the variable $x \in \mathbb{R}^n$ is referred to as the continuous state, or simply state, θ is the state of an underlying discrete-time homogeneous Markov chain $\Theta = \{\theta(k); k \geq 0\}$ having $\mathcal{N} = \{1, \ldots, N\}$ as state space and $\mathbb{P} = [p_{ij}]$, $i, j = 1, \ldots, N$ as the transition rate matrix. The matrices F_i and S_i, $i = 1, \ldots, N$, belong to the collections of N real constant matrix: $F = (F_1, \ldots, F_N)$, $\dim(F_i)= n \times n$, and $S = (S_1, \ldots, S_N)$, $\dim(S_i)= n \times n$, may be singular, with $\text{rank}(S_i) = r_{S_i} \leq n$.

For a set $\mathbb{A} \in \bar{\mathcal{F}}$ the indicator function $1_{\mathbb{A}}$ is defined in the usual way, that is, for $w \in \Omega$,

$$1_{\mathbb{A}}(w) = \begin{cases} 1 & \text{if} \quad w \in \mathbb{A} \\ 0 & \text{otherwise} \end{cases}$$

Notice that for any $i = 1, \ldots, N$, $1_{\{\theta(k)=i\}}(w) = 1$ if $\theta(k) = i$, and 0 otherwise.

At time step k, assuming $\theta(k) = i$ and $\theta(k+1) = j$, the dynamics are governed by the form

$$S_j x(k+1) = F_i x(k). \tag{3}$$

That is, we have random switches between descriptor systems. The aim of this paper is to study how to extend the regularity notion of descriptor system to SLSMJP. We present here three different concepts. One notion considers the behavior of the system for each possible realization of θ, as it considers (3) for each possible combination of $i, j \in \mathcal{N}$, and the other ones rely on the transition probabilities of the jumping system, in such a manner that realizations of θ with zero probability are disregarded.

3. THE CONCEPTS OF REGULARITY FOR SLSMJP

Regularity notions for dynamical systems are related to the idea of existence and uniqueness of solution for any given admissible initial condition. Since the SLSMJP is a stochastic system, there are many ways of characterizing existence, giving rise to different notions of regularity. The notions that we introduce in this section are related to the first and second moments of the process $x(k)$, via the quantities

$$q_i(k) := \mathcal{E}\left\{x(k)1_{\{\theta(k)=i\}} \,\middle|\, x_0,\, \theta_0\right\} \in \mathbb{R}^n;$$
$$Q_i(k) := \mathcal{E}\left\{x(k)x'(k)1_{\{\theta(k)=i\}} \,\middle|\, x_0,\, \theta_0\right\} \in \mathcal{R}^n.$$

At this point we are in position to introduce one regularity concept requiring the existence and uniqueness of SLSMJP.

Definition 3.1. The System Φ is said to be regular if $det(\lambda S_j - F_i)$ is not identically zero for each $j, i \in \mathcal{N}$, except a finite number of $\lambda \in \mathbb{C}$.

Note that the above definition does not fully take into account the stochastic nature of the SLSMJP, in the sense that it disregards the available information on the transition probabilities, $\mathbb{P}$. It basically requires that, for any realization of the Markov chain, $\omega \to \{\theta(0), \theta(1), \ldots\}$, the associated pairs $(S_{\theta(k+1)}, F_{\theta(k)})$, $k \geq 0$, are regular, even for ω such that $\mathbb{P}(\omega) = 0$.

In what follows we introduce the regularity notions associated with the quantities $q_i(k)$ and $Q_i(k)$. We first derive recursive equations for $q_i(k)$ and $Q_i(k)$. Let $\mathbb{I}_m$ denote the $m \times m$ identity matrix and, for $D_i \in \mathcal{M}^n$, $i = 1, \ldots, N$, $\text{diag}(D_i)$ be the $Nn \times Nn$ matrix with D_i in the diagonal and zero elsewhere. We define

(i) $\mathbb{F} := (\mathbb{P}' \otimes \mathbb{I}_n)\,\text{diag}(F_i) \in \mathcal{R}^{Nn}$;
(ii) $\mathbb{S} := \text{diag}\,(S_i) \in \mathcal{R}^{Nn}$;
(iii) $V := (\mathbb{P}' \otimes \mathbb{I}_{n^2}) \in \mathcal{R}^{Nn^2}$;
(iv) $H := \text{diag}\,(F_i \otimes F_i) \in \mathcal{R}^{Nn^2}$;
(v) $\mathcal{F} := V\,H \in \mathcal{R}^{Nn^2}$;
(vi) $\mathcal{S} := \text{diag}\,(S_i \otimes S_i) \in \mathcal{R}^{Nn^2}$;
(vii) $\hat{q}(k) = \begin{bmatrix} q_1(k) \\ q_2(k) \\ \vdots \\ q_N(k) \end{bmatrix} \in \mathbb{R}^{Nn}$;
(viii) $Q = (Q_1, \ldots, Q_N) \in \mathcal{M}^{n0}$.

Theorem 2. For $k = 0, 1, \ldots$

(a) $\mathbb{S}\hat{q}(k+1) = \mathbb{F}\hat{q}(k)$
(b) $\mathcal{S}\hat{\varphi}(Q(k+1)) = \mathcal{F}\hat{\varphi}(Q(k))$

Proof. (a).

$$\begin{aligned}
&\mathcal{E}\left\{S_{\theta(k+1)}x(k+1)1_{\theta(k+1)=j}\right\} \\
&\qquad = \mathcal{E}\left\{F_{\theta(k)}x(k)1_{\theta(k+1)=j}\right\} \\
&S_j\mathcal{E}\left\{x(k+1)1_{\theta(k+1)=j}\right\} \\
&\qquad = \sum_{i=1}^{N}\mathcal{E}\left\{F_{\theta(k)}x(k)1_{\theta(k+1)=j}1_{\theta(k)=i}\right\} \\
&S_jq_j(k+1) = \sum_{i=1}^{N}p_{ij}F_iq_i(k)
\end{aligned} \tag{4}$$

Equation (4) can be equivalently written as

$$\begin{bmatrix} S_1 & 0 & \ldots & 0 \\ 0 & S_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & S_N \end{bmatrix}\begin{bmatrix} q_1(k+1) \\ q_2(k+1) \\ \vdots \\ q_N(k+1) \end{bmatrix} = \begin{bmatrix} p_{11}F_1 & p_{21}F_2 & \ldots & p_{N1}F_N \\ p_{12}F_1 & p_{22}F_2 & \ldots & p_{N2}F_N \\ \vdots & \vdots & \ddots & \vdots \\ p_{1N}F_1 & p_{2N}F_2 & \ldots & p_{NN}F_N \end{bmatrix}\begin{bmatrix} q_1(k) \\ q_2(k) \\ \vdots \\ q_N(k) \end{bmatrix}, \tag{5}$$

which leads to the result in **(a)** with

$$\hat{q}(k) = \begin{bmatrix} q_1'(k) & q_2'(k) & \cdots & q_N'(k) \end{bmatrix}'.$$

(b). Consider

$$\begin{aligned}
W_j(k+1) &= \mathcal{E}\Big\{S_{\theta(k+1)}x(k+1)x'(k+1) \\
&\qquad S'_{\theta(k+1)}1_{\{\theta(k+1)=j\}}\Big\} \\
&= S_j\mathcal{E}\left\{x(k+1)x'(k+1)1_{\{\theta(k+1)=j\}}\right\}S_j' \\
&= S_jQ_j(k+1)S_j', \qquad j = 1, \ldots, N
\end{aligned} \tag{6}$$

or, equivalently,

$$\begin{bmatrix} W_1(k+1) \\ W_2(k+1) \\ \vdots \\ W_N(k+1) \end{bmatrix} = \begin{bmatrix} S_1Q_1(k+1)S_1' \\ S_2Q_2(k+1)S_2' \\ \vdots \\ S_NQ_N(k+1)S_N' \end{bmatrix}. \tag{7}$$

Employ Proposition 1 into (7) yields

$$\hat{\varphi}(W(k+1)) = \begin{bmatrix} S_1 \otimes S_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_N \otimes S_N \end{bmatrix} \times \hat{\varphi}(Q(k+1)) \tag{8}$$

i.e.,

$$\hat{\varphi}\big(W(k+1)\big) = \mathcal{S}\,\hat{\varphi}\big(Q(k+1)\big). \tag{9}$$

On the other hand, $W_j(k+1)$ can also be written as

$$\begin{aligned}
W_j(k+1) &= \mathcal{E}\Big\{\left(F_{\theta(k)}x(k)\right)\left(F_{\theta(k)}x(k)\right)' \\
&\qquad 1_{\{\theta(k+1)=j\}}\Big\} \\
&= \mathcal{E}\left\{F_{\theta(k)}x(k)x'(k)F'_{\theta(k)}1_{\{\theta(k+1)=j\}}\right\} \\
&= \sum_{i=1}^{N}\mathcal{E}\Big\{\left(F_{\theta(k)}x(k)x'(k)F'_{\theta(k)}\right)1_{\{\theta(k)=i\}} \\
&\qquad 1_{\{\theta(k+1)=j\}}\Big\} \\
&= \sum_{i=1}^{N}p_{ij}F_i\mathcal{E}\left\{x(k)x'(k)1_{\{\theta(k)=i\}}\right\}F_i' \\
&= \sum_{i=1}^{N}p_{ij}F_iQ_i(k)F_i'
\end{aligned} \tag{10}$$

i.e.,

$$\begin{bmatrix} W_1(k+1) \\ \vdots \\ W_N(k+1) \end{bmatrix} = \begin{bmatrix} p_{11}F_1Q_1(k)F_1' + \ldots + p_{N1}F_NQ_N(k)F_N' \\ \vdots \\ p_{1N}F_1Q_1(k)F_1' + \ldots + p_{NN}F_NQ_N(k)F_N' \end{bmatrix}. \tag{11}$$

Employing Proposition 1 into (11), we can check that

$$\hat{\varphi}(W(k+1)) = \mathcal{F}\hat{\varphi}(Q(k)) \tag{12}$$

and, identifying (9) and (12),

$$\mathcal{S}\,\hat{\varphi}\big(Q(k+1)\big) = \mathcal{F}\,\hat{\varphi}\big(Q(k)\big). \tag{13}$$

♢

Theorem 2 shows that the quantity q (related to the first moment of process $x(k)$) evolves in time as a state of a SS, satisfying the difference equation (1) with (S, F) replaced by $(\mathbb{S}, \mathbb{F})$. Similarly, Q (related to the second moment of $x(k)$) satisfies (1) with (S, F) replaced by $(\mathcal{S}, \mathcal{F})$. This provides a natural way for extending the Definition 2.1 to SLSMJP, as follows.

Definition 3.2. The System Φ is said to be first moment (f.m.) regular if $(\mathbb{S}, \mathbb{F})$ is regular.

Definition 3.3. The System Φ is said to be second moment (s.m.) regular if $(\mathcal{S}, \mathcal{F})$ is regular.

Theorem 1 can be applied to check regularity of the pencil $(\mathbb{S}+\lambda\mathbb{F})$, immediately leading to the next result, presented here without proof. See Yip and Sincovec [1981], Dai [1989], and references therein for similar proofs.

Theorem 3. The following statements are equivalent.

(i) The System Φ is f.m. regular;
(ii) If $X(0) = Ker(\mathbb{F}),\ X(k) = \{x|\mathbb{F}x \in \mathbb{S}X(k-1)\}$, it must be $Ker(\mathbb{S}) \cap X(k) = 0,\ k = 0, 1, \ldots$;
(iii) If $Y(0) = Ker(\mathbb{F}^T),\ Y(k) = \{x|\mathbb{F}^T x \in \mathbb{S}^T Y(k-1)\}$, $Ker(\mathbb{S}^T) \cap Y(k) = 0,\ k = 0, 1, \ldots$;
(iv) Let

$$\mathbb{G}(t) = \begin{bmatrix} \mathbb{S} & & & \\ \mathbb{F} & \mathbb{S} & & \\ & \mathbb{F} & \ddots & \\ & & \ddots & \mathbb{S} \\ & & & \mathbb{F} \end{bmatrix} \in \mathbb{R}^{(t+1)nN\times Nnt}.$$

Then $\text{rank}(\mathbb{G}(t)) = Nnt,\ t = 1, 2, \ldots$;

(v) Let

$$\mathbb{H}(t) = \begin{bmatrix} \mathbb{S} & \mathbb{F} & & & \\ & \mathbb{S} & \mathbb{F} & & \\ & & \ddots & \ddots & \\ & & & \mathbb{S} & \mathbb{F} \\ & & & & \mathbb{S}\ \mathbb{F} \end{bmatrix} \in \mathbb{R}^{Nnt\times Nn(t+1)}.$$

Then $\text{rank}(\mathbb{H}(t)) = Nnt,\ t = 1, 2, \ldots$;

(vi) rank $(\mathbb{G}(Nn)) = N^2n^2$;
(vii) rank $(\mathbb{H}(Nn)) = N^2n^2$.

Theorem 4. The following statements are equivalent

(i) The System Φ is s.m. regular;
(ii) If $X(0) = Ker(\mathcal{F}),\ X(k) = \{x|\mathcal{F}x \in \mathcal{S}X(k-1)\}$, it must be $Ker(\mathcal{S}) \cap X(k) = 0$;
(iii) If $Y(0) = Ker(\mathcal{F}^T),\ Y(k) = \{x|\mathcal{F}^T x \in \mathcal{S}^T Y(k-1)\}$, $Ker(\mathcal{S}^T) \cap Y(k) = 0$;
(iv) Let

$$\mathcal{G}(t) = \begin{bmatrix} \mathcal{S} & & & \\ \mathcal{F} & \mathcal{S} & & \\ & \mathcal{F} & \ddots & \\ & & \ddots & \mathcal{S} \\ & & & \mathcal{F} \end{bmatrix} \in \mathbb{R}^{(t+1)n^2N\times Nn^2t}.$$

Then $\text{rank}(\mathcal{G}(t)) = Nn^2t,\ t = 1, 2, \ldots$;

(v) Let

$$\mathcal{H}(t) = \begin{bmatrix} \mathcal{S} & \mathcal{F} & & & \\ & \mathcal{S} & \mathcal{F} & & \\ & & \ddots & \ddots & \\ & & & \mathcal{S} & \mathcal{F} \\ & & & & \mathcal{S}\ \mathcal{F} \end{bmatrix} \in \mathbb{R}^{Nn^2t\times Nn^2(t+1)}.$$

Then $\text{rank}(\mathcal{H}(t)) = Nn^2t,\ t = 1, 2, \ldots$;

(vi) rank $(\mathcal{G}(n^2N)) = N^2n^4$;
(vii) rank $(\mathcal{H}(n^2N)) = N^2n^4$.

Remark 1. Note that, when we consider $S_j = \mathbb{I}_n,\ i \in \mathcal{N}$, the System Φ reduces to a non-singular Markov jump linear system. Accordingly to Definitions 3.1 – 3.3, this system is regular, f.m. regular and s.m. regular, respectively. Indeed, it is a well-known fact that solutions for non-singular Markov jump systems (with finite Markov state space) exist and are unique, see e.g. Costa et al. [2005].

Remark 2. Consider the case when $N = 1$, sometimes referred to as a "degenerated" Markov jump case. It is a straightforward task to check that the SLSMJP reduces to a SS and that the Definitions 3.1 – 3.3 are equivalent to require that (S_1, F_1) is regular.

3.1 Relation between the first moment and second moment regularity

In this section we show that s.m. regularity is stronger than f.m. regularity, by employing the characterizations for regularity given in theorems 3 and 4.

Lemma 1. If $\text{rank}(\mathcal{H}(t))$ is full then $\text{rank}(\mathbb{H}(t))$ is full.

Proof. We present here only a sketch of the proof. We consider the case $n = 2$ and $N = 2$, for notational simplicity; systems with larger dimensions can be handled similarly. Let

$$S_1 = \begin{bmatrix} s_{11_1} & s_{12_1} \\ s_{21_1} & s_{22_1} \end{bmatrix},\ S_2 = \begin{bmatrix} s_{11_2} & s_{12_2} \\ s_{21_2} & s_{22_2} \end{bmatrix},$$

$$F_1 = \begin{bmatrix} f_{11_1} & f_{12_1} \\ f_{21_1} & f_{22_1} \end{bmatrix},\ F_2 = \begin{bmatrix} f_{11_2} & f_{12_2} \\ f_{21_2} & f_{22_2} \end{bmatrix}$$

$$\text{and} \quad \mathbb{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

By Theorem 3 we have

$$\mathbb{H}(t) = \begin{bmatrix} \mathbb{S} & \mathbb{F} & 0 & \cdots & 0 \\ 0 & \mathbb{S} & \mathbb{F} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbb{S} & \mathbb{F} \end{bmatrix}, \quad t = 1, 2, \ldots$$

and, by Theorem 4,

$$\mathcal{H}(t) = \begin{bmatrix} \mathcal{S} & \mathcal{F} & 0 & \cdots & 0 \\ 0 & \mathcal{S} & \mathcal{F} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{S} & \mathcal{F} \end{bmatrix}, \quad t = 1, 2, \ldots$$

The result can be proved by considering the negative implication, that is, we show that if $\mathbb{H}(t)$ is not full rank then $\mathcal{H}(t)$ is also not full rank for each $t = 1, 2, \ldots$ In this situation, there exists a linear combination of the rows of $\mathbb{H}(t)$ that equals to zero with scalar coefficients β_i, not all equal to zero. This leads, in particular, to the following set of equations

$$\begin{cases} \beta_1 s_{11_1} + \beta_2 s_{21_1} = 0 \\ \beta_3 s_{11_2} + \beta_3 s_{21_2} = 0 \\ \beta_1 p_{11} f_{11_1} + \beta_2 p_{11} f_{21_1} \\ \qquad + \beta_3 p_{12} f_{11_1} + \beta_4 p_{12} f_{21_1} = 0 \end{cases} \tag{14}$$

Taking the squares in the first equality in (14), we immediately get that

$$\beta_1^2 s_{11_1}^2 + \beta_1\beta_2 s_{11_1} s_{21_1} + \beta_1\beta_2 s_{11_1} s_{21_1} + \beta_2^2 s_{21_1}^2 = 0 \tag{15}$$

and multiplying the thirty equation of (14) by f_{11_1}

$$\begin{aligned} &\beta_1 p_{11} f_{11_1}^2 + \beta_2 p_{11} f_{11_1} f_{21_1} \\ &\qquad + \beta_3 p_{12} f_{11_1}^2 + \beta_4 p_{12} f_{11_1} f_{21_1} = 0 \end{aligned} \tag{16}$$

and by f_{21_1}

$$\begin{aligned} &\beta_1 p_{11} f_{11_1} f_{21_1} + \beta_2 p_{11} f_{21_1}^2 \\ &\qquad + \beta_3 p_{12} f_{11_1} f_{21_1} + \beta_4 p_{12} f_{21_1}^2 = 0 \end{aligned} \tag{17}$$

We need to analyze for which values of α_i, $i = 1, \ldots, Nn^2t$, the following equation holds

$$\alpha_1 \mathcal{H}_1 + \alpha_2 \mathcal{H}_2 + \ldots + \alpha_{Nn^2t} \mathcal{H}_{Nn^2t} = 0 \tag{18}$$

where $\mathcal{H}_l$ is the l-th row of matrix $\mathcal{H}(t)$, $t = 1, 2, \ldots$ In particular for $t = 1$ and we rewrite (18) as

$$\alpha_1 \mathcal{H}_1 + \alpha_2 \mathcal{H}_2 + \ldots + \alpha_8 \mathcal{H}_8 = 0. \tag{19}$$

From (19) we have that

$$\alpha_1 s_{11_1}^2 + \alpha_2 s_{11_1} s_{21_1} + \alpha_3 s_{21_1} s_{11_1} + \alpha_4 s_{21_1}^2 = 0. \tag{20}$$

Identifying (15) and (20) we are able to find α_i as function of β_j with $\alpha_1 = \beta_1^2$, $\alpha_2 = \alpha_3 = \beta_1\beta_2$ and $\alpha_4 = \beta_2^2$ however, from (19) we also have

$$\begin{aligned} &\alpha_1 p_{11} f_{11_1}^2 + \alpha_2 p_{11} f_{11_1} f_{21_1} + \alpha_3 p_{11} f_{11_1} f_{21_1} \\ &\quad + \alpha_4 p_{11} f_{21_1}^2 + \alpha_5 p_{12} f_{11_1}^2 + \alpha_6 p_{12} f_{11_1} f_{21_1} \\ &\qquad + \alpha_7 p_{12} f_{11_1} f_{21_1} + \alpha_8 p_{12} f_{21_1}^2 = 0. \end{aligned} \tag{21}$$

adding (16) and (17) we get

$$\begin{aligned} &\beta_1 p_{11} f_{11_1}^2 + \beta_2 p_{11} f_{11_1} f_{21_1} + \beta_1 p_{11} f_{11_1} f_{21_1} \\ &\quad + \beta_2 p_{11} f_{21_1}^2 + \beta_3 p_{12} f_{11_1}^2 + \beta_4 p_{12} f_{11_1} f_{21_1} \\ &\qquad + \beta_3 p_{12} f_{11_1} f_{21_1} + \beta_4 p_{12} f_{21_1}^2 = 0 \end{aligned} \tag{22}$$

and identifying in (21) and (22) we are able to find α_i as function of β_j with $\alpha_1 = \alpha_3 = \beta_1$, $\alpha_2 = \alpha_4 = \beta_2$, $\alpha_5 = \alpha_7 = \beta_3$, $\alpha_6 = \alpha_8 = \beta_4$. Note that $\alpha_1 = \beta_1^2$ and $\alpha_1 = \beta_1$, $\alpha_2 = \beta_1\beta_2$ and $\alpha_2 = \beta_2$, $\alpha_3 = \beta_1\beta_2$ and $\alpha_3 = \beta_1$, and $\alpha_4 = \beta_2^2$ and $\alpha_4 = \beta_2$ so $\beta_1 = \beta_2$ with $\beta_1 = 0$ or $\beta_1 = 1$. Thus we can find non zero α_i satisfing (19), hence the claim.

$\diamondsuit$

4. NUMERICAL EXAMPLES

Example 4.1. Consider the system

$$S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbb{P} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}.$$

Note that

$$\det(\lambda S_1 - F_2) = 0,$$

so, by Definition 3.1, we have that this system is not regular, but by Theorem 3 this system is f.m. regular even though by Theorem 4 this system is not s.m. regular.

Note that, in Example 4.1 the system is neither regular by Definition 3.1 nor s.m. regular, although it is f.m. regular. This example show that regularity of f.m. not imply in regularity of s.m., i.e. the converse of Lemma 1 does not hold.

The next example presents one case when the system is regular, according to Definition 3.1, but it is neither f.m. nor s.m. regular.

Example 4.2. Consider the system

$$S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad F_1 = F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{and} \quad \mathbb{P} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

Note that

$$\det(\lambda S_1 - F_1) = \det(\lambda S_1 - F_2) \neq 0$$
$$\det(\lambda S_2 - F_1) = \det(\lambda S_2 - F_2) \neq 0,$$

so, by Definition 3.1, this system is regular, and by Theorem 4 we have that this system is not s.m. regular. This shows that regularity by Definition 3.1 does not imply in regularity by the second moment.

5. CONCLUSION

In this paper we provide regularity concepts for SLSMJP. In Definition 3.1 we present a concept that is based on a collection of matrices which defines the system, but it does not take into account the transition probabilities of the jump process. Statistical information should be relevant to characterize the system behavior, and in Definitions 3.2 and 3.3 we present f.m. and s.m. regularity notions that rely on this information, via the conditional moments of Theorem 2. We show in Lemma 1 that s.m. regularity implies f.m. regularity and, in Example 4.1, that the converse of Lemma 1 does not hold. Example 4.2 shows that regularity by Definition 3.1 does not imply s.m. regularity.

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