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Synchronization of Four Identical Nonlinear Systems with Time-delay ⋆

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Abstract: This paper considers the synchronization problem for four identical nonlinear systems coupled with time-delay. We have already studied the synchronization problem for bidirectional two coupled systems with delays and derived sufficient conditions to synchronize the systems. In this paper, these approaches are extended for four identical chaotic systems unidirectionally or bidirectionally coupled using state or output feedback with time-delays. Firstly, we show, using the small-gain theorem, that trajectories of coupled strictly semi-passive systems converge to a bounded region. Then we derive sufficient conditions for synchronization of coupled systems. The derived conditions are based on the delay-dependent Lyapunov-Krasovskii approach, and the criteria are obtained in the form of linear matrix inequalities (LMIs). The effectiveness of the derived conditions is illustrated by numerical examples.

1. INTRODUCTION

Synchronization phenomena are of interest to researchers in applied physics, biology, social science, engineering(Pecora and Carroll (1990.); Strogatz and Stewart (1993.); Nijmeijer and Mareels (1997); Pikovsky et al. (2001)). More recently, applications of these phenomena to engineering have also been considered and analyzed via control theory (Huijberts et al. (2007); Nijmeijer and Rodriguez-Angeles (2003); Oguchi and Nijmeijer (2005); Pogromsky et al. (2002)) . On the other hand, in practical situations, time-delays caused by signal transmission affect the behavior of coupled systems. It is therefore important to study the effect of time-delay in existing synchronization schemes. Although the effect of time-delay in the synchronization of coupled systems has been investigated both numerically and theoretically by a number of researchers, these works concentrate on synchronization of systems with a coupling term typically described by $K(x_i(t-\tau) - x_j(t-\tau))$ or $K(Cx_i(t-\tau) - Cx_j(t-\tau))$(Amano et al. (2006)) and there are few results for the case in which the coupling term is described by $K(x_i(t) - x_j(t-\tau))$. The former requires that each system has a feedback with the same length of time-delay as the transmittal delay, while the latter does not need such a delayed feedback. For the latter case, however, as the coupling term does not vanish when the systems synchronize, even if uncoupled each system is bounded, the coupled systems are not necessarily bounded. Therefore the synchronization problemin this case needs further study. We have already considered the problem for two chaotic systems bidirectionally coupled with the coupling term $K(x_i(t) - x_j(t-\tau))$ and derived sufficient conditions for synchronization (Yamamoto et al. (2007); Oguchi et al. (2007)).

In this paper we consider synchronization of identical chaotic systems unidirectionally or bidirectionally coupled using state or output feedback described by $K(Cx_i(t) - Cx_j(t-\tau))$. First, we introduce the notion of strict semi-dissipativity and show the boundedness of trajectories of coupled systems provided that each system is strictly semi-passive. Then we derive sufficient conditions for synchronization of the systems coupled unidirectionally or bidirectionalyl by using the stability criterion for delay systems.

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2. PRELIMINARIES

Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm. For a vector function $v(t) : [0,\infty) \to \mathbb{R}^n$, if $\|v\|_\infty \triangleq \sup_{t\geq 0} \|v(t)\| < \infty$, then we denote $v \in \mathcal{L}_\infty^n$.

In the following subsections, we review some results derived in our previous work (Yamamoto et al. (2007)).

2.1 Semi-passivity and semi-dissipativity

Consider the nonlinear system

$$\dot{x}_i(t) = f_i(x_i, u_i) \quad , \quad y_i(t) = C_i x_i \qquad (t \geq 0) \tag{1}$$

with state $x_i \in \mathbb{R}^n$, input $u_i \in \mathbb{R}^p$, output $y_i \in \mathbb{R}^m$, $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $C_i \in \mathbb{R}^{m\times n}$.

According to semi-passivity as defined in Pogromsky et al. (2002), we introduce strict semi-passivity and strict semi-dissipativity as follows.

Definition 1. (strict semi-passivity). Assume that $p = m$. System (1) is said to be strictly semi-passive, if there exist

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a C^1-class function $V_i : \mathbb{R}^n \to \mathbb{R}$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_i(\cdot)$ and $\overline{\alpha}_i(\cdot)$ satisfying

$$\underline{\alpha}_i(\|x_i\|) \le V_i(x_i) \le \overline{\alpha}_i(\|x_i\|)$$
$$\dot{V}_i(x_i) \le -H_i(x_i) + y_i^T u_i$$

for all $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m, y_i \in \mathbb{R}^m$, where the function $H_i(x_i)$ satisfies the following condition:

$$\|x_i\| \ge \eta_i \quad \Rightarrow \quad H_i(x_i) \ge 0$$

for a positive real number η_i.

Definition 2. (strict semi-dissipativity). System (1) is said to be strictly semi-dissipative with respect to the supply rate $w_i(u_i, y_i)$, if there exist a C^1-class function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}$ and class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_i(\cdot)$ and $\overline{\alpha}_i(\cdot)$ satisfying

$$\underline{\alpha}_i(\|x_i\|) \le V(x_i) \le \overline{\alpha}_i(\|x_i\|) \tag{2}$$
$$\dot{V}_i(x_i) \le w_i(u_i, y_i) - H_i(x_i) \tag{3}$$

for all $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^p, y_i \in \mathbb{R}^m$, where the function $H(x_i)$ satisfies

$$\|x_i\| \ge \eta_i \quad \Rightarrow \quad H_i(x_i) \ge 0 \tag{4}$$

for some positive real number η_i.

Remark 3. The system is strictly semi-passive if the supply rate $w_i(u_i, y_i) = y_i^T u_i$ for all $u_i \in \mathbb{R}^m$.

For a strictly semi-dissipative system, the following lemma can be proved in a similar way as the argument of the input-to-state stability (ISS) in Isidori (1999). For later use, we decompose u_i into l_i blocks such as $u_i = \mathrm{col}(u_{i1}, \ldots, u_{il_i})$ with $u_{ij} \in \mathbb{R}^{p_j}$ and $\sum_{j=1}^{l_i} p_j = p$, and consider the case of $y_i(t) = x_i(t)$.

Lemma 4. Suppose that system (1) is strictly semi-dissipative with respect to $w(u_i, y_i) \le \sum_{j=1}^{l_i} \beta_{ij}(\|u_{ij}\|)$ where $\alpha_i \in \mathcal{K}_\infty$ and $\beta_{ij} \in \mathcal{K}$, i.e. there exist a C^1-class function $V_i : \mathbb{R}^n \to \mathbb{R}$ such that

$$\underline{\alpha}_i(\|x_i\|) \le V_i(x_i) \le \overline{\alpha}_i(\|x_i\|) \tag{5}$$
$$\dot{V}_i(x_i) \le -\alpha_i(\|x_i\|) - H_i(x_i) + \sum_{j=1}^{l_i} \beta_{ij}(\|u_{ij}\|) \tag{6}$$

where $H_i(x_i)$ satisfies property (4), then the trajectories $x_i(t)$ of the system (1) satisfy the following inequality for any $u_{ij} \in \mathcal{L}_\infty^{p_j}$ and the initial state x_{i0}.

$$\limsup_{t\to\infty} \|x_i(t)\| \le \max_{1\le j\le l_i} \{\gamma_{ij}(\limsup_{t\to\infty} \|u_{ij}(t)\|), \rho_i(\eta_i)\} \tag{7}$$

where

$$\rho_i(\cdot) = \underline{\alpha}_i^{-1} \circ \overline{\alpha}_i(\cdot), \tag{8}$$
$$\gamma_{ij}(\cdot) = \underline{\alpha}_i^{-1} \circ \overline{\alpha}_i \circ \alpha_{ij}^{-1} \circ \kappa\beta_{ij}(\cdot) \qquad (j = 1, \ldots, l_i) \tag{9}$$

with $\sum_{j=1}^{l_i} \alpha_{ij}(\cdot) = \alpha_i(\cdot)$ and $\kappa > 1$.

2.2 Small-gain theorem

Now, we consider the case in which the two systems are coupled by the following inputs containing time-delay,

$$\begin{aligned} u_{11}(t) =& y_2(t-\tau) = C_2 x_2(t-\tau) \\ u_{21}(t) =& y_1(t-\tau) = C_1 x_1(t-\tau). \end{aligned} \tag{10}$$

where τ is a constant delay and the initial conditions of x_i for $i = 1, 2$ are respectively given by

$$\begin{aligned} x_i(\theta) &= \phi_i(\theta) \qquad (-\tau \le \theta \le 0) \\ x_i(0) &= \phi_i(0) = x_{0i} \end{aligned} \tag{11}$$

where $\phi_i : [-\tau, 0] \to \mathbb{R}^n$. In addition, we suppose that each system (1) is strictly semi-dissipative. Then from Lemma 4, each system satisfies the properties (7).

Define class-$\mathcal{K}$ functions as

$$\begin{aligned} \pi_{12}(r) &= \gamma_{11}(\sigma_{\max}(C_2) \cdot r) \\ \pi_{21}(r) &= \gamma_{21}(\sigma_{\max}(C_1) \cdot r) \end{aligned} \qquad (r \ge 0) \tag{12}$$

where $\gamma_{ij}(\cdot)$ are defined by (9) and $\sigma_{\max}(\cdot)$ denotes the maximum singular value of a matrix.

Then we obtain the following lemma.

Lemma 5. For two coupled systems (1) with coupling term (10), if the functions $\pi_{12}(\cdot)$ and $\pi_{21}(\cdot)$ in (12) satisfy

$$\pi_{12} \circ \pi_{21}(r) < r \qquad \text{for all } r > 0, \tag{13}$$

then the trajectories $x_1(t)$ and $x_2(t)$ satisfy

$$\limsup_{t\to\infty} \|x_1(t)\| \le \max\{\zeta_1, \pi_{12}(\zeta_2)\} \tag{14}$$
$$\limsup_{t\to\infty} \|x_2(t)\| \le \max\{\zeta_2, \pi_{21}(\zeta_1)\} \tag{15}$$

where $\zeta_i = \max_{2\le j\le l_i}\{\gamma_{ij}(\limsup_{t\to\infty}(\|u_{ij}\|), \rho_i(\eta_i)\}$ for any inputs $u_{ij} \in \mathcal{L}_\infty^{p_j}$.

This lemma can be also extended for systems with time-varying delays (Oguchi et al. (2007)).

3. 4-COUPLED SYSTEMS

We consider four identical systems:

$$\Sigma_i : \begin{cases} \dot{x}_i(t) = Ax_i + f(x_i) + Bu_i \\ y_i(t) = Cx_i \qquad (t \ge 0) \\ x_i(\theta) = \phi_i(\theta) \qquad (-\tau \le \theta \le 0) \end{cases} \tag{16}$$

for $i = 1, \ldots, 4$, where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$. In addition, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and $\phi_i : [-\tau, 0] \to \mathbb{R}^n$ with $\tau > 0$.

Now we assume that each system (16) is strictly semi-passive and these systems are coupled by the following controller

$$u_i(t) = \sum_{j=1, j\ne i}^{4} K_{ij}(y_i(t) - y_j(t - \tau_{ij})) \tag{17}$$

where the time-delays $0 < \tau_{ij} \le \tau$ are constants. This description includes the following two cases which will be considered later.

(i) $K_{21} = K_{32} = K_{43} = K_{14} \le 0$ and $K_{ij} = 0$
(ii) $K_{ij} = K_{ji} \le 0$, $\tau_{ij} = \tau_{ji}$

The former means that the network has an unidirectional ring structure and the latter that the coupling for each system is bidirectional. In addition, if $m = n$ and C is nonsingular, the coupling represents a full state coupling. While, if $m < n$, it is an output coupling. Here we formulate synchronization of coupled systems as follows.

Definition 6. If there exist a positive real number r such that the trajectories $x_i(t)$ of the systems (16) with initial conditions ϕ_i such that $\|\phi_i - \phi_j\|_c \le r$ satisfy $\|x_i(t) - x_j(t)\| \to 0 \ \ as \ \ t \to \infty$ for all i, j, then the coupled systems (16) and (17) are asymptotically synchronized.

The goal of this work is to derive conditions for coupled systems (16) and (17) to synchronize asymptotically. In

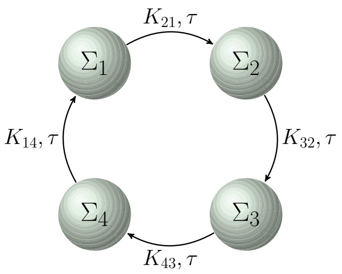

Fig. 1. An unidirectional ring network

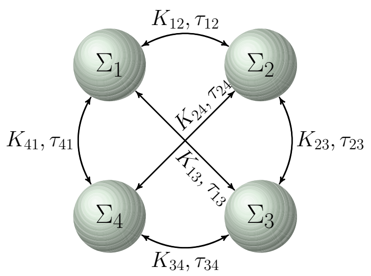

Fig. 2. A bidirectional coupled system

this paper, we investigate synchronization conditions for unidirectionally or bidirectionally coupled systems to be synchronized asymptotically.

3.1 Boundedness of Coupled Systems

We show under suitable assumptions the boundedness of the coupled systems (16) and (17). Firstly, we consider the case $m = n$. In addition, we assume the nonsingularity of the matrix C. Note that strictly semi-passive systems with coupling term (17) are strictly semi-dissipative. From the property, we obtain the following result by using Lemma 4.

Theorem 7. Define class-$\mathcal{K}$ functions as

$$\pi_{ij}(r) := \gamma_{ij}(\sigma_{max}(C)r)$$

for $r \geq 0$, where γ_{ij} are defined in Lemma 4. If the functions $\pi_{ij}(\cdot)$ satisfy

$$\pi_{ij}(r) < r \quad \text{for all} \quad r > 0,$$

then the trajectories of the coupled systems converge to the bounded set

$$\Omega = \{x \in \mathbb{R}^n | \|x_i\| \leq \rho(\eta)\}. \tag{18}$$

Remark 8. If the coupling gains satisfy $K_{21} = K_{32} = K_{43} = K_{14} \leq 0$ and $K_{ij} = 0$ for others, the coupled system has an unidirectional ring network structure. In this case, the functions γ_{ij} are identical, and the above condition is simplified as $\pi(r) < r$ for all $r > 0$.

Next, we consider the case of output coupling, that is $m < n$. Then we assume that CB is non-singular.

From the non-singularity of CB, the system (16) can be transformed to the following normal form Lozano et al. (2000).

$$\dot{y}_i(t) = a(y_i, z_i) + CBu_i \tag{19}$$
$$\dot{z}_i(t) = q(y_i, z_i) \tag{20}$$

for $i = 1, 2, 3, 4$, where $z_i \in \mathbb{R}^{n-m}$ and $\begin{bmatrix} y_i^T & z_i^T \end{bmatrix}^T = \Phi x_i$ for a nonsingular matrix $\Phi \triangleq \begin{bmatrix} C^T & N^T \end{bmatrix}^T$ with $N \in \mathbb{R}^{(n-m)\times n}$ such that $NB = 0$ and the functions $a : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and $q : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ are Lipschitz continuous.

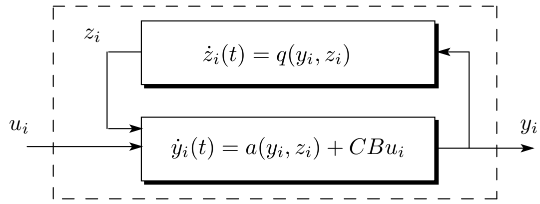

Fig. 3. Decomposition of each system

Here the first equation can be recognized as a system with with input $v_i^T = (u_i^T, z_i^T)$ and output y_i. At this point, we assume that

- The system (19) is strictly semi-dissipative with respect to the supply rate $w(v_i, y_i) \leq \beta_y(\|z_i\|) + y_i^T u_i$, where $\beta_y \in \mathcal{K}$, i.e. there exist a positive definite $\mathcal{C}^1$-class functions V_y, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_y$, $\overline{\alpha}_y$ and ϵ_y satisfying
$$\underline{\alpha}_y(\|y_i\|) \leq V_y(y_i) \leq \overline{\alpha}_y(\|y_i\|)$$
$$\dot{V}_y(y_i) \leq -\epsilon_y(\|y_i\|) - H_y(y_i) + \beta_y(\|z_i\|) + y_i^T u_i \tag{21}$$
for all $y_i \in \mathbb{R}^m, u_i \in \mathbb{R}^m, z_i \in \mathbb{R}^{n-m}$, where the function $H_y(y_i)$ satisfies that $H_y(y_i) \geq 0$ if $\|y_i\| \geq \eta_y$ for some positive real number η_y.
- The system (20) is strictly semi-dissipative with respect to the supply rate $w(y_i, z_i) \leq \beta_z(\|y_i\|)$, where $\beta_z \in \mathcal{K}$, i.e. there exist a positive definite $\mathcal{C}^1$-class functions V_z, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_z$, $\overline{\alpha}_z$ and α_z satisfying
$$\underline{\alpha}_z(\|z_i\|) \leq V_z(z_i) \leq \overline{\alpha}_z(\|z_i\|)$$
$$\dot{V}_z(z_i) \leq -\alpha_z(\|z_i\|) - H_z(z_i) + \beta_z(\|y_i\|)$$
for all $z_i \in \mathbb{R}^{n-m}, y_i \in \mathbb{R}^m$, where the function $H_z(z_i)$ satisfies that $H_z(z_i) \geq 0$ if $\|z_i\| \geq \eta_z$ for some positive real number η_z.

Substituting (17) for (21), we obtain

$$\begin{aligned}\dot{V}_y(y_i) &\leq -\epsilon_y(\|y_i\|) - H_y(y_i) + \beta_y(\|z_i\|) + y_i^T u_i \\ &\leq -\alpha_i(\|y_i\|) - H_y(y_i) + \beta_y(\|z_i\|) \\ &\qquad + \sum_{j=1, j\neq i}^{4} \beta_{ij}(\|y_{j,\tau_{ij}}\|)\end{aligned}$$

where $\alpha_i(r) = \epsilon_y(r) - \frac{1}{2}\sum_{j=1,j\neq i}^{4} \lambda_{max}(K_{ij})r^2$ and $\beta_{ij}(r) = -\frac{1}{2}\lambda_{min}(K_{ij})r^2$.

Then, by applying controller (17), from lemma 4, we know that

$$\begin{aligned}\limsup_{t\to\infty} \|y_i(t)\| \leq &\max_{1\leq j\leq 4, j\neq i} \{\gamma_y(\limsup_{t\to\infty} \|z_i\|), \\ &\gamma_{ij}(\limsup_{t\to\infty} \|y_j\|), \rho_y(\eta_y)\}\end{aligned}$$
$$\limsup_{t\to\infty} \|z_i(t)\| \leq \max\{\gamma_z(\limsup_{t\to\infty} \|y_i\|), \rho_z(\eta_z)\}$$

hold, where

$\rho_y(\cdot) = \underline{\alpha}_y^{-1} \circ \overline{\alpha}_y(\cdot),\ \gamma_y(\cdot) = \rho_y \circ \alpha_y^{-1} \circ \kappa\beta_y(\cdot),$
$\rho_z(\cdot) = \underline{\alpha}_z^{-1} \circ \overline{\alpha}_z(\cdot),\ \gamma_z(\cdot) = \rho_z \circ \alpha_z^{-1} \circ \kappa\beta_z(\cdot),$

$$\gamma_{ij}(\cdot) = \rho_y \circ \alpha_{ij}^{-1} \circ \kappa\beta_{ij}(\cdot),\ \alpha_y(r) + \sum_{j=1, j\neq i}^{4} \alpha_{ij}(r) = \alpha_i(r)$$

and $\alpha_{ij}(r) = \frac{1}{3}(\epsilon_y(r) - \alpha_y(r)) - \frac{1}{2}\lambda_{max}(K_{ij})r^2$ such that $\alpha_{ij} \in \mathcal{K}_\infty$.

Then we obtain the following theorem.

Theorem 9. For all $r > 0$ and $i, j (1 \le i < j \le 4)$, if

$$\gamma_y \circ \gamma_z(r) < r \quad \text{and} \quad \gamma_{ij}(r) < r \tag{22}$$

hold, then the trajectories of the system (16) converge to the set

$$\Omega \triangleq \{x_i \in \mathbb{R}^n \mid \|y_i\| \le s_y \text{ and } \|z_i\| \le s_z\} \tag{23}$$

where $s_y \triangleq \max\{\rho_y(\eta_y),\ \gamma_y \circ \rho_z(\eta_z)\}$,
$s_z \triangleq \max\{\rho_z(\eta_z), \gamma_z \circ \rho_y(\eta_y)\}$.

4. SYNCHRONIZATION

In this section, we derive conditions for synchronization of the coupled systems (16) with (17).

Define the synchronization error $e(t)$ as

$$e(t) := [e_{12}(t)^T, e_{13}(t)^T, e_{14}(t)^T]^T \tag{24}$$

with $e_{12}(t) = x_1(t) - x_2(t)$, $e_{13}(t) = x_1(t) - x_3(t)$, $e_{14}(t) = x_1(t) - x_4(t)$. Then $\lim_{t\to\infty} \|e(t)\| \to 0$ means all systems asymptotically synchronize. Therefore we consider a condition in which the dynamics of e has $e = 0$ as an asymptotically stable equilibrium. In general, however, the dynamics of e does not have $e = 0$ as a solution for arbitrary coupling gains and delays in (17). In order to give the network a symmetric structure, we consider the following two cases:

(i) **Unidirectional coupling:** $K_{21} = K_{32} = K_{43} = K_{14} \le 0$ and $K_{ij} = 0$, $\tau_{ij} = \tau$
(ii) **Bidirectional coupling:** $K_{12} = K_{34} := K_1$, $K_{13} = K_{24} := K_2$, $K_{23} = K_{41} := K_3$ and $\tau_{12} = \tau_{34} := \tau_1$, $\tau_{13} = \tau_{24} := \tau_2$, $\tau_{23} = \tau_{41} := \tau_3$

In these cases, the dynamics of the error $e(t)$ has $e = 0$ as an equilibrium solution. Therefore the synchronization problem for these cases can be reduced to the stability problem of the origin of the error dynamics.

4.1 Unidirectional Coupling

To begin with, we consider the case in which the network has an unidirectional ring structure. For this case, the synchronization error dynamics is given by

$$\dot{e}(t) = A_0(x_1)e(t) + A_1 e(t - \tau) \tag{25}$$

where $A_0(x_1) = I \otimes \hat{A}(x_1)$ with $\hat{A}(x_1) = A + BKC + D(x_1)$ for $D(x_1) \triangleq \left(\frac{\partial f(x)}{\partial x}\right)|_{x=x_1}$ and A_1 is defined as

$$A_1 = \begin{bmatrix} 0 & 0 & BKC \\ -BKC & 0 & BKC \\ 0 & -BKC & BKC \end{bmatrix}$$

in which $\otimes$ denotes the Kronecker product.

If $e = 0$ is asymptotically stable, $e(t)$ asymptotically converges to zero and synchronization is accomplished. Therefore the synchronization problem can be reduced to the stability problem for the above retarded system. By using the Lyapunov-Krasovskii theorem, we obtain the following synchronization condition.

Theorem 10. For all $x_1 \in \Omega$ given by (18) or (23), if there exist positive definite matrices $P, Q, Z \in \mathbb{R}^{3n\times 3n}$ and matrices $Y, W \in \mathbb{R}^{3n\times 3n}$ satisfying the following LMI, then $e = 0$ of the error dynamics is asymptotically stable.

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & -\tau Y & \tau A_0^T Z \\ \Gamma_{12}^T & \Gamma_{22} & -\tau W & \tau A_1^T Z \\ -\tau Y^T & -\tau W^T & -\tau Z & 0 \\ \tau Z A_0 & \tau Z A_1 & 0 & -\tau Z \end{bmatrix} < 0 \tag{26}$$

where

$$\Gamma_{11} = PA_0 + A_0^T P + Y + Y^T + Q$$
$$\Gamma_{12} = PA_1 - Y + W^T,\quad \Gamma_{22} = -Q - W - W^T$$

As the LMI (26) is affine with respect to the system matrices $A_0(x_1)$ and A_1, this result can be extended to a stability criterion for the polytopic systems.

Since x_1 is bounded, each element of $D(x_1)$ is also bounded. As a result, the approximated error dynamics (25) can be rewritten by the following polytopic system:

$$\dot{e}(t) = \sum_{i=1}^{m} p_i A_0^i e(t) + A_1 e(t - \tau)$$

where $A_0^i = A_0 + D_i$ are constant matrices and $p_i(x_1) \in [0, 1]$ are polytopic coordinates satisfying the convex sum property $\sum_{i=1}^m p_i(x_1) = 1$. Using the "vertex systems", we can obtain the following polytopic linear differential inclusion (PLDI)

$$\dot{e}(t) \in \mathrm{Co}\Big\{A_0^1 e(t) + A_1 e(t-\tau), \cdots, A_0^m e(t) + A_1 e(t - \tau)\Big\} \tag{27}$$

where Co denotes a convex hull. Therefore we can obtain the following stability criterion.

Theorem 11. Consider the PLDI (27). If there exist positive definite matrices $P, Q^i, Z \in \mathbb{R}^{3n\times 3n}$ and matrices $Y^i, W^i \in \mathbb{R}^{3n\times 3n}$ for $i = 1, \ldots, m$ satisfying the following LMI, then $e = 0$ of the error dynamics is asymptotically stable.

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12}^i & -\tau Y^i & \tau A_0^T Z \\ \Gamma_{12}^{iT} & \Gamma_{22}^i & -\tau W^i & \tau A_1^T Z \\ -\tau Y^{iT} & -\tau W^{iT} & -\tau Z & 0 \\ \tau Z A_0 & \tau Z A_1 & 0 & -\tau Z \end{bmatrix} < 0 \tag{28}$$

where

$$\Gamma_{11}^i = PA_0 + A_0^T P + Y^i + Y^{iT} + Q^i$$
$$\Gamma_{12}^i = PA_1 - Y^i + W^{iT},\quad \Gamma_{22}^i = -Q^i - W^i - W^{iT}$$

Using Theorem 11, we can check the stability of $e = 0$ by solving the finite number of LMIs.

Example 12. Consider a network of 4 coupled Lorenz systems.

$$\dot{x}_i(t) = \begin{bmatrix} \sigma(x_{i2} - x_{i1}) \\ r x_{i1} - x_{i2} - x_{i1}x_{i3} \\ -b x_{i3} + x_{i1}x_{i2} \end{bmatrix} + Bu_i\ ,\ y_i = Cx_i \tag{29}$$

where $\sigma = 10$, $r = 28$, $b = 8/3$ and $B^T = C$. At this stage, we assume that $m = 3$ and $C = I_{3\times 3}$ and the coupled system forms an unidirectional ring network with $K_{21} = K_{32} = K_{43} = K_{14} < 0$, $K_{ij} = 0$ and $\tau_{ij} = \tau$ in

(17). Now, take a storage function $V(\tilde{x}_i) = \frac{1}{2}\tilde{x}_i^T\tilde{x}_i$, where $\tilde{x}_i = [x_{i1}\ x_{i2}\ x_{i3} - \sigma - r]^T$. Then the derivative along the trajectory of each system is given by

$$\dot{V}(\tilde{x}_i) = -\alpha(\|\tilde{x}_i\|) - H(\tilde{x}_i) + \beta(\|\tilde{x}_{j\tau}\|)$$

for $(i,j) \in \{(1,4),(2,1),(3,2),(4,3)\}$, where the functions $\alpha(\cdot), \beta(\cdot)$ and $H(x)$ are defined as $\alpha(\|\tilde{x}_i\|) = (\frac{k}{2}+\epsilon)\|\tilde{x}_i\|^2$, $\beta(\|\tilde{x}_i\|) = \frac{k}{2}\|\tilde{x}_i\|^2$ and

$$\begin{aligned} H(\tilde{x}_i) =& (\sigma-\epsilon)\tilde{x}_{i1}^2 + (1-\epsilon)\tilde{x}_{12}^2 \\ &+ (b-\epsilon)(\tilde{x}_{13} - \frac{b-2\epsilon}{2(b-\epsilon)}(\sigma+r))^2 - \frac{b^2(\sigma+r)^2}{4(b-\epsilon)}, \end{aligned}$$

and $0 < \epsilon < 1$. Then $\gamma_{ij}(\cdot)$ is given by

$$\gamma_{ij}(r) = \sqrt{\frac{\kappa\frac{k}{2}}{(k/2+\epsilon)}}\ r, \tag{30}$$

where $\kappa > 1$. Since $C = I$, $\pi(\cdot) = \gamma(\cdot)$. Therefore, setting κ sufficiently close to 1, $\pi(r) < r$ holds for all $r > 0$ and any finite number $k > 0$. Furthermore, for $\epsilon = 0.01$, the minimum η satisfying $H(\tilde{x}_i) \geq 0$ is given by $\eta = 39.4$ and $\rho(\cdot)$ is the identity map. So the bounded set is estimated by

$$\Omega = \{\tilde{x}_i \in \mathbb{R}^3 | \|\tilde{x}_i\| \leq 39.4\}$$

and each trajectory $x_i(t)$ converges to the set Ω. Setting $k = 30$ and $\tau = 0.01$, the LMI criterion is satisfied for all $x_1 \in \Omega$. Figure 4 shows that the norm of $e(t)$ converges to zero and synchronization is achieved, and Figure 5 shows that the coupled systems behave chaotically under synchronization.

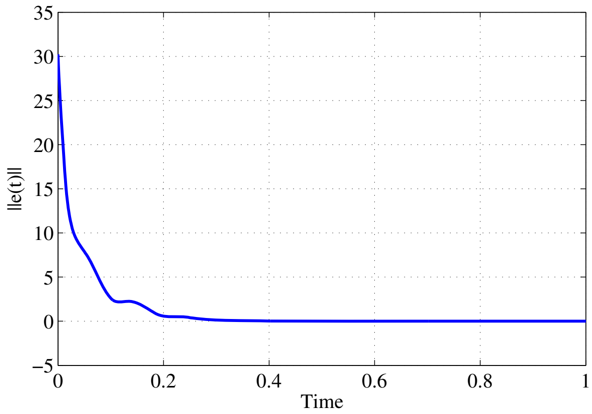

Fig. 4. The norm of the error $\|e(t)\|$

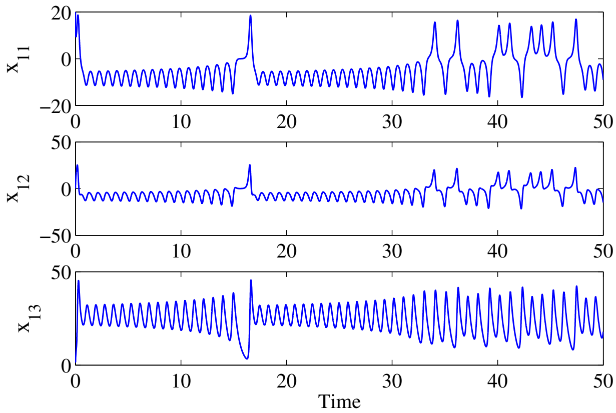

Fig. 5. Behavior of the state x_1

4.2 Bidirectional Coupling

Next we derive a synchronization condition for bidirectionally coupled systems. In a similar way as the unidirectional coupling case, the error dynamics is given by

$$\dot{e}(t) = \tilde{A}_0(x_1)e(t) + \sum_{j=1}^{3}\tilde{A}_j e(t-\tau_j)$$

where $\tilde{A}_0(x_1) = I \otimes \hat{A}(x_1)$ with $\hat{A}(x_1) = A + B(\sum_{j=1}^3 K_j)C + D(x_1)$ and $\tilde{A}_j$ for $j = 1,2,3$, are defined as

$$\tilde{A}_1 = \begin{bmatrix} \Lambda_1 & 0 & 0 \\ \Lambda_1 & 0 & -\Lambda_1 \\ \Lambda_1 & -\Lambda_1 & 0 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} 0 & \Lambda_2 & -\Lambda_2 \\ 0 & \Lambda_2 & 0 \\ -\Lambda_2 & \Lambda_2 & 0 \end{bmatrix}$$

$$\tilde{A}_3 = \begin{bmatrix} 0 & -\Lambda_3 & \Lambda_3 \\ -\Lambda_3 & 0 & \Lambda_3 \\ 0 & 0 & \Lambda_3 \end{bmatrix} \quad \text{where } \Lambda_i = BK_iC.$$

Then the synchronization condition is given by the following theorem.

Theorem 13. For all $x_1 \in \Omega$ given by (18) or (23), if there exist a positive definite matrix $P \in \mathbb{R}^{3n\times 3n}$ satisfying the following LMI, then $e = 0$ of the error dynamics is asymptotically stable.

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} & J_{12} & J_{13} \\ J_{12}^T & -J_{22} & 0 \\ J_{13}^T & 0 & -J_{33} \end{bmatrix} < 0 \tag{31}$$

where $\bar{A} = \tilde{A}_0(x_1) + \sum_{i=1}^3 \tilde{A}_i^T$, $J_{12} = [\tilde{A}_0(x_1)^T P, \tilde{A}_1^T P, \tilde{A}_2^T P, \tilde{A}_3^T P]$, $J_{22} = \frac{1}{\tau_1+\tau_2+\tau_3} diag\{P,P,P,P\}$, $J_{13} = [P\tilde{A}_1, P\tilde{A}_2, P\tilde{A}_3]$, and $J_{33} = \frac{1}{4}diag\{\frac{1}{\tau_1}P, \frac{1}{\tau_2}P, \frac{1}{\tau_3}P\}$.

This LMI criterion can be also rewritten by using PLDI:

$$\dot{e}(t) \in \mathrm{Co}\Big\{\tilde{A}_0^1 e(t) + \sum_{j=1}^3 \tilde{A}_j e(t-\tau_j), \cdots, \\ \tilde{A}_0^m e(t) + \sum_{j=1}^3 \tilde{A}_1 e(t-\tau_j)\Big\} \tag{32}$$

Theorem 14. If there exist a positive definite matrix $P \in \mathbb{R}^{3n\times 3n}$ satisfying the following LMIs, then $e = 0$ of the error dynamics is asymptotically stable.

$$\begin{bmatrix} \bar{A}^{iT} P + P\bar{A}^i & J_{12} & J_{13} \\ J_{12}^T & -J_{22} & 0 \\ J_{13}^T & 0 & -J_{33} \end{bmatrix} < 0,\ i = 1,\dots,m \tag{33}$$

where $\bar{A}^i = \tilde{A}_0^i + \sum_{i=1}^3 \tilde{A}_i^T$, $J_{12} = [\tilde{A}_0^{iT} P, \tilde{A}_1^T P, \tilde{A}_2^T P, \tilde{A}_3^T P]$, $J_{22} = \frac{1}{\tau_1+\tau_2+\tau_3} diag\{P,P,P,P\}$, $J_{13} = [P\tilde{A}_1, P\tilde{A}_2, P\tilde{A}_3]$, and $J_{33} = \frac{1}{4}diag\{\frac{1}{\tau_1}P, \frac{1}{\tau_2}P, \frac{1}{\tau_3}P\}$.

Example 15. Consider four Lorenz systems (29) bidirectionally coupled using output feedback with delays. For (29), we assume that $B^T = C = [0\ 1\ 0; 0\ 0\ 1]$.

For real numbers $k_1, k_2, k_3 > 0$, set coupling gains K_j as $K_1 = -k_1 I_{2\times 2}$, $K_2 = -k_2 I_{2\times 2}$ and $K_3 = -k_3 I_{2\times 2}$ and time-delays $\tau_1 = \tau_3 = 2\times 10^{-3}$ and $\tau_2 = 5\times 10^{-3}$.

Now letting $\tilde{y}_i = [x_{i2}, x_{i3} - r]^T$ and $z_i = x_{i1}$, define storage functions $V_y(\tilde{y}_i) \triangleq \frac{1}{2}\tilde{y}_i^T\tilde{y}_i$ and $V_z(z_i) \triangleq \frac{1}{2}z_i^2$. Then the derivative of $V_y(\tilde{y}_i)$ along the trajectory of (29) satisfies

$$\dot{V}_y(\tilde{y}_i) \le -\alpha_i(\|\tilde{y}_i\|) - H_y(\tilde{y}_i) + \sum_{j=1, j\neq i}^{4} \beta_{ij}(\|\tilde{y}_{j,\tau_{ij}}\|)$$

where $\alpha_i(\cdot) = \sum_{j=1}^{3} \alpha_{ij}(\cdot)$, $\alpha_{ij}(r) = \left(\frac{\epsilon}{3} - \frac{\lambda_{max}(K_{ij})}{2}\right) r^2$, $\beta_{ij}(r) = -\frac{\lambda_{min}(K_{ij})}{2} r^2$ and

$$H_y(\tilde{y}_i) = (1-\epsilon)\tilde{y}_{i1}^2 + (b-\epsilon)\tilde{y}_{i2}^2 + br\tilde{y}_{i2}$$

with $\epsilon = 0.01$. Here, the minimum η_y satisfying $H_y(\tilde{y}_i) \ge 0$ is 28.9. Similarly, for the function $V_z(z_i)$,

$$\dot{V}_z(z_i) \le -\alpha_z(\|z_i\|) + \beta_z(\|\tilde{y}_i\|)$$

holds, where $\alpha_z(r) = \frac{\sigma}{2} r^2$, $\beta_z(r) = \frac{\sigma}{2} r^2$. This means that the system of z_i is strictly dissipative with respect to $\beta_z(\|\tilde{y}_i\|)$ and therefore $\eta_z = 0$. In addition, from the definitions of the storage functions $V_y(\tilde{y}_i)$ and $V_z(z_i)$, $\rho_y(r) = \rho_z(r) = r$ for any $r \ge 0$. As a result, γ_{ij} is given by

$$\gamma_{ij}(r) = \sqrt{\frac{\kappa \frac{k_l}{2}}{\frac{k_l}{2} + \frac{\epsilon}{3}}}\, r \quad , l \in \{1, 2, 3\}$$

and satisfies (22) with κ sufficiently close to 1. Furthermore, $\gamma_z(r) = r$ for any $r \ge 0$ and since $\beta_y(\|z\|) = 0$, $\gamma_y(r) = 0$ which means (22) holds. Finally, we obtain $s_y = s_z = \eta_y = 28.9$. Therefore, from Theorem 9, the trajectories $x_i(t)$ converge to the set

$$\Omega = \{x_i \in \mathbb{R}^3 | (x_{i2}^2 + (x_{i3} - r)^2)^{\frac{1}{2}} \le 28.9 \text{ and } \|x_{i1}\| \le 28.9\}.$$

Setting $k_1 = 28, k_2 = 20$ and $k_3 = 28$, the LMI condition holds for all $x_1 \in \Omega$. Figure 6 shows the behavior of $x_1(t)$. In this figure, the cylinder denotes the estimated boundary of the set Ω. We know that the trajectories converge to the set Ω. Figure 7 shows the behavior of the norm of $e(t)$. Since the norm converges to zero, the synchronization of these systems is perfectly accomplished.

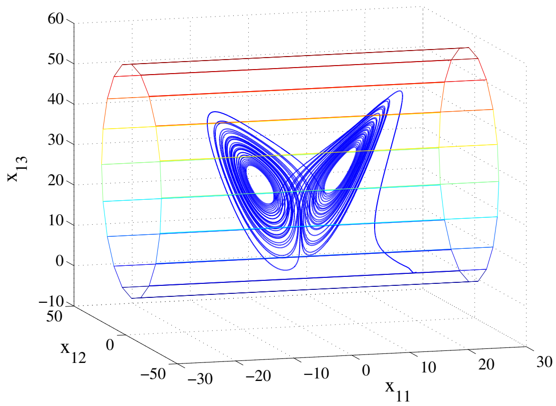

Fig. 6. Phase portrait of system 1

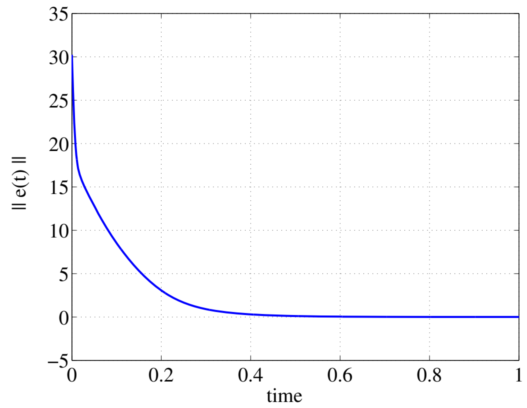

Fig. 7. The norm of the error $\|e(t)\|$

5. CONCLUSION

In this paper, we have considered conditions for synchronization of four nonlinear systems unidirectionally or bidirectionally coupled using state feedback or output feedback with time-delays cased by the signal exchange. For systems unidirectionally or bidirectionally coupled, we showed the boundedness of the strictly semi-dissipative systems by the small-gain theorem and estimated the region to which all trajectories converge. Then, for the coupled systems with symmetric structures, we derived sufficient conditions for synchronization of the systems by the Lyapunov-Krasovskii theorem. Finally, we like to mention that the boundedness of the trajectories of the coupled systems is global property but the stability of the origin of the error dynamics is local property along the trajectory of system 1 due to the derivation process. Therefore, the derivation of the global condition for synchronization should be addressed in our future work.

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