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Estimate of Attractive Regions for Systems Satisfying Polytopic Uncertainties in Given Regions

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Abstract: To estimate stability region of systems satisfying polytopic uncertainties in given regions is very important since such systems are given as models of linear systems with saturating control or nonlinear systems with nonlinear elements which satisfy sector conditions in given regions. In this paper, we propose a method to estimate the maximal robust attractive region of such systems using polytope Lyapunov functions. To demonstrate the usefulness of the proposed method we show some numerical examples.

1. INTRODUCTION

Linear control systems with input saturations appears frequently in practice since most of actuators display saturation characteristic. Saturation can have complicated effects on control system performance and it therefore becomes essential to determine the domain of attraction of the system. There have been continual efforts in addressing this issue. In the last decade, the issue of computing estimates of attractive regions for linear systems with control saturation has been extensively studied by many authors. See Romanchuk [1996], Pittet et al. [1997], Blanchini et al. [1997], Hindi & Boyd [1998], Gomes & Tarbouriech [1999], Fong & Hsu [2000], Hu & Lin [2000], Hu & Lin [2000], Hu & Lin [2001], Gomes et al. [2002], Gomes et al. [2002], Johansson [2002], Ohta [2002], Alamo et al. [2005], and the references therein. See also the survey by Genesio et al. [1985], Hu & Lin [2001] and Blanchini et al. [2007]. Most of results use a Lyapunov function to estimate attractive regions: Attractive regions are obtained using quadratic, Luré-type, piecewise quadratic, polytope, and piecewise-linear Lyapunov functions. In most of results, linear systems with control saturation are treated as systems satisfying polytopic uncertainties in given regions. Such systems are not only models of linear systems with control saturation but also models of nonlinear systems with nonlinear elements satisfying sector conditions in given regions.

For discrete time systems satisfying polytopic uncertainties in given regions, Blanchini et al. [1997] proposed a method to compute the maximal robust attractive region in the given region. However, for continuous time systems, such a method has not been proposed.

In this paper, we consider continuous time systems satisfying polytopic uncertainties in given regions, and propose a method to estimate the maximal robust stability region for continuous time systems satisfying polytopic uncertainties in given regions. The usefulness of the proposing method is demonstrated by a numerical example.

Notation. Let $\mathbf{N}$, $\mathbf{Z}_+$ and $\mathbf{R}_+$ denote the set of natural numbers, nonnegative integers and nonnegative real numbers. For a set A in $\mathbf{R}^n$, int A, bd A, and co A denote interior, boundary, and convex hull of A. For a set $\mathcal{X} \subseteq \mathbf{R}^n$, $\mathcal{X} \pm \hat{x} = \{x' = x \pm \hat{x},\ x \in \mathcal{X}\}$. For a matrix A and a vector y, $[A]_i$ and y_i denote the i-th row vector of A and i-the element of y. For a polytope P, $\mathcal{F}(P)$ and $\mathcal{N}(P)$ denote the set of all facets of P and the set of all nodes of P. A vector h_i is the normalized normal vector (NNV) of the facet $F_i \in \mathcal{F}(P)$ if $h_i^\top(x - x') = 0$ and $h_i^\top x = 1$ for all $x, x' \in F_i$.

2. PROBLEM STATEMENT

Consider a system given by

$$\Sigma^C: \quad \dot{x} = f(x), \quad x(0) = x_0 \in X \tag{1}$$

where $x \in X \subseteq \mathbf{R}^n$, $f(x)$ is a nonlinear function satisfying $f(0) = 0$, and $X \subseteq \mathbf{R}^n$ is a polytope which we are concerned about the behavior of solutions of Σ^C.

Let $x(t; x_0)$ be the solution of Σ^C with initial condition $x(0; x_0) = x_0$. Let Ω^C_∞ and $\mathcal{D}^C$ be closed sets including 0 as its interior point. We say that Ω^C_∞ is a positively invariant set (PIS) for Σ^C in X if

$$\Omega^C_\infty \subseteq X \quad \wedge \quad (x_0 \in \Omega^C_\infty \ \Rightarrow\ x(t; x_0) \in \Omega^C_\infty \quad \forall\, t \geqq 0). \tag{2}$$

and that $\mathcal{D}^C$ is an attractive region (AR) for Σ^C in X if it is a PIS, and if

$$\mathcal{D}^C \subseteq X \quad \wedge \quad (x_0 \in \mathcal{D}^C \ \Rightarrow\ \lim_{t\to\infty} x(t; x_0) = 0). \tag{3}$$

In this paper, we assume that the function f in Σ^C satisfies the polytopic uncertainty in X:

$$\exists\, \{A_q \in \mathbf{R}^{n\times n}\}_{q=1}^{Q} \ :\ f(x) \in \text{co}\ \{A_q x\}_{q=1}^{Q}, \quad x \in X. \tag{4}$$

Let $A : \mathbf{R}_+ \to \mathbf{R}^{n\times n}$ be a piecewise continuous matrix function satisfying

$$A(t) \in \text{co}\ \{A_1, A_2, \cdots, A_Q\}, \quad t \geqq 0 \tag{5}$$

and consider a linear time varying system given by

$$\Sigma^C(A): \quad \dot{x} = A(t)x. \tag{6}$$

We denote a PIS and a AR for $\Sigma^C(A)$ in X by $\Omega^C_\infty(A)$ and $\mathcal{D}^C(A)$, respectively, Moreover, we define a robust PIS (RPIS) $\tilde{\Omega}^C_\infty$ and a robust AR (RAR) $\tilde{\mathcal{D}}^C$ of the system $\Sigma^C(A)$ in X by

$$\tilde{\Omega}^C_\infty = \bigcap_{A(t)\in \text{co}\ \{A_1,A_2,\cdots,A_Q\},\ \forall\, t} \Omega^C_\infty(A) \subseteq X, \tag{7}$$

$$\tilde{\mathcal{D}}^C = \bigcap_{A(t)\in \text{co}\ \{A_1,A_2,\cdots,A_Q\},\ \forall\, t} \mathcal{D}^C(A) \subseteq X. \tag{8}$$

Finally, we say that $\tilde{\Omega}_\infty^{C^*}$ and $\tilde{\mathcal{D}}^{C^*}$ are the maximal RPIS (MRPIS) and the maximal RAR (MRAR) for $\Sigma^C(A)$ in X if for any $\tilde{\Omega}_\infty^{C}$ and any $\tilde{\mathcal{D}}^{C}$ are subsets of $\tilde{\Omega}_\infty^{C^*}$ and $\tilde{\mathcal{D}}^{C^*}$, respectively. We note that the MRPIS $\tilde{\Omega}_\infty^{C^*}$ is the maximal admissible set (MAS, see Ohta & Tanizawa [2007]) for $\Sigma^C(A)$ since we are considering a very special constraints for the system Σ^C, that is, the variables $z(t)$ to be constrained for this system is $z(t) = x(t)$.

Let $x(t;x_0)$ be a solution of Σ^C. Then, there exists a matrix function $A(t;x_0)$ such that

$$A(t;x_0)x(t;x_0) = f[x(t;x_0)] \quad \forall\, t \geqq 0. \tag{9}$$

If $x_0 \in \Omega_\infty^C$, then, by the assumption, $A(t;x_0)x(t;x_0) \in \text{co}\,\{A_q x(t;x_0)\}_{q=1}^{Q}$ for all t, and, hence, $x(t;x_0)$ coincides with $x(t;x_0,A(\cdot,x_0))$, where $x(t;x_0,A(\cdot,x_0))$ is the solution of $\Sigma^C(A)$ with $A(t) = A(t,x_0)$ and the initial condition $x(0;x_0,A(\cdot,x_0)) = x_0$. Therefore, $\tilde{\Omega}_\infty^{C^*}$ and $\tilde{\mathcal{D}}^{C^*}$ are a PIS and a AR for Σ^C in X, respectively.

The issue we consider in the following is to compute a RAR $\tilde{\mathcal{D}}^C \subseteq X$, which almost coincides with the MRAR $\tilde{\mathcal{D}}^*$.

3. MAIN RESULT

3.1 Euler approximation and the MRPIS

Let us consider the Euler approximation of the system $\Sigma^C(A)$

$$\Sigma_\Delta^E(A): \quad x[k+1] = A_\Delta[k]x[k], \tag{10}$$

where

$$A_\Delta[k] = I + \Delta A(t_k), \quad t_k = k\Delta, \quad k = 0,1,\cdots, \tag{11}$$

and $\Delta > 0$ is the step size of the Euler approximation.

Let γ be a positive number, and let us consider a modified system of $\Sigma_\Delta^E(A)$.

$$\Sigma_\Delta^E(A,\gamma): \quad x[k+1] = A_{\Delta,\gamma}[k]x[k], \tag{12}$$

where

$$A_{\Delta,\gamma}[k] = I + \Delta(A(t_k) + \gamma I), \quad t_k = k\Delta, \quad k = 0,1,\cdots. \tag{13}$$

By (5) and (13), the following relation holds.

$$A_{\Delta,\gamma}[k] \in \text{co}\,\{A_{1,\Delta,\gamma}, A_{2,\Delta,\gamma}, \cdots, A_{Q,\Delta,\gamma}\}, \quad \forall\, k, \tag{14}$$

$$A_{q,\Delta,\gamma} = 1 + \Delta(A_q + \gamma I), \quad q \in \mathcal{Q} = \{1,2,\cdots,Q\}. \tag{15}$$

We define the MRPIS for uncertain discrete time systems in a quite similar way for uncertain continuous time systems. We denote the MRPIS for $\Sigma_\Delta^E(A,\gamma)$ in X by $\tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$. Blanchini et al. [1997] showed that $\tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$ is a polyhedral set (see also Gilbert & Tan [1991]), and, hence, it is a polytope if it is bounded.

Let X be given by

$$X = \{x \in \mathbf{R}^n : Mx \leqq \mathbf{1}\}, \tag{16}$$

where $\mathbf{1}$ is the vector whose elements are all 1 and the inequality in (16) is the componentwise inequalities, that is, $M_i x \leqq 1$ for all $i \in \{1,2,\cdots,N_M\}$, where M_i is the i-th row vector of M and N_M is the number of rows of M. Then, $\tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$ is characterized as follows (see Blanchini et al. [1997], Pluymers et al. [2005]).

$$\tilde{\Omega}_\infty^{E^*}(\Delta,\gamma) = \lim_{k\to\infty} \tilde{\Omega}_k^{E}(\Delta,\gamma) \tag{17}$$

$$\tilde{\Omega}_k^{E}(\Delta,\gamma) = \{x_0 \in X : M\prod_{i=1}^{k} A_{q_i,\Delta,\gamma}x_0 \leqq \mathbf{1}, \quad \forall\, q, q_i \in \mathcal{Q}, \ i \in [1\cdots k]\}. \tag{18}$$

From these equations, it is easy to see that the following results hold (see Blanchini et al. [1997], Pluymers et al. [2005])

$$\tilde{\Omega}_{k+1}^{E}(\Delta,\gamma) \subseteq \tilde{\Omega}_k^{E}(\Delta,\gamma), \quad \forall\, k = 0,1,\cdots, \tag{19}$$

$$\tilde{\Omega}_{k+1}^{E}(\Delta,\gamma) = \tilde{\Omega}_k^{E}(\Delta,\gamma) \ \Rightarrow\ \tilde{\Omega}_\infty^{E^*}(\Delta,\gamma) = \tilde{\Omega}_k^{E}(\Delta,\gamma). \tag{20}$$

Based on the above results, a procedure to compute the dual polytope P^D of $\tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$ is the following:

Procedure make_PD;
1. P^D: a very small polytope such that $0 \in \text{int}\, P^D$;
   Q_D: an empty queue; $\gamma > 0$: a very small number;
2. for $i \in \{1,2,\cdots,N_M\}$ do
   begin
2.1 if $(M_i^\top \notin P^D)$ then
   begin
2.1.1 co $(P^D,\ M_i^\top)$; append(Q_D, M_i);
   end
   end
3. while $(Q_D \neq \emptyset)$ do
   begin
3.1 $\eta :=$ pop(Q_D);
3.2 for $q \in \{1,2,\cdots,Q\}$ do
   begin
3.2.1 $h = \eta\, A_{q,\Delta,\gamma}$;
3.2.2 if $(h^\top \notin P^D)$ then
   begin
3.2.2.1 co $(P^D,\ h^\top)$; append(Q_D, h);
   end
   end
   end
4. return P^D;

Remark 1. Note that a node h_i of P^D correspond to the normalized normal vector of a facet F_i of $\tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$ and that the normalized normal vector x_j of a facet F_j' of P^D correspond to a node of $\tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$.

Remark 2. In make_PD, pop(Q_D) means to get the first element from the queue Q_D and to remove it from Q_D, and append(Q_D, M_i) means that append M_i at the end of Q_D.

Remark 3. Procedure make_PD executes the same job with Algorithm 1 in Pluymers et al. [2005], in which 3.2.2 (determining $h^\top \notin P^D$ holds or not) is executed by solving a LP. In Procedure make_PD, we apply Beneath-Beyond method (see **?**]) to determine it and it is much more efficient according to our experience. Blanchini et al. [1997] proposed a method using $\tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$, which is more time consuming according to our experience.

3.2 PLF and Inner Approximation of the MRAR

To derive our main result, let us introduce a candidate of Polytope Lyapunov Function (PLF) which corresponds to a polytope P_1 including 0 as an interior point and is defined by the following (see Ohta et al. [1993])

$$V(x;P_1) = \max_{F_i\in\mathcal{F}(P_1)} h_i^\top x, \tag{21}$$

where h_i is the normal vector of a facet F_i of P_1 and h_i is normalized in the sense that

$$h_i^\top x = 1 \quad \forall\, x \in F_i. \tag{22}$$

Remark 4. We note here that (21) is different from the definition of PLF in Ohta et al. [1993], but it is an equivalent definition (see Ohta et al. [1993], B.4). A Lyapunov function proposed in Blanchini [1995] is equivalent with (21). In Molchanov & Pyatnitskii [1988], Kiendl et al. [1992] and Polański [1995], quite similar definitions are given, but in these papers, it is required that P_1 is balanced. On the other hand, in (21) it is not required that P_1 is balanced. In general, if X is not balanced then P_1 is so, and, hence, it is better not to require that P_1 is balanced (see for example, Example 3).

The function $V(x; P_1)$ defined by (21) has the following properties.

Lemma 1. Let P_1 be a polytope including 0 as an interior point. Define $V(x; P_1)$ by (21). Then, we have

$$V(x; P_1) \leqq 1 \quad \Leftrightarrow \quad x \in P_1, \tag{23}$$

$$V(\alpha x; P_1) = \alpha V(x; P_1), \quad \forall\, x \in \mathbf{R}^n \quad \forall\, \alpha \geqq 0, \tag{24}$$

$$V(x_1 + x_2; P_1) \leqq V(x_1; P_1) + V(x_2; P_1), \quad \forall x_1, x_2 \in \mathbf{R}^n. \tag{25}$$

Lemma 1 is direct consequence that $V(x; P_1)$ is the Minkowski function (or gauge function) of a convex set Blanchini et al. [2007], Luenberger [1969].

Lemma 2. Suppose that $\tilde{\Omega}_\infty^{E^*}(\Delta, \gamma)$ is bounded and that $\tilde{\Omega}_\infty^{E^*}(\Delta, \gamma)$ includes 0 as an interior point. Let $P_1 = \tilde{\Omega}_\infty^{E^*}(\Delta, \gamma)$ in (21). Then, we have

$$V(x; P_1) = 1 \quad \Rightarrow \quad V(A_{q,\Delta,\gamma} x; P_1) \leqq 1, \quad \forall\, q \in Q. \tag{26}$$

Proof. See Appendix.

From this fact, we have our main result.

Theorem 1. Let γ be a positive number, and let us consider an uncertain continuous time system (6) and an uncertain discrete time system (12). If $\tilde{\Omega}_\infty^{E^*}(\Delta_1, \gamma)$ is bounded, then the following relation holds.

$$0 < \Delta_2 \leqq \Delta_1 \quad \Rightarrow \quad \tilde{\Omega}_\infty^{E^*}(\Delta_1, \gamma) \subseteq \tilde{\Omega}_\infty^{E^*}(\Delta_2, \gamma) \subseteq \tilde{\mathcal{D}}_\infty^{C^*}. \tag{27}$$

Proof. See Appendix.

The first inclusion relation is new. About the second inclusion relation, a closely related but different result was shown in Blanchini et al. [2007], where asymptotic stability is considered.

4. EXAMPLES

In this section, we will show several examples to illustrate the usefulness of our proposing method.

Example 1. Let us consider a system given by (6) where $n = 2$, $X = [-7, 7] \times [-7, 7]$, $Q = 2$,

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.06 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1.94 & -1 \end{bmatrix}.$$

This system is not quadratically stable (see Olas [1991]).

We compute $\tilde{\Omega}_\infty^{E^*}(\Delta, \gamma)$ for $\gamma = 10^{-5}$ and $\Delta = 0.0025, 0.005, 0.001, 0.02, 0.04, 0.08, 0.16, 0.32, 0.34.$. When $\Delta \geqq 0.35$, we could not have $\tilde{\Omega}_\infty^{E^*}(\Delta, \gamma)$. It seems the Euler approximation becomes robustly unstable when $\Delta \geqq 0.35$. In Fig. 1, we show 6 polytopes $\tilde{\Omega}_\infty^{E^*}(\Delta, \gamma)$, where $\gamma = 10^{-5}$, $\Delta = 0.02, 0.04, 0.08, 0.16, 0.32, 0.34$. We do not show $\tilde{\Omega}_\infty^{E^*}(0.001, \gamma)$, $\tilde{\Omega}_\infty^{E^*}(0.005, \gamma)$, and $\tilde{\Omega}_\infty^{E^*}(0.0025, \gamma)$, because they are quite similar to $\tilde{\Omega}_\infty^{E^*}(0.02, \gamma)$. The relation (27) holds. The smallest one is $\tilde{\Omega}_\infty^{E^*}(0.34, \gamma)$, which is drawn using black lines. The maximal one is $\tilde{\Omega}_\infty^{E^*}(0.02, \gamma)$, which is drawn using red lines.

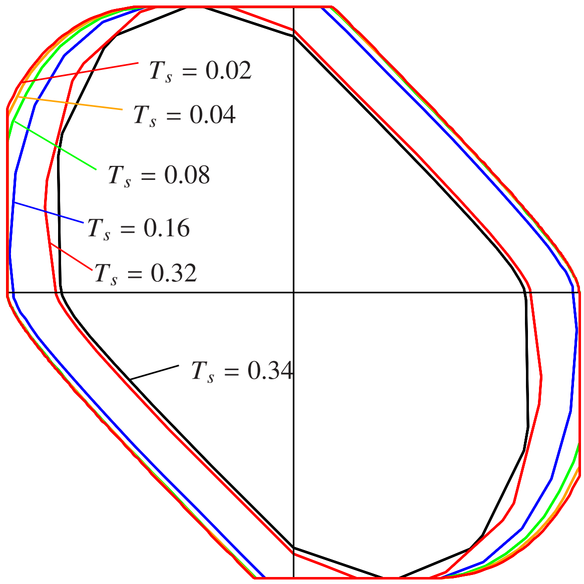

Fig. 1. $\tilde{\Omega}_\infty^{E^*}(\Delta, 10^{-5})$, where $\Delta = 0.02,\ 0.04,\ 0.08,\ 0.16,\ 0.32,$ and 0.34.

In Table 1, we summarize data about computing $\tilde{\Omega}_\infty^{E^*}(\Delta, 10^{-5})$'s.

Table 1.
N_n: the number of nodes of $\tilde{\Omega}_\infty^{E^*}(\Delta, 10^{-5})$, N_L: the number of execution of 2.1 and/or 3.2.2 in Procedure make_PD, and T_C: the user CPU time (1 CPU time = 0.016(sec), * means $T_C = 0$ or 1)

Δ	0.16	0.08	0.04	0.02	0.01	0.005	0.0025
N_n	48	92	180	703	702	1398	2794
N_L	167	183	359	352	1403	2795	5587
T_C	*	*	*	4	12	47	302

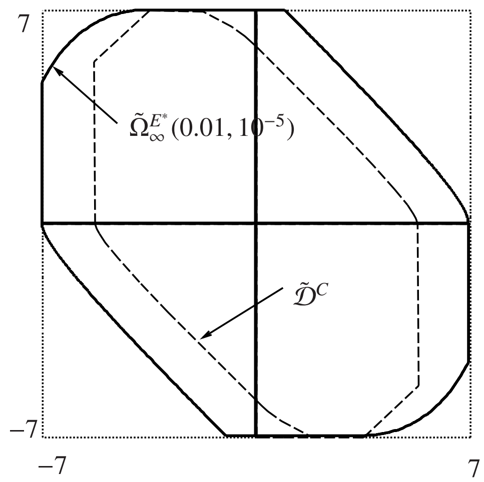

Fig. 2. A $\tilde{\mathcal{D}}^C$ in X and $\tilde{\Omega}_\infty^{E^*}(0.01, 10^{-5})$ in X, where $X = [-7, 7]^2$.

Example 2. Let us consider again the system treated in Example 1. In Fig. 2, the polytope (convex polygon) denoted by the dashed line is a $\tilde{\mathcal{D}}^C$ obtained applying Piecewise Linear

Lyapunov Function (see Ohta [2001]) and the polytope denoted by solid line is a $\tilde{\Omega}^{E^*}_{\infty}(0.01, 10^{-5})$ obtained applying the method proposed in this paper. We can conclude that the proposing method gives a much larger estimate of the MRAR than the method in Ohta [2001]. We note that the number of facets of $\tilde{\mathcal{D}}^C$ is 28 and $T_C = 15$ for computing $\tilde{\mathcal{D}}^C$, and that the number of facets of $\tilde{\Omega}^{E^*}_{\infty}(0.01, 10^{-5})$ is 702 and $T_C = 12$ for computing it.

Example 3. Let us consider a simple manipulator described by

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\omega} = -\cos(\theta) + M_\theta, \quad M_\theta = -(\theta - \hat{\theta}) - (\omega - \hat{\omega}) + \theta^*. \end{cases} \tag{28}$$

When $\theta^* = 1$, the equilibrium $[\hat{\theta}\ \hat{\omega}]^\top$ is $[1\ 0]^\top$. Let $x_1 = \theta - \hat{\theta}$, $x_2 = \omega - \hat{\omega}$, and $x = [x_1\ x_2]^\top$. Then, we have

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -[\cos(x_1 + \hat{\theta}) - \cos(\hat{\theta})] - x_1 - x_2. \end{cases} \tag{29}$$

The function $\psi(x_1) = -[\cos(x_1 + \hat{\theta}) - \cos(\hat{\theta})] - x_1$ satisfies the sector condition that $\psi(x_1) \in \text{co}\ \{-10^{-3}x_1, -x_1\}$ in $x_1 \in [-2, 4]$ as shown in Fig. 3, and, hence, we have

$$f(x) = \begin{bmatrix} x_2 \\ \psi(x_1) - x_2 \end{bmatrix} \in \text{co}\ \left\{ \begin{bmatrix} 0 & 1 \\ -10^{-3} & -1 \end{bmatrix} x, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x \right\},$$
$$x \in X = [-2, 4] \times [-2, 2]. \tag{30}$$

We compute $\Omega_U = \tilde{\Omega}^{E^*}_{\infty}(0.01, 10^{-5})$ in $X = [-2, 4] \times [-2, 2]$ and $\Omega_S = \tilde{\Omega}^{E^*}_{\infty}(0.01, 10^{-5})$ in $X' = [-2, 2] \times [-2, 2]$. As we can see in Fig. 4, Ω_U is much larger than Ω_S, and, hence, using unbalanced X brings much better result for this example.

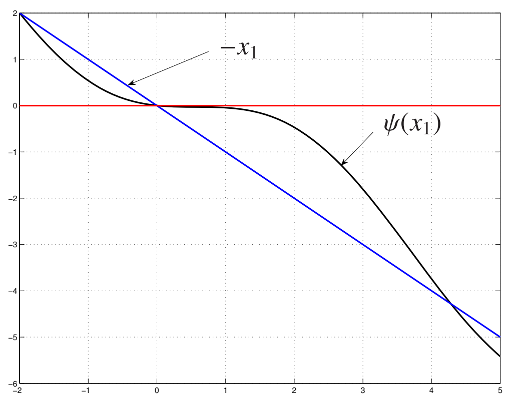

Fig. 3. The function $\psi(x_1) = -[\cos(x_1 + \hat{\theta}) - \cos(\hat{\theta})] - x_1$ satisfies a sector condition in $x_1 \in [-2, 4]$.

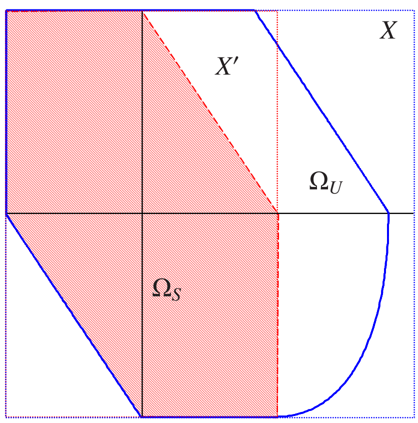

Fig. 4. $\Omega_U = \tilde{\Omega}^{E^*}_{\infty}(0.01, 10^{-5})$ in $X = [-2, 4] \times [-2, 2]$ and $\Omega_S = \tilde{\Omega}^{E^*}_{\infty}(0.01, 10^{-5})$ in $X' = [-2, 2] \times [-2, 2]$.

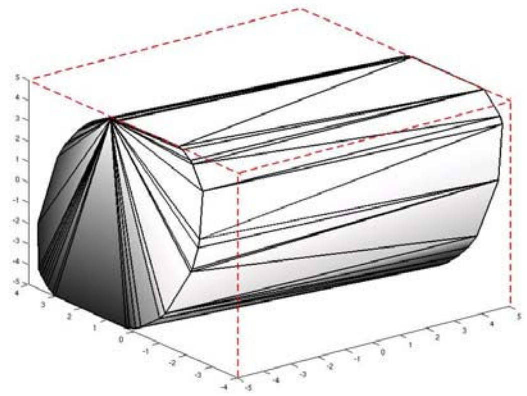

Fig. 5. $\tilde{\Omega}^{E^*}_{\infty}(0.05, 10^{-5})$.

Example 4. Let us consider the case when $Q = 2$ and A_1 and A_2 in (4) are given by

$$A_1 = \begin{bmatrix} -1 & -8.5 & -8.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{31}$$

We compute $\tilde{\Omega}^{E^*}_{\infty}(0.05, 10^{-5})$ shown in Fig. 5, which has 470 facets and 250 nodes, and the user CPU time is 337.

Example 5. Pittet et al. [1997] Consider Σ^C with

$$f(x) = Ax - B\,\text{sat}(Fx), \quad x \in X \subseteq \mathbf{R}^2 \tag{32}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad F = \begin{bmatrix} -10 & -5 \end{bmatrix}, \tag{33}$$

$$\text{sat}(u) = \begin{cases} 1 & \text{if } u \geqq 5, \\ u & \text{if } |u| \leqq 5, \\ -1 & \text{if } u \leqq -5, \end{cases} \tag{34}$$

$$X = \{x \in \mathbf{R}^2 : |x_1| \leqq 8.5, \ |x_2| \leqq 10, \ |x_1 + x_2| \leqq 5, \ |Fx| \leqq 5/0.18\}. \tag{35}$$

Therefore, $Q = 2$ and

$$A_1 = A - BF, \quad A_2 = A - 0.18BF. \tag{36}$$

Fig. 6 shows a $\tilde{\mathcal{D}}^C$ obtained using Quadratic Lyapunov Function (QLF) and $\tilde{\Omega}^{E}_{\infty}(0.01, 10^{-7})$ computed by proposing method. $\tilde{\mathcal{D}}^C$ is computed by using maxdet.

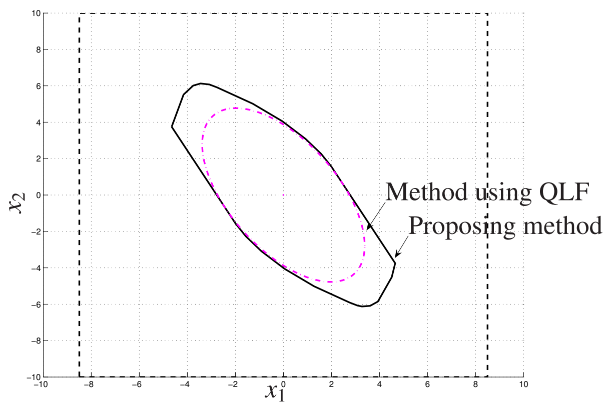

Fig. 6. $\tilde{\mathcal{D}}^C$ (dot-dashed line) and $\tilde{\Omega}^E_\infty(0.01, 10^{-7})$ (solid line).

Example 6. Pittet et al. [1997] Let us consider again the system treated in Example 5. In Fig. 7, RARs $\tilde{\mathcal{D}}^C$ computed using PLF and $\tilde{\mathcal{D}}^C$ computed using Popov Criterion, and $\tilde{\Omega}^E_\infty(0.01, 10^{-7})$ are shown. We note that $\tilde{\mathcal{D}}^C$ computed using Popov Criterion has area which is out side $\tilde{\Omega}^E_\infty(0.01, 10^{-7})$. This is not surprise since a Lure type Lyapunov function, which derived by Popov criterion, utilizes the fact that the system considered in this example is a piecewise linear system while the proposing method treats a motorized system of it, and, hence, it causes conservativeness of stability conditions. This example suggests us we need more efforts to compute the maximal positive invariant set for piecewise linear systems.

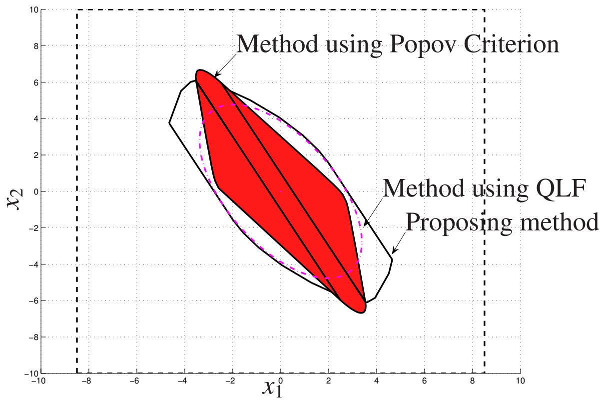

Fig. 7. $\tilde{\mathcal{D}}^C$ computed using PLF (dot-dashed line), $\tilde{\Omega}^E_\infty(0.01, 10^{-7})$ (solid line), and $\tilde{\mathcal{D}}^C$ computed using Popov Criterion (shaded area).

5. CONCLUSION

In this paper, using polytope Lyapunov functions, we proposed a method to estimate the MRAR in a given region X of continuous time systems satisfying polytopic uncertainty in X. We examined the usefulness of the proposed method through some numerical examples.

The construction method of Polytope Lyapunov Function (PLF) proposed by Brayton & Tong [1979] and Ohta et al. [1993] was the method adding nodes to current polytope and it can construct a attractive region if and only if the considering system is robustly stable. However, in general, the resulting polytope is not the maximal robust attractive region, and the method requires huge computing time to get larger robust attractive region. On the other hand, Blanchini et al. [1997] gave a method to compute the MRAR in a given region X of discrete time systems satisfying polytopic uncertainty in X. Our result is a corresponding result for continuous time systems. Moreover, as long as moderate dimensional cases, say $n \leqq 10$, we would like to say that Procedure make_PD is more efficient than the method proposed in Blanchini et al. [1997] according to our experience.

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Proof of Lemma 2.

Since $\tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$ is a RPIS for the system $\Sigma_\Delta^E(A,\gamma)$, $A_{q,\Delta,\gamma}x \in \tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$ whenever $x \in \tilde{\Omega}_\infty^{E^*}(\Delta,\gamma)$. Then, (26) is follows from (23). ■

Proof of Theorem 1. Let $P_1 = \tilde{\Omega}_\infty^{E^*}(\Delta_1,\gamma)$. Given any $x \in P_1$ such that $V(x;P_1) = 1$, where $V(e;P_1)$ is defined by (21). Then, (25) holds and, hence,

$$A_{q,\Delta_1,\gamma}x = [1 + \Delta_1(A_q + \gamma I)]x \in \tilde{\Omega}_\infty^{E^*}(\Delta_1), \quad \forall\, q \in Q. \tag{37}$$

Since X is convex, $x, A_{q,\Delta_1,\gamma}x \in X$ means that

$$\begin{aligned}\tilde{A}_{q,\Delta_2,\gamma}x &= x + \Delta_2(A_q + \gamma I)x \\ &= (1-\tau)x + \tau(x + \Delta_1(A_q+\gamma I)x) \\ &= (1-\tau)x + \tau A_{q,\Delta_1,\gamma}x \in X, \\ &\qquad \tau = \Delta_2/\Delta_1 \in [0,1] \;\; \forall\, \Delta_2 \in [0,\Delta_1].\end{aligned}$$

Therefore,

$$V(\tilde{A}_{q,\Delta_2}x;P_1) \leqq 0, \quad \forall\, x \in \partial P_1,$$

and, hence, P_1 is positively invariant under $\tilde{A}_{\Delta_2,\gamma}[k]$ satisfying $\tilde{A}_{\Delta_2,\gamma}[k] \in$ co $\{I + \Delta_2(A_q+\gamma I)\}_{q=1}^{Q}$. Since $\tilde{\Omega}_\infty^{E^*}(\Delta_2)$ is the MRPIS for (10) with uncertainty (14), where $\Delta = \Delta_2$, we have $P_1 \subseteq \tilde{\Omega}_\infty^{E^*}(\Delta_2,\gamma)$, which shows the first relation in (27).

Next we shows the second relation in (27). It suffices to show that

$$\begin{aligned}V'_{(6)}(x;P_1) &\leqq \sup_{A' \in \mathrm{co}\,\{A_q\}_{q=1}^{Q}} \overline{\lim_{\Delta\downarrow 0}} \frac{V(x+\Delta A'x;P_1) - V(x;P_1)}{\Delta} \\ &\leqq -\gamma V(x;P_1) \;\; \forall\, x \in \tilde{\Omega}_\infty^{E^*}(\Delta_1.\end{aligned} \tag{38}$$

We note that $x \in P_1$ if and only if $h_i x \leqq 1$ for all h_i, where h_i is the NNV of $F_i \in \mathcal{F}(P_1)$.

Suppose that $\hat{x}_0 \in$ bd P_1. Let $\Delta_0 \in (0,\Delta_1)$ be sufficiently small so that there exists $F_{i_0} \in \mathcal{F}(P_1)$ such that

$$\begin{aligned}&\hat{x}_0,\, \hat{x}_0 + \Delta A'\hat{x}_0,\, \hat{x}_0 + \Delta(A'+\gamma I)\hat{x}_0 \\ &\quad \in \mathrm{cc}\; F_{i_0} = \{\rho e:\; \rho \geqq 0, \;\; e \in F_i\} \;\; \forall\, \Delta \in [0,\Delta_0],\end{aligned} \tag{39}$$

where cc F is the convex cone determined by a facet F and is defined by cc $F = \{\rho x:\; \rho \geqq 0, \;\; x \in F\}$,

Then,

$$V(\hat{x}_0;P_1) = h_{i_0}^\top \hat{x}_0 = 1, \tag{40}$$

$$V(\hat{x}_0 + \Delta A'\hat{x}_0;P_1) = h_{i_0}^\top(\hat{x}_0 + \Delta A'\hat{x}_0) \leqq 1, \tag{41}$$

$$V(\hat{x}_0 + \Delta(A'+\gamma I)\hat{x}_0;P_1) = h_{i_0}^\top(\hat{x}_0 + \Delta(A'+\gamma I)\hat{x}_0) \leqq 1 \tag{42}$$

since $(\hat{x}_0 + \Delta(A'+\gamma I)\hat{x}_0) \in P_1$ for all $\Delta \in [0,\Delta_1]$ by the first relation in (27).

Therefore, we have

$$\begin{aligned}&V(\hat{x}_0 + \Delta A'\hat{x}_0;P_1) - V(\hat{x}_0;P_1) \\ &\;= V(\hat{x}_0 + \Delta(A'+\gamma I)\hat{x}_0;P_1) - V(\hat{x}_0;P_1) \\ &\quad - [V(\hat{x}_0 + \Delta(A'+\gamma I)\hat{x}_0;P_1) - V(\hat{x}_0 + \Delta A'\hat{x}_0;P_1)] \\ &\;= \Delta h_{i_0}^\top(A'+\gamma I)\hat{x}_0 - \Delta\gamma h_{i_0}^\top \hat{x}_0 \\ &\;\leqq -\Delta\gamma V(\hat{x}_0;P_1),\end{aligned} \tag{43}$$

and, hence, we have $V'_{(6)}(\hat{x}_0;P_1) \leqq -\gamma V(\hat{x}_0;P_1)$.

If $x \in$ int P_1, then there exists $\rho \in [0,1)$ and $\hat{x}_0 \in$ bd P_1 such that $x = \rho\hat{x}_0$. Then,

$$V'_{(6)}(x;P_1) = \rho V'_{(6)}(\hat{x}_0;P_1) \leqq -\gamma\rho V(\hat{x}_0;P_1) = -\gamma V(x;P_1).$$

Therefore, we obtained (38). This completes the proof. ■