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Set-valued state estimation for uncertain continuous-time systems via limited capacity communication channels ⋆

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Abstract: This paper addresses a problem of set-valued state estimation for uncertain continuous-time systems via limited capacity communication channels. The uncertainty of the systems satisfies an integral quadratic constraint. Using results from the robust Kalman filtering, we design a coder/decoder-estimator pair that allows us to construct set-valued state estimate of the systems via communication channels.

Keywords: Networked systems; Estimation under uncertainty; Linear uncertain systems

1. INTRODUCTION

A standard assumption in the classical control theory is that the data transmission required by the control algorithm can be performed with infinite precision. However, it is becoming more common to employ digital limited-capacity communication networks for exchange of information between system components. The resources available in such systems for communication between sensors, controllers and actuators can be severely limited due to size or cost. This problem may arise where the large number of mobile units need to be controlled remotely by a single controller. Since the radio spectrum is limited, communication constraints are a real concern. For example, the paper Stilwell and Bishop [2000] shows that the major difficulty in controlling a platoon of autonomous underwater vehicles is the bandwidth limitation on communication between the vehicles. Another classes of examples are offered by complex networked sensor systems containing a very large number of low power sensors and micro-electromechanical systems. In all these problems, classical optimal control and estimation theory cannot be applied since the control signals and the state information are sent via a limited capacity digital communication channel, hence, the controller or estimator only observes the transmitted sequence of finite-valued symbols.

Due to the enormous growth in communication technology, there has been a significant interest in the problem of control and state estimation via limited capacity communication channels in recent years (see, e.g., Delchamps [1990], Brockett and Liberzon [2000], Elia and Mitter [2001], Petersen and Savkin [2001], Ishii and Francis [2002], Liberzon [2003], Savkin and Petersen [2003], De Persis and Isidori [2004], Nair and Evans [2004], Matveev and Savkin [2005], Malyavej and Savkin [2005], Liberzon and Hespanha [2005], Savkin [2006], Savkin and Cheng [2007], Cheng and Savkin [2007], Matveev and Savkin [2007, 2008]). Minimum capacity of the communication channels required for state estimation and control has been investigated in, e.g., Nair and Evans [2004], Savkin [2006].

In terms of robust state estimation, the works of Savkin and Petersen [2003], Malyavej and Savkin [2005] provide algorithms that allow one to reliably estimate states of an uncertain system through communication networks. The algorithms were developed based on the robust Kalman filtering technique of Petersen and Savkin [1999].

This paper continues the research of Malyavej and Savkin [2005]. The principal difference with Malyavej and Savkin [2005] is that here we encode the output rather than the state estimate of the uncertain systems. A major advantage in encoding the measured output is that the dimension of the quantisation region can be reduced, and hence the required date rate may be reduced. It is because the dimension of the measured output vector is normally less than that of the state vector. Also, if the sensors that measure the output vector are spatially distributed, it is more convenience to transmit the measurements directly, rather than collecting all the measurements and processing them at an encoder before transmission.

The paper is organised as follows. In Section 2, we formulate a problem of set-valued state estimation via limited communication channels. In Section 3, by using a static quantisation scheme, a design of a coder/decoder-estimator pair that solves the proposed problem is presented. For a special class of systems, Section 4 presents a design of a coder/decoder-estimator pair that utilises a dynamic quantisation scheme.

The proofs of the results will be given in the full version of the paper.

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2. PROBLEM STATEMENT

Consider the time-varying uncertain system defined over the finite time interval $[0,\ NT]$:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)w(t) \\ z(t) &= K(t)x(t) \\ y(t) &= C(t)x(t) + v(t) \end{aligned} \tag{1}$$

where $N > 0$ is an integer, $T > 0$ is a given constant, $x \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^p$ and $v(t) \in \mathbb{R}^l$ are the *uncertainty inputs*, $z(t) \in \mathbb{R}^q$ is the *uncertainty output*, and $y(t) \in \mathbb{R}^l$ is the *measured output*; and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $K(\cdot)$ are bounded piecewise continuous matrix functions.

Notation 2.1. Let $x = [x_1\ x_2\ \cdots\ x_n]'$ be a vector from $\mathbf{R}^n$. Then $\|x\|_\infty := \max_{j=1,\dots,n} |x_j|$. Furthermore, $\|\cdot\|$ denotes the standard Euclidean vector norm: $\|x\| := \sqrt{\sum_{j=1}^n x_j^2}$.

The uncertainty $[w(t)\ \ v(t)]'$ vector in system (1) satisfies the following integral quadratic constraint (IQC). Let $Y_0 = Y_0' > 0$ be a given matrix, $x_0 \in \mathbb{R}^n$ be a given vector, and $Q(\cdot) = Q(\cdot)'$ and $R(\cdot) = R(\cdot)'$ be given bounded piecewise continuous matrix weighting functions satisfying the following condition. There exists a constant $\delta > 0$ such that $Q(t) \geq \delta I$, $R(t) \geq \delta I$ for all t. Then for a given time interval $[0,\ s]$, $s \leq NT$, we will consider the uncertainty inputs $w(\cdot)$ and $v(\cdot)$ and initial condition $x(0)$ such that

$$\begin{aligned} &(x(0) - x_0)'Y_0(x(0) - x_0) + \int_0^s (w(t)'Q(t)w(t) + \\ &v(t)'R(t)v(t))dt \leq d + \int_0^s \|z(t)\|^2 dt. \end{aligned} \tag{2}$$

Besides the IQC (2), we assume that there exists a known bound $\alpha > 0$ for $v(\cdot)$ such that $\|v(s)\| \leq \alpha$ for $s \leq NT$.

In our estimation problem, a sensor measures the output $y(t)$ of the system (1) and the information of $y(t)$ is sent to a remote location. The only way of communicating information from the sensor to that remote location is via a digital communication channel which carries one discrete-valued symbol $h(kT)$ at time kT, selected from a coding alphabet $\mathcal{H}$ of size ν. Here $T > 0$ is a given period and $k = 0, 1, 2, 3, \ldots$. This restricted number ν of codewords $h(kT)$ is determined by the transmission data rate of the channel. For example, if μ is the number of bits that our channel can transmit at any time instant, then $\nu = 2^\mu$ is the number of admissible codewords. We assume that the channel is a perfect noiseless channel and there is no time delay.

We consider the problem of set-valued estimation of the state of system (1) via a limited capacity communication channel. Our estimating scheme consists of two components: a coder $\mathcal{F}$ and a decoder-estimator $\mathcal{G}$. The coder is developed at the measurement location by taking the measured output $y(\cdot)$ and coding to the codeword $h(kT)$. Then the codeword $h(kT)$ is transmitted via a limited capacity communication channel to the decoder-estimator that is remotely located. The decoder-estimator takes the codeword $h(kT)$ and produces a set X_{kT} that overbounds the true set of possible state $x(kT)$ at the remote location. This situation is illustrated in Figure 1. The coder and the decoder-estimator are of the following form: for $k = 0, 1, 2, 3, \ldots, N$,

$$\begin{aligned} &\textbf{Coder: } h(kT) = \mathcal{F}\left(y(\cdot)|_0^{kT}\right); \\ &\textbf{Decoder-estimator: } X_{kT} = \mathcal{G}\left(h(T), h(2T), ..., h(kT)\right). \end{aligned} \tag{3}$$

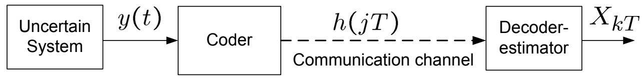

Fig. 1. Set-valued state estimation via digital communication channel

Notation 2.2. Let $y(t) = y_0(t)$ be a fixed measured output of the uncertain system (1) and let the finite time interval $[0, s]$ be given. Furthermore, let $\mathcal{F}$ and $\mathcal{G}$ be given coder and decoder-estimator. Then, $X_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}]$ denotes the set produces by the coder/decoder-estimator pair that captures all possible state $x(s)$ at time s for the uncertain system (1) with uncertain inputs satisfying the constraint (2).

The problem of set-valued state estimation via limited capacity communication channels considered in this paper is the problem of constructing the coder/decoder-estimator pair $(\mathcal{F}, \mathcal{G})$ and the set $X_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}]$.

In the next section, we present an algorithm for constructing a coder/decoder-estimator pair using a static quantisation scheme that solves the above state estimation problem.

3. CODER/DECODER-ESTIMATOR WITH STATIC QUANTISATION SCHEME

In order to construct the coder/decoder-estimator pair using a static quantisation scheme, we first require the solution to the following Riccati equation:

$$\begin{aligned} -\dot{S}(t) &= S(t)A(t) + A(t)'S(t) \\ &\quad + S(t)B(t)Q(t)^{-1}B(t)S(t) + K(t)'K(t), \\ S(0) &= Y_0, t \in [0,\ NT]. \end{aligned} \tag{4}$$

Also, we need the solution of the state equation: for $t \in [0,\ NT]$,

$$\dot{\eta}(t) = -[A(t) + B(t)Q(t)^{-1}B(t)S(t)]'\eta(t), \quad \eta(0) = Y_0 x_0. \tag{5}$$

Let $X_s[x_0, d]$ be the set of all possible states $x(s)$ at time s of the uncertain system (1) with uncertain inputs satisfying the constraint (2). The following robust prediction result was obtained in Moheimani et al. [1998] and it is useful to determine the size of a quantisation region.

Theorem 1. Let $Y = Y_0 > 0$ be a given matrix, and $Q(\cdot) = Q(\cdot)'$ and $R(\cdot) = R(\cdot)'$ be given matrix functions such that condition (2) holds on the time interval $[0,\ NT]$. Then, for a given vector $x_0 \in \mathbb{R}^n$, a constant $d > 0$ and any time $s \in [0, NT]$, the set $X_s[x_0, d]$ is bounded if and only if the Riccati equation (4) has a solution over $[0, NT]$ such that $S(\cdot) > 0$. Furthermore, the set $X_s[x_0, d]$ is given by

$$X_s[x_0, d] = \{x_s \in \mathbb{R}^n : x_s'S(s)x_s - 2x_s'\eta(s) + h_s \leq d\} \tag{6}$$

where

$$h_s = x_0'S_0x_0 - \int_0^s \eta(t)'B(t)Q(t)^{-1}B(t)'\eta(t)dt.$$

Proof: See Moheimani et al. [1998].

By using Theorem 1, it can be shown that if the Riccati equation (4) has a solution over $[0, NT]$ such that $S(\cdot) > 0$, then for all $s \in [0, NT]$,

$$\|x(s)\|_\infty \le \|\zeta(s)\|_\infty + \max_{i=1,2,\ldots,n} \sqrt{[S^{-1}(s)]_{i,i}} \times \sqrt{d + \zeta(s)' S(s) \zeta(s) - h_s} \tag{7}$$

where $[S^{-1}(s)]_{i,i}$ is the (i,i) diagonal element of the matrix $S^{-1}(s)$ and $\zeta(s) := S^{-1}(s)\eta(s)$. Then, using (7), we can define a bound for $y(\cdot)$ over the time interval $[0, NT]$ as follows:

$$L := \max_{s \le NT} \{\|C(s)\|_\infty \|x(s)\|_\infty + \|v(s)\|_\infty\}. \tag{8}$$

Such a bound L exists since $C(\cdot)$, $x(\cdot)$ and $v(\cdot)$ are all bounded over the time interval $[0, NT]$. Also, L can be pre-computed without the knowledge of the actual output $y(\cdot)$. This bound is then used to define the quantisation region, as we are to going to encode the output measurement $y(kT)$, $k = 0, 1, 2, \ldots, N$.

Our proposed coder/decoder-estimator $(\mathcal{F}, \mathcal{G})$ uses uniform quantisation of the output $y(\cdot)$ of system (1). Also, we let the set $\mathcal{B}_L := \{y \in \mathbf{R}^l : \|y\|_\infty \le L\}$ be the quantisation region. We propose to quantise the output y by dividing the quantisation region $\mathcal{B}_L$ uniformly into q^l hypercubes where q is a specified integer. For example, for $l = 2$, $q = 3$, $L = a$, the region $\mathcal{B}_L$ would be divided into nine regions as shown in Figure 2.

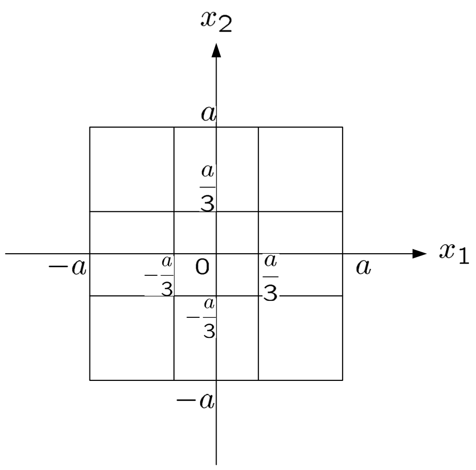

Fig. 2. Uniform quantisation of the state space.

Indeed, for each $i \in \{1, 2, \ldots, l\}$, we divide the corresponding component of the vector y into q intervals as follows:

$$\begin{aligned} I_1^i(L) &:= \left\{ y_i : y_i \in [-L,\ -L + \frac{2L}{q}) \right\}; \\ I_2^i(L) &:= \left\{ y_i : y_i \in [-L + \frac{2L}{q},\ -L + \frac{4L}{q}) \right\}; \cdots \\ I_q^i(L) &:= \left\{ y_i : y_i \in [L - \frac{2L}{q}, L] \right\}. \end{aligned} \tag{9}$$

Then for any $y \in \mathcal{B}_L$, there exist unique integers $i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, q\}$ such that $y \in I_{i_1}^1(L) \times I_{i_2}^2(L) \times \ldots \times I_{i_l}^l(L)$. Also, corresponding to the integers $i_1, i_2, \ldots, i_l$, we define the vector η as follows:

$$\eta(i_1, i_2, \ldots, i_l) := -L + \left[\frac{L(2i_1 - 1)}{q} \ \ \frac{L(2i_2 - 1)}{q} \ \cdots \ \frac{L(2i_l - 1)}{q} \right]'. \tag{10}$$

The vector $\eta(\cdot)$ is the centre of the hypercube $I_{i_1}^1(L) \times I_{i_2}^2(L) \times \ldots \times I_{i_l}^l(L)$ containing the original point y. Note the regions $I_{i_1}^1(L) \times I_{i_2}^2(L) \times \ldots \times I_{i_l}^l(L)$ partition the region $\mathcal{B}_L$ into q^l regions. In our proposed coder/decoder-estimator, each one of these regions or hypercubes will be assigned a codeword and the coder will transmit the codeword corresponding to the current output vector $y(kT)$. The transmitted codeword will correspond to the integers $i_1, i_2, \ldots, i_l$. By defining $\bar{y}(kT) := \eta(i_1, i_2, \ldots, i_l)$, for a given $\epsilon > 0$, we can choose $q > 0$ such that

$$\|y(kT) - \bar{y}(kT)\|_\infty \le L/q \le \epsilon, \qquad \forall k = 0, 1, 2, \ldots, N. \tag{11}$$

In other words, ϵ gives the quantisation error and it can be controlled by varying the parameter q.

Before introducing the coder/decoder-estimator, we consider the following jump Riccati equation:

$$\begin{aligned} \dot{P}(t) &= A(t)P(t) + P(t)A(t)' + B(t)Q^{-1}(t)B(t)' \\ &\quad + P(t)K(t)'K(t)P(t), \qquad \text{for } t \ne kT \\ P(kT) &= [P^{-1}(kT^-) + C(kT)'\bar{R}C(kT)]^{-1}, \\ &\quad \text{for } k = 1, 2, \ldots, N. \end{aligned} \tag{12}$$

where $\bar{R} = r^{-1}I$ and $r > 0$ is a given scalar. Here $P(t^-)$ denotes the limit of the matrix function $P(\cdot)$ at the point t from the left, i.e., $P(t^-) := \lim_{\epsilon > 0, \epsilon \to 0} P(t - \epsilon)$. The jump Riccati differential equation (12) behaves like a standard Riccati differential equation between sampling instants. However, at the sample times, its solution exhibits finite jump.

Now, we are in position to introduce our proposed coder/decoder-estimator that uses the static quantisation scheme:

Coder $\mathcal{F}$

$$\begin{aligned} &h(kT) = \{i_1, i_2, \ldots, i_l\}, \ \text{for } k = 0, 1, 2, \ldots, N \\ &\quad \text{and } y(kT) \in I_{i_1}^1(L) \times I_{i_2}^2(L) \times \ldots \times I_{i_l}^l(L). \end{aligned} \tag{13}$$

Decoder-estimator $\mathcal{G}$

$$\begin{aligned} \dot{\hat{x}}(t) &= [A(t) + P(t)K(t)'K(t)]\hat{x}(t), \quad \text{for } t \ne kT \\ \hat{x}(kT) &= \hat{x}(kT^-) - P(kT^-)C(kT)'\bar{R}C(kT)\hat{x}(kT^-) \\ &\quad + P(kT^-)C(kT)'\bar{R}\bar{y}(kT), \quad \text{for } k = 1, 2, \ldots, N, \\ \bar{y}(kT) &= \eta(i_1, i_2, \ldots, i_l), \quad \text{for } h(kT) = \{i_1, i_2, \ldots, i_l\}, \\ \hat{x}(0) &= x_0, \end{aligned} \tag{14}$$

The result of this section is stated as follows:

Theorem 2. Consider the uncertain system (1), (2) and the coder/decoder-estimator pair $(\mathcal{F}, \mathcal{G})$ (13), (14). Let $Y = Y_0 > 0$ be a given matrix, $x_0 \in \mathbb{R}^n$ be a given vector, and $Q(\cdot) = Q(\cdot)'$ and $R(\cdot) = R(\cdot)'$ be given matrix functions such that $Q(t) \ge \delta I$, $R(t) \ge \delta I$ on time interval $[0, NT]$ for some $\delta > 0$. Also, let $\bar{R} = r^{-1}I > 0$ be a given diagonal matrix, $d > 0$, $T > 0$, $\alpha > 0$, and $\epsilon > 0$

be given constants, and $s \in (0, NT]$ be given. Suppose that the solution $S(\cdot)$ to the Riccati Equation (4) with initial condition $S(0) = Y_0$ is defined and positive-definite on the interval $[0, NT]$, and the solution $P(\cdot)$ to the jump Riccati equation (12) is defined and positive-definite on the interval $[0, NT]$ with initial condition $P(0) = Y_0^{-1}$. Furthermore, suppose that the quantisation parameter q satisfying

$$q \geq L/\epsilon, \tag{15}$$

where L is defined in (8). Then,

$$X_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}] = \{x_s \in \mathbb{R}^n : (x_s - \hat{x}(s))' P(s)^{-1}(x_s - \hat{x}(s)) \leq d + \rho(s)\} \tag{16}$$

where

$$\rho(s) := \int_0^s \|K(t)\hat{x}(t)\|^2 dt + \frac{N(\alpha + \epsilon\sqrt{l})^2}{r} - \sum_{kT \leq s} \|\bar{R}^{1/2}(C(kT)\hat{x}(kT) - \bar{y}_0(kT))\|^2,$$

the state $\hat{x}(\cdot)$ is defined by the jump state equation (14) with initial condition x_0, the signal $\bar{y}_0(\cdot)$ is the sampled and quantised fixed measurement output $y_0(\cdot)$ of the uncertain system (1), (2).

Proof: The proof of Theorem 2 will be given in the full version of the paper.

Remark 3.1. The set $X_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}]$ is an ellipsoid and the centroid of this ellipsoid $\hat{x}(s)$ can be used to provide a point-valued state estimate.

The proposed coder/decoder-estimator with static quantisation scheme in this section requires us to determine the bound L (8) *a priori*. One can imagine that the constant L can be very large, requiring a large q to keep the condition $L/q \leq \epsilon$ holds for a given ϵ. As a result, this scheme requires large data rate communication channels. In the next section, a dynamic quantisation scheme is proposed for a special class of system (1), and this scheme may require a lesser data rate.

4. CODER/DECODER-ESTIMATOR WITH DYNAMIC QUANTISATION SCHEME

In this section, we consider the time-varying uncertain system defined over the finite time interval $[0, \ NT]$:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)w(t) \\ y(t) &= C(t)x(t) + v(t) \end{aligned} \tag{17}$$

where $N > 0$ is an integer, $T > 0$ is a given constant, $x \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^p$ and $v(t) \in \mathbb{R}^l$ are the *uncertainty inputs* from $L_2[0, NT]$, and $y(t) \in \mathbb{R}^l$ is the *measured output*; and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are bounded piecewise continuous matrix functions.

The uncertainty $[w(t) \ \ v(t)]'$ vector in system (17) and the initial condition $x(0)$ satisfies the following condition. Let $Y_0 = Y_0' > 0$ be a given matrix, $x_0 \in \mathbb{R}^n$ be a given vector, and $Q(\cdot) = Q(\cdot)'$ and $R(\cdot) = R(\cdot)'$ be given bounded piecewise continuous matrix weighting functions satisfying the following condition: there exists a constant $\delta > 0$ such that $Q(t) \geq \delta I$, $R(t) \geq \delta I$ for all t. Then for a given time interval $[0, \ \ s]$, $s \leq NT$, we will consider the uncertainty inputs $w(\cdot)$ and $v(\cdot)$ and initial condition $x(0)$ such that

$$(x(0) - x_0)' Y_0 (x(0) - x_0) + \int_0^s (w(t)'Q(t)w(t) + v(t)'R(t)v(t))dt \leq d. \tag{18}$$

Moreover, the bound α of the uncertainty input $v(\cdot)$ is known, i.e., $\|v(s)\| \leq \alpha$ for $s \leq NT$.

Essentially, the uncertain system (17), (18) is a special case of system (1), (2) with $K(\cdot) \equiv 0$.

We first consider the following jump Riccati equation:

$$\begin{aligned} \dot{P}(t) &= A(t)P(t) + P(t)A(t)' \\ &\quad + B(t)Q^{-1}(t)B(t)', \quad \text{for } t \neq kT; \\ P(kT) &= [P^{-1}(kT^-) + C(kT)'\bar{R}C(kT)]^{-1}, \\ &\quad \text{for } k = 1, 2, \ldots, N. \end{aligned} \tag{19}$$

where $\bar{R} = r^{-1}I$ and $r > 0$ is a given scalar. Suppose the solution $P(\cdot)$ to the jump Riccati equation (19) is defined and positive-definite on the interval $[0, NT]$ with initial condition $P(0)$. Then there exists a constant $\beta > 0$ such that the solution $P(\cdot)$ satisfies

$$\max_{i=1,2,\ldots,n} \sqrt{[P(kT)]_{i,i}} \leq \beta, \quad \text{for } k = 0, 1, 2, \ldots, N. \tag{20}$$

Since the matrix $C(\cdot)$ is a bounded piecewise continuous matrix function, there exists a constant $\gamma > 0$ such that

$$\max_{k=0,1,2,\ldots,N} \|C(kT)\|_\infty \leq \gamma, \tag{21}$$

where $\|C(\cdot)\|_\infty$ denotes the maximum row sum matrix norm of the matrix $\|C(\cdot)\|_\infty$, i.e., $\|C(\cdot)\|_\infty := \max_i \sum_{j=1}^n |c(\cdot)_{ij}|$ and $c(\cdot)_{ij}$ is the ij-th element of the matrix $C(\cdot)$.

In this section, our proposed coder/decoder-estimator $(\mathcal{F}, \mathcal{G})$ uses uniform quantisation of the difference between the output $y(kT)$ of the system (17) and the predicted output $\hat{y}(kT^-)$, for $k = 0, 1, 2, \ldots, N$, (as defined in the coder/decoder-estimator $(\mathcal{F}, \mathcal{G})$ (23) and (24)), instead of only $y(kT)$ as in Section 3. We let $a > 0$ be a given constant and the set

$$\mathcal{B}_a := \left\{(y(kT) - \hat{y}(kT^-)) \in \mathbf{R}^l : \|(y(kT) - \hat{y}(kT^-))\|_\infty \leq a\right\} \tag{22}$$

be the quantisation region. Again, we quantise the difference $(y(kT) - \hat{y}(kT^-))$ by dividing the quantisation region $\mathcal{B}_a$ uniformly into q^l hypercubes where q is a specified integer.

Now, we introduce the coder/decoder-estimator that uses the dynamic quantisation scheme:

Coder $\mathcal{F}$

$$
\begin{aligned}
\dot{\hat{x}}(t) &= A(t)\hat{x}(t), \quad \text{for } t \neq kT;\\
\hat{x}(kT) &= \hat{x}(kT^-) - P(kT^-)C(kT)'\bar{R}C(kT)\hat{x}(kT^-)\\
&\quad + P(kT^-)C(kT)'\bar{R}\bar{y}(kT), \quad \text{for } k = 1,2,\ldots,N;\\
\hat{y}(kT^-) &= C(kT^-)\hat{x}(kT^-); \qquad \hat{x}(0) = x_0;\\
h(kT) &= \{i_1, i_2, \ldots, i_l\}, \text{ for } k = 1,2,\ldots,N\\
&\text{and } (y(kT) - \hat{y}(kT^-)) \in I^1_{i_1}(a) \times I^2_{i_2}(a) \times \ldots \times I^l_{i_l}(a);\\
\bar{y}(kT) &= \hat{y}(kT^-) + \eta(i_1, i_2, \ldots, i_l),\\
&\quad \text{for } h(kT) = \{i_1, i_2, \ldots, i_l\};
\end{aligned} \tag{23}
$$

Decoder-estimator $\mathcal{G}$

$$
\begin{aligned}
\dot{\hat{x}}(t) &= A(t)\hat{x}(t), \quad \text{for } t \neq kT;\\
\hat{x}(kT) &= \hat{x}(kT^-) - P(kT^-)C(kT)'\bar{R}C(kT)\hat{x}(kT^-)\\
&\quad + P(kT^-)C(kT)'\bar{R}\bar{y}(kT), \quad \text{for } k = 1,2,\ldots,N;\\
\bar{y}(kT) &= \hat{y}(kT^-) + \eta(i_1, i_2, \ldots, i_l),\\
&\quad \text{for } h(kT) = \{i_1, i_2, \ldots, i_l\};\\
\hat{y}(kT^-) &= C(kT^-)\hat{x}(kT^-); \qquad \hat{x}(0) = x_0.
\end{aligned} \tag{24}
$$

We are now in position to state the result of this section.

Theorem 3. Consider the uncertain system (17), (18) and the coder/decoder-estimator pair $(\mathcal{F}, \mathcal{G})$ (23), (24). Let $Y = Y_0 > 0$ be a given matrix, $x_0 \in \mathbb{R}^n$ be a given vector, and $Q(\cdot) = Q(\cdot)'$ and $R(\cdot) = R(\cdot)'$ be given matrix functions such that $Q(t) \geq \delta I$, $R(t) \geq \delta I$ on the time interval $[0, NT]$ for some $\delta > 0$. Also, let $\bar{R} = r^{-1}I > 0$ be a given diagonal matrix, $d > 0$, $T > 0$, $\alpha > 0$, $\gamma > 0$, $\beta > 0$, and $\epsilon > 0$ be given constants, and $s \in (0, NT]$ be given. Suppose that the solution $P(\cdot)$ to the jump Riccati equation (19) is defined and positive-definite on the interval $[0, NT]$ with initial condition $P(0) = Y_0^{-1}$. Furthermore, suppose that the quantisation parameter q satisfying

$$q \geq a/\epsilon \tag{25}$$

where

$$a := \gamma\beta\sqrt{\bar{d}} + \alpha, \qquad \bar{d} := d + N(\alpha + \epsilon\sqrt{l})^2/r. \tag{26}$$

Then,

$$
\begin{aligned}
&X_s[x_0, y_0(\cdot)|_0^s, d, \mathcal{F}, \mathcal{G}] =\\
&\quad \{x_s \in \mathbb{R}^n : (x_s - \hat{x}(s))'P(s)^{-1}(x_s - \hat{x}(s)) \leq d + \rho(s)\}
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
\rho(s) &:= N(\alpha + \epsilon\sqrt{l})^2/r\\
&\quad - \sum_{kT \leq s} \|\bar{R}^{1/2}(C(kT)\hat{x}(kT) - \bar{y}_0(kT))\|^2,
\end{aligned} \tag{28}
$$

the state $\hat{x}(\cdot)$ is defined by the jump state equation (24) with initial condition x_0, the signal $\bar{y}_0(\cdot)$ is the sampled and quantised fixed measurement output $y_0(\cdot)$ of the uncertain system (17), (18).

Proof: The proof of Theorem 3 will be given in the full version of the paper.

5. CONCLUSION

A problem of set-valued state estimation for uncertain continuous-time systems via limited capacity communication channels was studied in this paper. To solve the problem, we designed a coder and a decoder-estimator, by employing the robust Kalman filtering technique, that allow us to estimate the states of the uncertain systems via communication channels. The decoder-estimator generates set-valued state estimate that is an ellipsoid containing all the possible states of the uncertain system. The results presented in this work will be useful in estimating states of spatially distributed systems. Since the set that captures all the possible states may be overly conservative, the tightness of this set deserves further investigation.

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