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Delay-dependent Robust H_∞ Control for Uncertain Singular Systems with Time-varying Delay ⋆

Huijiao WANG [*,**,***] Anke XUE [*] Renquan LU [*]
Xiaodong ZHAO [*] Xiaohui ZHOU [*]

[*] *Institute of Information & Control,Hangzhou Dianzi University, Hangzhou 310018, P.R.China(e-mail: hjwang@hdu.edu.cn; akxue@hdu.edu.cn; rqlu@hdu.edu.cn; xdzhao@hdu.edu.cn).*
[**] *Institute of Automation, Zhejiang Sci-Tech University, Hangzhou 310018, P.R.China*
[***] *National Laboratory of Industrial Control Technology, Zhejiang University,Hangzhou, Zhejiang, 310027, P.R.China*

Abstract: The problem of delay-dependent robust H_∞ control for uncertain singular systems with time-varying delay is addressed in this paper. The uncertainty is assumed to be norm bounded. By establishing an *integral inequality* based on quadratic terms, a new delay-dependent bounded real lemma is derived and expressed in terms of linear matrix inequality(LMI). A suitable robust H_∞ state feedback control law is presented, which guarantees that the resultant closed-loop system is regular, impulse-free and stable with disturbance attenuation level γ for all admissible uncertainties. Two numerical examples are given to demonstrate the applicability of the proposed method.

1. INTRODUCTION

Time-delays are frequently encountered in many fields of science and engineering(Hale and Lunel [1993], Gu et al. [2003]). Many significant results have been reported in the literature, see Cao et al. [1998], Gao and Chen [2007], Gao et al. [2004], Han [2004], Han [2005], Jiang and Han [2006], He et al. [2007], Zhang et al. [2006] and references therein. In the past few years, there have been various approaches to reduce the conservatism of delay-dependent conditions. For a system with small delay, a model transformation technique or bounding cross terms technique is often used to reduce the conservatism. But the model transformation may introduce additional dynamics(Gu and Niculescu [2000], Gu and Niculescu [2001]). Using bounding technique requires that some matrix variables should be limited to a certain structure to obtain controller synthesis conditions in terms of LMIs(Park et al. [1998], Park [1999]). This limitation introduces some conservatism.

On the other hand, singular systems, which are known as descriptor systems, implicit systems, generalized state-space systems or semi-state systems, have received much attention since singular model can preserve the structure of practical systems and can better describe a large class of physical systems than regular ones (Dai [1989], Lewis [1986]). The objective of robust H_∞ control for uncertain singular systems is to design a state feedback control law such that the resultant closed-loop system is regular, impulse-free(for continuous singular systems) and causal(for discrete singular systems), and stable with a given disturbance attenuation level for all admissible parameter uncertainties. For continuous singular time-delay systems, some sufficient conditions were obtained for the problem of robust H_∞ control (Xu et al. [2002], Shi et al. [2000], Yang and Zhang [2005], Fridman and Shaked [2002a]). However, the conditions obtained in Xu et al. [2002] are delay independent, which are conservative, especially for small delay. Shi et al. [2000] assumes that the nominal system is regular, impulse-free and stable, which limits its application. The criteria obtained in (Yang and Zhang [2005], Fridman and Shaked [2002a]) were under the assumption that the delay was constant, when the delay is time-varying, they are inapplicable. In practical systems, the time-delay is usually time-varying such as in networked control systems (Yue et al. [2004], Yue et al. [2005]). To the best of our knowledge, the class of uncertain singular time-varying delay systems has not yet been fully investigated. Particularly delay-dependent sufficient conditions of robust H_∞ control are few even not existing in the literature.

In this paper, the problem of robust H_∞ control is considered for a class of singular systems with time-varying delay and norm-bounded uncertainties. With the introduction of a new *integral inequality*, which is used in obtaining controller synthesis condition for singular systems for the first time, a strict LMI delay-dependent bounded real lemma for singular time-varying delay systems is obtained. The robust H_∞ control problem is also solved and an explicit expression of the desired state feedback control law is given, which can be obtained by solving the feasibility problem of a strict LMI. Two examples are given to show the effectiveness of the proposed method.

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2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the uncertain singular system with time-varying delay described by

$$\begin{cases} E\dot{x}(t) =(A+\Delta A)x(t)+(A_d+\Delta A_d)x(t-d(t)) \\ \qquad\qquad +(B+\Delta B)u(t)+B_{\omega 1}\omega(t) \\ z(t) =Cx(t)+Du(t)+B_{\omega 2}\omega(t) \\ x(t) =\phi(t), t=[-\bar{d},0] \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector. $\omega(t) \in \mathbb{R}^p$ is the disturbance input vector and $z(t) \in \mathbb{R}^q$ is the controlled output vector. E,A,A_d,B,$B_{\omega 1}$,$B_{\omega 2}$,C and D are constant matrices of appropriate dimensions, where E may be singular and we assume that rank$E = r \leq n$. $\Delta A, \Delta A_d$ and ΔB are unknown and possibly time-varying matrices representing norm-bounded parameter uncertainties and are assumed to be of the following form,

$$[\,\Delta A\ \ \Delta A_d\ \ \Delta B\,] = MF(t)\,[\,N_a\ \ N_d\ \ N_b\,] \quad (2)$$

where M, N_a, N_d, N_b are known constant matrices of appropriate dimensions, and $F(t)$ is an unknown matrix function satisfying $F^T(t)F(t) \leq I$. $d(t)$ is time-varying delay with known bound in system (1) such that

$$0 < d(t) \leq \bar{d},\ \dot{d}(t) \leq d < \infty \quad (3)$$

$\phi(t)$ is a compatible vector valued initial function.

The nominal unforced singular system of (1) can be written as

$$E\dot{x}(t) = Ax(t) + A_d x(t-d(t)) \quad (4)$$

Definition 1. (Dai [1989], Lewis [1986], Xu et al. [2002])
1) The pair (E, A) is said to be regular if $det(sE-A)$ is not identically zero.
2) The pair (E, A) is said to be impulse-free if $deg(det(sE-A)) = \text{rank}\ E$.

Definition 2. (Xu et al. [2002])
1) The singular system (4) is said to be regular and impulse free if the pair (E, A) is regular and impulse free.
2) The singular system (4) is said to stable if for any $\epsilon > 0$, there exists a scalar $\delta(\epsilon) > 0$ such that for any compatible initial conditions $\phi(t)$ satisfying $\sup_{-d(t)\leq t\leq 0} \|\phi(t)\|\leq\delta(\epsilon)$, the solution $x(t)$ of the system (4) satisfies $\|x(t)\| \leq \epsilon$ for $t \geq 0$. Furthermore, $\lim_{t\to\infty} x(t) = 0$.

For the system (1), we consider the following memoryless linear state feedback control law,

$$u(t) = Kx(t), K \in \mathbb{R}^{m\times n} \quad (5)$$

Then the resultant closed-loop system is

$$\begin{cases} E\dot{x}(t) =(A_k+\Delta A_k)x(t)+(A_d \\ \qquad\qquad +\Delta A_d)x(t-d(t))+B_{\omega 1}\omega(t) \\ z(t) =C_k x(t)+B_{\omega 2}\omega(t) \end{cases} \quad (6)$$

where $A_k = A + BK$, $\Delta A_k = \Delta A + \Delta BK$ and $C_k = C + DK$.

The robust H_∞ control problem to be addressed in this paper is to design a state feedback control law (5) such that, for all admissible parameter uncertainties satisfying (2) and (3), the following criteria are satisfied:

1) The closed-loop system (6) is regular, impulse-free and stable for all admissible uncertainties when $\omega(t) = 0$.

2) For zero initial condition of $x(t)$ and a prescribed scalar $\gamma > 0$, $J = \int_0^\infty (z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t))dt < 0$.

We conclude this section by presenting several preliminary results, which will be used in the proof of our main results.

Lemma 1. (Petersen [1987]) Given matrices Γ, Λ and symmetric matrix Ω, we have $\Omega+\Gamma F\Lambda+\Lambda^T F^T\Gamma^T < 0$ for any $F^T F \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that $\Omega + \epsilon^{-1}\Gamma\Gamma^T + \epsilon\Lambda^T\Lambda < 0$.

Lemma 2. (Kristic and Deng [1998]) Consider the function $\varphi : R^+ \to R$, if $\dot{\varphi}$ is bounded on $[0, \infty)$, that is, there exists a scalar $\alpha > 0$ such that $\mid \dot{\varphi}(t) \mid\leq \alpha$ for all $t \in [0, \infty)$, then $\varphi(t)$ is uniformly continuous on $[0, \infty)$.

Lemma 3. (Barbalat's Lemma)(Kristic and Deng [1998]) Consider the function $\varphi :\ R^+ \to R$, if φ is uniformly continuous and $\int_0^\infty \varphi(s)ds < \infty$, then $\lim_{t\to\infty} \varphi(t) = 0$.

3. MAIN RESULTS

In this section, we give a solution to the problem of robust H_∞ control for the system (1) formulated previously by using strict LMI approach.

3.1 Delay-dependent Bounded Real Lemma for nominal singular system

We first consider the nominal singular time-varying delay system (1) with $u(t) = 0$, that is

$$\begin{cases} E\dot{x}(t) = Ax(t) + A_d x(t-d(t)) + B_{\omega 1}\omega(t) \\ z(t) = Cx(t) + B_{\omega 2}\omega(t). \end{cases} \quad (7)$$

For the nominal system (7), we introduce two vectors as follows

$$\xi(t) = \left[\, x^T(t)\ \ x^T(t-d(t)\ \ \omega^T(t))\,\right]^T, y(t) = E\dot{x}(t).$$

The following lemma gives the relationship between the vectors $\xi(t)$ and $\dot{x}(t)$, which will play a key role in achieving delay-dependent bounded real lemma.

Lemma 4. (Integral Inequality) For any constant matrices $N_1 \in \mathbb{R}^{n\times n}$, $N_2 \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times p}$, a positive-definitive symmetric matrix $Z \in \mathbb{R}^{n\times n}$, and a time-varying delay $d(t)$, then

$$-\int_{t-d(t)}^{t} \dot{x}^T(s)E^T ZE\dot{x}(s)ds \leq \xi^T(t)\{\Pi + d(t)Y^T Z^{-1}Y\}\xi(t) \quad (8)$$

where

$$\Pi = \begin{bmatrix} N_1^T E + E^T N_1 & E^T N_2 - N_1^T E & E^T W \\ * & -N_2^T E - E^T N_2 & -E^T W \\ * & * & 0 \end{bmatrix} \quad (9)$$

$$Y = [\,N_1\ \ N_2\ \ W\,]$$

Proof. Let $C = \begin{bmatrix} Z^{1/2} & Z^{-1/2}Y \\ 0 & 0 \end{bmatrix}$, then

$$\begin{bmatrix} Z & Y \\ Y^T & Y^T Z^{-1}Y \end{bmatrix} = C^T C \geq 0.$$

It follows

$$\int_{t-d(t)}^{t} \begin{bmatrix} E\dot{x}(s) \\ \xi(t) \end{bmatrix}^T \begin{bmatrix} Z & Y \\ Y^T & Y^T Z^{-1} Y \end{bmatrix} \begin{bmatrix} E\dot{x}(s) \\ \xi(t) \end{bmatrix} ds \geq 0 \quad (10)$$

Notice that

$$\int_{t-d(t)}^{t} 2\xi^T(t) Y^T E\dot{x}(s) ds = 2\xi^T(t) Y^T \begin{bmatrix} E & -E & 0 \end{bmatrix} \xi(t)$$

Rearranging (10) yields (8).

Based on Lemma 4, the following theorem presents a delay-dependent bounded real lemma for the nominal singular time-varying delay system (7).

Theorem 1. The nominal singular time-varying delay system (7) is regular, impulse-free and stable with disturbance attenuation level γ, if there exist positive-definite symmetric matrices P, Q, Z and matrices $S, S_d, S_\omega, N_1, N_2, W$ with appropriate dimensions such that

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \bar{d}N_1^T & \Xi_{14} & C^T \\ * & \Xi_{22} & \Xi_{23} & \bar{d}N_2^T & \Xi_{24} & 0 \\ * & * & \Xi_{33} & \bar{d}W^T & \Xi_{34} & B_{\omega 2}^T \\ * & * & * & -\bar{d}Z & 0 & 0 \\ * & * & * & * & -\bar{d}Z & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \quad (11)$$

where

$$\begin{aligned} \Xi_{11} =& A^T PE + SR^T A + E^T PA + A^T RS^T \\ &+ N_1^T E + E^T N_1 + Q, \\ \Xi_{12} =& A^T RS_d^T + SR^T A_d + E^T PA_d + E^T N_2 - N_1^T E, \\ \Xi_{13} =& E^T PB_{\omega 1} + SR^T B_{\omega 1} + A^T RS_\omega^T + E^T W, \\ \Xi_{14} =& \bar{d}A^T Z,\ \Xi_{24} = \bar{d}A_d^T Z,\ \Xi_{34} = \bar{d}B_{\omega 1}^T Z, \\ \Xi_{22} =& -(1-d)Q + A_d^T RS_d^T + S_d R^T A_d \\ &- N_2^T E - E^T N_2 \\ \Xi_{23} =& S_d R^T B_{\omega 1} + A_d^T RS_\omega^T - E^T W, \\ \Xi_{33} =& -\gamma^2 I + B_{\omega 1}^T RS_\omega^T + S_\omega R^T B_{\omega 1}, \end{aligned}$$

and $R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^T R = 0$.

Proof. Since rank$E = r \leq n$, there must exist two invertible matrices G and $H \in \mathbb{R}^{n\times n}$ such that

$$\bar{E} = GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (12)$$

Then, R can be parameterized as $R = G^T \begin{bmatrix} 0 \\ \bar{\Phi} \end{bmatrix}$, where $\bar{\Phi} \in \mathbb{R}^{(n-r)\times(n-r)}$ is any nonsingular matrix.

Similar to (12), we define

$$\bar{A} = GAH = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$$

$$\bar{P} = G^{-T} PG^{-1} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}$$

$$\bar{N}_1 = G^{-T} N_1 H = \begin{bmatrix} \bar{N}_{1,11} & \bar{N}_{1,12} \\ \bar{N}_{1,21} & \bar{N}_{1,22} \end{bmatrix}$$

$$\bar{S} = H^T S = \begin{bmatrix} \bar{S}_{11} \\ \bar{S}_{21} \end{bmatrix},\ \bar{R} = G^{-T} R = \begin{bmatrix} 0 \\ \bar{\Phi} \end{bmatrix}$$

Since $\Xi_{11} < 0$ and $Q > 0$, we can formulate the following inequality easily,

$$\begin{aligned} \Psi =& A^T PE + SR^T A + E^T PA + A^T RS^T \\ &+ N_1^T E + E^T N_1 < 0 \end{aligned}$$

Pre- and post-multiplying $\Psi < 0$ by H^T and H, respectively, yields

$$\begin{aligned} \bar{\Psi} =& H^T \Psi H \\ =& \bar{A}^T \bar{P}\bar{E} + \bar{S}\bar{R}^T \bar{A} + \bar{E}^T \bar{P}\bar{A} + \bar{A}^T \bar{R}\bar{S}^T \\ &+ \bar{N}_1^T \bar{E} + \bar{E}^T \bar{N}_1 \\ =& \begin{bmatrix} \bar{\Psi}_{11} & \bar{\Psi}_{12} \\ * & \bar{A}_{22}^T \bar{\Phi}\bar{S}_{21}^T + \bar{S}_{21}\bar{\Phi}^T \bar{A}_{22} \end{bmatrix} \\ <& 0 \end{aligned} \quad (13)$$

Since $\bar{\Psi}_{11}$ and $\bar{\Psi}_{12}$ are irrelevant to the results of the following discussion, the real expression of these two variables are omitted here. From (10), it is easy to see that

$$\bar{A}_{22}^T \bar{\Phi}\bar{S}_{21}^T + \bar{S}_{21}\bar{\Phi}^T \bar{A}_{22} < 0 \quad (14)$$

and thus $\bar{A}_{22}$ is nonsingular. Otherwise, supposing $\bar{A}_{22}$ is singular, there must exist a non-zero vector $\zeta \in \mathbb{R}^{n-r}$, which ensures $\bar{A}_{22}\zeta = 0$. And then we can conclude that $\zeta^T(\bar{A}_{22}^T \bar{\Phi}\bar{S}_{21}^T + \bar{S}_{21}\bar{\Phi}^T \bar{A}_{22})\zeta = 0$, and this contradicts (14). So $\bar{A}_{22}$ is nonsingular. Then, the pair of (E, A) is regular and impulse-free, which implies from Definition 2 that the system (7) is regular and impulse-free. In the following, we will prove that the system (7) is also stable when $\omega(t) = 0$.

Considering the (7) with $\omega(t) = 0$, we define the functional

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)), \quad (15)$$

where

$$V_1(x(t)) = x^T(t) E^T PEx(t),$$

$$V_2(x(t)) = \int_{t-d(t)}^{t} x^T(s) Qx(s) ds,$$

$$V_3(x(t)) = \int_{-\bar{d}}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) E^T ZE\dot{x}(s) ds d\theta.$$

Differentiating $V(x(t))$ with respect to t, we have

$$\dot{V}_1 = \dot{x}^T(t) E^T PEx(t) + x^T(t) E^T PE\dot{x}(t),$$

$$\dot{V}_2 = x^T(t) Qx(t) - (1 - \dot{d}(t)) x^T(t - d(t)) Qx(t - d(t)),$$

$$\begin{aligned} \dot{V}_3 =& \bar{d}\dot{x}^T(t) E^T ZE\dot{x}(t) - \int_{t-\bar{d}}^{t} \dot{x}^T(s) E^T ZE\dot{x}(s) ds \\ \leq& \bar{d}\dot{x}^T(t) E^T ZE\dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s) E^T ZE\dot{x}(s) ds. \end{aligned}$$

Furthermore, noting $E^T R = 0$, we can deduce

$$0 = 2\dot{x}^T(t) E^T R(S^T x(t) + S_d^T x(t - d(t))) \quad (16)$$

Moreover, LMI (11) obviously implies that

$$\tilde{\Xi} = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \bar{d}N_1^T & \bar{d}A^T Z \\ * & \Xi_{22} & \bar{d}N_2^T & \bar{d}A_d^T Z \\ * & * & -\bar{d}Z & 0 \\ * & * & * & -\bar{d}Z \end{bmatrix} < 0, \quad (17)$$

Then it follows from (17) and Lemma 4 that

$$\dot{V}(x(t)) \leq \xi^T(t) \tilde{\Xi} \xi(t) < 0$$

and

$$\begin{aligned}\lambda_1\|x(t)\|^2 - V(x(0)) &\le x^T(t)E^TPEx(t) - V(x(0)) \\ &\le V(x(t)) - V(x(0)) \\ &= \int_0^t \dot{V}(x(s))ds \\ &\le -\lambda_2\int_0^t \|x(s)\|^2 ds \\ &< 0\end{aligned} \tag{18}$$

where $\lambda_1 = \lambda_{min}(E^TPE) > 0, \lambda_2 = -\lambda_{max}(\Xi) > 0$.

Taking into account (18), we can deduce that

$$\lambda_1\|x(t)\|^2 + \lambda_2\int_0^t \|x(s)\|^2 ds \le V(x(0))$$

Therefore

$$0 < \|x(t)\|^2 \le \frac{1}{\lambda_1}V(x(0)),$$
$$0 < \int_0^t \|x(s)\|^2 ds \le \frac{1}{\lambda_2}V(x(0)).$$

Thus, $\|x(t)\|$ and $\int_0^t \|x(s)\|^2 ds$ are bounded. Similarly, we have that $\|\dot{x}(t)\|$ is bounded. By Lemma 2, we obtain $\|\dot{x}(t)\|^2$ is uniformly continuous. Therefore, noting that $\int_0^t \|x(s)\|^2 ds$ is bounded, and using Lemma 3, we get

$$\lim_{t\to\infty} x(t) = 0.$$

Then, according to Definition 2, the singular delay system (7) is stable when $\omega(t) = 0$.

On the other hand, when $\omega(t) \neq 0$, using the same functional as in (15), and noting $E^TR = 0$, we can deduce

$$0 = 2\dot{x}^T(t)E^TR(S^Tx(t) + S_d^Tx(t-d(t) + S_\omega^T\omega(t)) \tag{19}$$

Then noting the zero initial condition of $x(t)$, we have

$$\begin{aligned}J &= \int_0^\infty (z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t))dt \\ &\le \int_0^\infty (z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t))dt + V(x_\infty) - V(x_0) \\ &= \int_0^\infty \xi^T(t)\Xi\xi(t)dt\end{aligned}$$

It is easy to see that (11) guarantees $J < 0$. This completes the proof.

3.2 Robust H_∞ state feedback controller design

In this sequel, we give a strict LMI design algorithm for the system (1). For notational simplicity, we first consider the system (1) with $\Delta A = \Delta A_d = \Delta B = 0$, which, with the control law (5), results in the following closed-loop system,

$$\begin{cases} E\dot{x}(t) = A_kx(t) + A_dx(t-d(t)) + B_{\omega1}\omega(t) \\ z(t) = C_kx(t) + B_{\omega2}\omega(t)\end{cases} \tag{20}$$

For this system, we have the following theorem.

Theorem 2. The singular system (20) with time-varying delay is regular, impulse-free and stable with disturbance attenuation level γ if there exist positive-definite symmetric matrices P, Q, Z and matrices S, N_1, N_2, W, X, L with appropriate dimensions such that

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} & \bar{d}N_1^T & 0 & B_{\omega1} \\ * & \Upsilon_{22} & \Upsilon_{23} & \Upsilon_{24} & 0 & \bar{d}Z & 0 \\ * & * & \Upsilon_{33} & -EW & \bar{d}N_2^T & 0 & 0 \\ * & * & * & -\gamma^2I & \bar{d}W^T & 0 & B_{\omega2} \\ * & * & * & * & -\bar{d}Z & 0 & 0 \\ * & * & * & * & * & -\bar{d}Z & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{21}$$

where

$$\begin{aligned}\Upsilon_{11} &= AX + X^TA^T + BL + L^TB^T + N_1^TE^T + EN_1 \\ &\quad + Q,\ \Upsilon_{14} = (CX + DL)^T + EW \\ \Upsilon_{12} &= EP + SR^T - X^T + AX + BL,\ \Upsilon_{23} = X^TA_d^T \\ \Upsilon_{13} &= X^TA_d^T + EN_2 - N_1^TE^T,\ \Upsilon_{22} = -X - X^T \\ \Upsilon_{24} &= (CX + DL)^T,\ \Upsilon_{33} = -(1-d)Q - EN_2 - N_2^TE^T\end{aligned}$$

and $R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $ER = 0$. Furthermore, a suitable state feedback control law is given by $u(t) = LX^{-1}x(t)$.

Proof. Following the same philosophy as that in Fridman and Shaked [2002b], we represent the system (20) as the following form,

$$\begin{cases} \bar{E}\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{A}_d\bar{x}(t-d(t)) + \bar{B}_{\omega1}\omega(t), \\ z(t) = \bar{C}\bar{x}(t) + \bar{B}_{\omega2}\omega(t)\end{cases} \tag{22}$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & 0\end{bmatrix}, \bar{x}(t) = \begin{bmatrix} x(t) \\ y(t)\end{bmatrix}, \bar{C} = [\, C_k \ \ 0\,],$$
$$\bar{A} = \begin{bmatrix} 0 & I \\ A_k & -I\end{bmatrix}, \bar{A}_d = \begin{bmatrix} 0 & 0 \\ A_d & 0\end{bmatrix}, \bar{B}_{\omega1} = \begin{bmatrix} 0 \\ B_{\omega1}\end{bmatrix}$$
$$\bar{B}_{\omega2} = B_{\omega2}$$

Then, by the result of Theorem 1, we have that the system (22) is regular, causal and stable with disturbance attenuation level γ, if (11) holds, where E, A, A_d, $B_{\omega1}$, C, $B_{\omega2}$, P, Q, Z, R, S, S_d, S_ω, N_1, N_2, W are replaced by $\bar{E}$, $\bar{A}$, $\bar{A}_d$, $\bar{B}_{\omega1}$, $\bar{C}$, $\bar{B}_{\omega2}$, $\bar{P}$, $\bar{Q}$, $\bar{Z}$, $\bar{R}$, $\bar{S}$, $\bar{S}_d$, $\bar{S}_\omega$, $\bar{N}_1$, $\bar{N}_2$, $\bar{W}$ respectively. Especially, we select

$$\bar{P} = \begin{bmatrix} P & 0 \\ 0 & \beta I\end{bmatrix}, \bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & \beta I\end{bmatrix}, \bar{Z} = \begin{bmatrix} Z & 0 \\ 0 & \beta I\end{bmatrix},$$
$$\bar{R} = \begin{bmatrix} R & 0 \\ 0 & X\end{bmatrix}, \bar{S} = \begin{bmatrix} S & I \\ 0 & I\end{bmatrix}, \bar{N}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & \beta I\end{bmatrix},$$
$$\bar{N}_2 = \begin{bmatrix} N_2 & 0 \\ 0 & \beta I\end{bmatrix}, \bar{W} = \begin{bmatrix} W \\ \beta I\end{bmatrix}, \bar{S}_d = \bar{S}_\omega = 0$$

where $P \in \mathbb{R}^{n\times n}, Q \in \mathbb{R}^{n\times n}, Z \in \mathbb{R}^{n\times n}$ are positive-definite symmetric matrices, $R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^TR = 0$, $X \in \mathbb{R}^{n\times n}$ is any nonsingular matrix, $S \in \mathbb{R}^{n\times(n-r)}$, $N_1 \in \mathbb{R}^{n\times n}$, $N_2 \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times p}$ is any matrices. It is easy to see that $\bar{R}$ is with full column rank and satisfies $\bar{E}^T\bar{R} = 0$. Then, the following condition can be obtained by using Schur complement and letting $\beta \longrightarrow 0$,

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \bar{d}N_1^T & 0 & \Lambda_{17} \\ * & \Lambda_{22} & X^TA_d & X^TB_{\omega1} & 0 & \bar{d}Z & 0 \\ * & * & \Lambda_{33} & -E^TW & \bar{d}N_2^T & 0 & 0 \\ * & * & * & -\gamma^2I & \bar{d}W^T & 0 & B_{\omega2}^T \\ * & * & * & * & -\bar{d}Z & 0 & 0 \\ * & * & * & * & * & -\bar{d}Z & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{23}$$

where

$$\begin{aligned}\Lambda_{11} &= A_k^T X + X^T A_k + N_1^T E + E^T N_1 + Q\\ \Lambda_{12} &= E^T P + SR^T - X^T + A_k^T X,\ \Lambda_{22} = -X - X^T\\ \Lambda_{13} &= X^T A_d + E^T N_2 - N_1^T E,\ \Lambda_{14} = X^T B_{\omega 1} + E^T W\\ \Lambda_{17} &= C_k^T,\ \Lambda_{33} = -(1-d)Q - E^T N_2 - N_2^T E\end{aligned}$$

Now, consider the following singular time-varying delay system,

$$\begin{cases} E^T \dot{\varsigma}(t) = A_k^T \varsigma(t) + A_d^T \varsigma(t-d(t)) + C_k^T \eta(t) \\ z(t) = B_{\omega 1}^T \varsigma(t) + B_{\omega 2}^T \eta(t) \end{cases} \quad (24)$$

where $\varsigma(t) \in \mathbb{R}^n$ is the state vector.

Note that $\det(sE - A_k) = \det(sE^T - A_k^T)$, then the pair (E, A_k) is regular, impulse-free and stable if and only if the pair (E^T, A_k^T) is regular, impulse-free and stable and thus, the system (20) is regular, impulse-free and stable if and only if the system (24) is regular, impulse-free and stable. Furthermore, following the similar line as that in the proof of Theorem 1, the performance index of (24) satisfies $J = \int_0^\infty (z^T(t)z(t) - \gamma^2 \eta^T(t)\eta(t)) < 0$.

Therefore, as long as the regularity, free of impulse and stability and H_∞ performance are concerned, we can consider the system (24) instead of (20). Then, LMI (21) can be obtained by replacing $E, A_k, A_d, B_{\omega 1}$, C_k and $B_{\omega 2}$ in (23) by E^T, A_k^T, A_d^T, C_k^T, $B_{\omega 1}^T$ and $B_{\omega 2}^T$ respectively and introducing a matrix $L = KX$.

The solution of robust H_∞ control problem is presented in the following theorem.

Theorem 3. Consider the uncertain singular system (1) with time-varying delay, if there exist positive-definite symmetric matrices P, Q, Z, matrices S, N_1, N_2, W, X, L and scalars $\epsilon_1 > 0, \epsilon_2 > 0$ such that

$$\left[\begin{matrix} \Theta_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} & \bar{d}N_1^T \\ * & \Upsilon_{22} & X^T A_d^T & \Upsilon_{24} & 0 \\ * & * & \Theta_{33} & -EW & \bar{d}N_2^T \\ * & * & * & -\gamma^2 I & \bar{d}W^T \\ * & * & * & * & -\bar{d}Z \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{matrix}\right. \quad \left.\begin{matrix} 0 & B_{\omega 1} & \Theta_{18} & X^T N_d^T \\ \bar{d}Z & 0 & \Theta_{28} & X^T N_d^T \\ 0 & 0 & 0 & 0 \\ 0 & B_{\omega 2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\bar{d}Z & 0 & 0 & 0 \\ * & -I & 0 & 0 \\ * & * & -\epsilon_1 I & 0 \\ * & * & * & -\epsilon_2 I \end{matrix}\right] < 0 \quad (25)$$

then, we can construct a robust state feedback control law $u(t) = LX^{-1}x(t)$, such that the resultant closed-loop system is regular, impulse-free and stable with disturbance attenuation level γ for all admissible uncertainties satisfying (2) and (3), where $\Theta_{11} = \Upsilon_{11} + \epsilon_1 MM^T$, $\Theta_{33} = \Upsilon_{33} + \epsilon_2 MM^T$, $\Theta_{18} = \Theta_{28} = (N_a X + N_b L)^T$, $R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $ER = 0$ and $\Upsilon_{11}, \Upsilon_{12}, \Upsilon_{13}, \Upsilon_{14}, \Upsilon_{22}, \Upsilon_{24}, \Upsilon_{33}$ follow the same definition as those in (21).

Proof. Replacing A by $A + MF(k)N_a$, A_d by $A_d + MF(k)N_d$ and B by $B + MF(k)N_b$ in (21) respectively result in the following condition,

$$\begin{aligned}\Upsilon &+ \Gamma_1 F(t)\Phi_1 + \Phi_1^T F^T(t)\Gamma_1^T \\ &+ \Gamma_2 F(t)\Phi_2 + \Phi_2^T F^T(t)\Gamma_2^T < 0\end{aligned} \quad (26)$$

where

$$\begin{aligned}\Gamma_1 &= \left[\, M^T\ 0\ 0\ 0\ 0\ 0\ 0 \,\right]^T,\\ \Phi_1 &= [\, N_a X + N_b L\ \ N_a X + N_b L\ \ 0\ 0\ 0\ 0\ 0 \,].\\ \Gamma_2 &= \left[\, 0\ 0\ M^T\ 0\ 0\ 0\ 0 \,\right]^T,\\ \Phi_2 &= [\, N_d X\ \ N_d X\ \ 0\ 0\ 0\ 0\ 0 \,].\end{aligned}$$

By Lemma 1, it follows that (26) holds for any $F(t)$ satisfying $F^T(t)F(t) \le I$ if there exists scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\begin{aligned}\Upsilon &+ \varepsilon_1^{-1}\Gamma_1\Gamma_1^T + \varepsilon_1 \Phi_1^T \Phi_1 \\ &+ \varepsilon_2^{-1}\Gamma_2\Gamma_2^T + \varepsilon_2 \Phi_2^T \Phi_2 < 0,\end{aligned} \quad (27)$$

which is equal to (25) in the sense of Schur complement.

Remark 1. If the disturbance attenuation level γ is given, we can directly solve the feasibility problem of LMI (25) to obtain a suitable feedback control law. Otherwise, we can solve the following optimization problem

$$\begin{aligned}&\min \gamma \\ &s.t. \text{LMI}(25), P > 0, Q > 0, Z > 0, \epsilon_1 > 0, \epsilon_2 > 0\end{aligned} \quad (28)$$

to obtain a minimal disturbance attenuation level.

Remark 2. The delay-dependent robust H_∞ control problem for singular systems is solved only the case of state feedback. When output feedback is concerned, the results in this paper is easily to be extended for output feedback case, which maybe supplies a further research topic for singular systems with time-varying delay.

4. NUMERICAL EXAMPLE

In this section, we give two examples to demonstrate the applicability of the proposed design algorithm.

Example 1. Consider the system (1) with the following parameters

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.5 & -1 \\ 2 & 0 \end{bmatrix}, A_d = \begin{bmatrix} -1.1 & 1 \\ 0 & 0.5 \end{bmatrix},$$

$$B = B_{\omega 1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = [\, 0.1\ 0.5 \,], D = 0.3, B_{\omega 2} = 0.1,$$

$$M = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, N_a = N_d = [\, 0.2\ 0.2 \,], N_b = 0.2.$$

It is clearly that (E, A) has one impulsive mode, then the criteria in Shi et al. [2000] is inapplicable to this example. According to Theorem 3, by selecting $R = [\, 0\ 1 \,]^T$, and solving the feasibility problem of LMI (25), for a given $\gamma = 0.25$ and $d = 0.3$, we obtain that the maximum allowable delay bound $\bar{d} = 3.5047$, and the resulted state feedback gain $K = [\, -0.2678\ -1.8145 \,]$.

On the other hand, when γ is unknown, from Remark 1, for the known $\bar{d} = 5$, $d = 0.3$ the achieved H_∞ performances, $\gamma_{min} = 0.1001$.

Table 1. γ_{min} and K obtained by different methods when $d(t) = 1.2$

Method	Fridman and Shaked [2002a]	this paper
γ_{min}	11	0.5874
K	$\begin{bmatrix} 42.6854 & -0.3426 \end{bmatrix}$	$\begin{bmatrix} -9.9460 & 0 \end{bmatrix}$

Example 2. When the time-delay is constant, consider the nominal system (20) with the following parameters(Fridman and Shaked [2002a])

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A = 0, A_d = \begin{bmatrix} -1 & 0 \\ -0.5 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix},$$

$$B_{\omega 1} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, C = \begin{bmatrix} 1 & 0.2 \end{bmatrix}, D = 0.1, B_{\omega 2} = 0,$$

Fridman and Shaked [2002a] employed model transformation technique to analysis the robust H_∞ performance for this system, but the model transformation introduces additional dynamics and adds conservatism. Since the time-delay is constant, only letting $d = 0$ in Theorem 2, we can obtain the robust H_∞ controller. When $d(t) = 1.2$, the obtained γ_{min} and K by different methods is presented in Table 1, which shows the less conservation of our method.

5. CONCLUSION

The delay-dependent robust H_∞ control for uncertain singular systems with time varying delay is studied in this paper. By establishing an *integral inequality* based on quadratic terms, a new LMI based delay-dependent bounded real lemma is derived. Meanwhile, a control law design algorithm is also given, which guarantees that the resultant closed-loop system is regular, impulse-free and stable with disturbance attenuation level γ for all admissible uncertainties and time-delay. Finally, two numerical examples are given to show the effectiveness of the proposed approach.

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