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On Positive Real Lemma for Non-minimal Realization Systems

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Abstract: In this paper, we state the positive real lemma and the strictly positive real lemma (KYP lemma) for non-minimal realization systems. First we show the positive real lemma for stabilizable and observable systems under only the constraint on the regularity of the systems, by using the generalized algebraic Riccati equation. Moreover we show that the solution of the Lyapunov equation in the positive real lemma is positive definite. Next we similarly derive the KYP lemma for stabilizable and observable systems with only the above constraint and show that the corresponding solution in the KYP Lemma is positive definite. Finally, as the duals of these problems, we show that the positive real and KYP lemmas for controllable and detectable systems have both positive definite solutions.

1. INTRODUCTION

The positive real lemma is well-known as a useful criterion for determining positive realness in the state space representation [2, 9]. On the other hand, the strictly positive real lemma (or Kalman-Yakubovich-Popov (KYP) lemma) is also well-known as a significant criterion for determining strictly positive realness of transfer functions [2, 8, 9]. There are many researches on the KYP Lemma [3, 5, 14]. Both the positive real lemma and the KYP lemma are used in analysis and synthesis of control systems [4, 7]. In particular, Lur'e systems are made stable by using observer based control in Reference [6, 10], where the KYP lemma was required for stabilizable and observable systems since the observer is uncontrollable. In Reference [3], Collado et al. gave the KYP Lemma for stabilizable and observable systems. However, the systems that they discussed had the constraint that the set of controllable modes and the set of uncontrollable modes do not intersect. On the other hand, In Reference [14], Zhang et al. provided the conditions that descriptor systems without infinite zeros are regular, stable, impulse-free and strictly positive real. In the present paper, we derive the positive real and KYP lemmas for stabilizable and observable state space systems without the above mode constraint, by using the descriptor form.

We state the following four cases. First, we discuss the Positive Real Lemma in the case where $\sigma(A) \subset \mathbf{C}_-$, (A, B) is stabilizable and (C, A) is observable. Second, we discuss that lemma in the case where $\sigma(A) \subset \mathbf{C}_- \cup \mathbf{\Omega}$, (A, B) is stabilizable and (C, A) is observable. Next, we discuss the Strictly Positive Real Lemma in the case where $\sigma(A) \subset \mathbf{C}_-$, (A, B) is stabilizable and (C, A) is observable. Finally, Finally, we discuss the duals of the above three cases.

The notation is fairly standard in this paper. In particular the descriptor form of a transfer function matrix is denoted by

$$\left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right] := C(sE - A)^{-1}B + D$$

When $E = I$, the system has a proper transfer function matrix and can be represented in the state space form. However, to avoid any confusion, the above notation with $E = I$ will be used.The notations $\mathbf{C}_-$ and $\mathbf{C}_+$ represent the open left and right half complex plane, respectively; $\mathbf{\Omega}$ denotes the imaginary axis. Furthermore, $\sigma(A)$ is the set of the eigenvalues of A, and $\sigma_f(sE - A)$ denotes the set of the finite eigenvalues of $sE - A$. A generalized eigenvalue of (E, A) is defined to be a scalar λ satisfying $|\lambda E - A| = 0$. Const denotes some constant matrix.

2. PRELIMINARY

In this section, we consider a proper square transfer matrix $G(s)$ with a minimal realization as follows:

$$G(s) := \left[\begin{array}{c|c} A - sI & B \\ \hline C & D \end{array}\right] \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{m\times n}$, and $D \in \mathbf{R}^{m\times m}$. Note in this paper that we do not assume the nonsingularity of $D + D^T$, namely $G(s) + G^T(-s)$ can have infinite zeros.

2.1 Positive Realness

Definition 1. A $m \times m$ transfer matrix $G(s)$ is positive real if the following inequality is satisfied.

$$G(s) + G^T(\bar{s}) \geq 0 \quad {}^\forall \mathrm{Re}(s) > 0 \tag{2}$$

Lemma 1. (Positive Real Lemma[2])
Assume (A, B) is controllable and (C, A) is observable. Then a proper matrix $G(s)$ is positive real if and only if there exist real matrices $P > 0$, L, and W that satisfy the Positive Real Lemma Equations (PRLE):

$$PA + A^T P = -L^T L \tag{3a}$$
$$C - B^T P = W^T L \tag{3b}$$
$$D + D^T = W^T W \tag{3c}$$

Then $G(s)$ has the following spectral factorization:

$$G(s) + G^T(-s) = V^T(-s)V(s) \tag{4}$$

where

$$V(s) := \left[\begin{array}{c|c} A - sI & B \\ \hline L & W \end{array}\right] \tag{5}$$

2.2 Strictly Positive Realness

Definition 2. A $m \times m$ transfer matrix $G(s)$ is strictly positive real if there exists a scalar $\varepsilon > 0$ such that $G(s-\varepsilon)$ is positive real.

Lemma 2. (Strictly Positive Real Lemma[2])
Assume (A, B) is controllable and (C, A) is observable. Then a proper matrix $G(s)$ is strictly positive real if and only if there exist real matrices $P > 0$, L, W and a scalar $\varepsilon > 0$ that satisfy the Strictly Positive Real Lemma Equations:

$$PA + A^T P = -L^T L - \varepsilon P \tag{6a}$$
$$C - B^T P = W^T L \tag{6b}$$
$$D + D^T = W^T W \tag{6c}$$

3. MAIN RESULT

Instead of $G(s)$ given by (1), the following descriptor form of $G(s)$ with D_e *nonsingular* is used in the sequel:

$$G(s) = \left[\begin{array}{cc|c} A - sI & 0 & B \\ 0 & I & \alpha I - D \\ \hline C & I & \alpha I \end{array}\right] =: \left[\begin{array}{c|c} A_e - sE_e & B_e \\ \hline C_e & D_e \end{array}\right] \tag{7}$$

where α is a positive scalar such that $B_2 := \alpha I - D$ is nonsingular.

3.1 A new characterization of Positive Realness

We propose a new characterization of positive realness using a generalized algebraic Riccati equation (GARE) instead of the PRLE (3).

Theorem 3. Assume that (A, B) is controllable and (C, A) is observable. Then a proper transfer matrix $G(s)$ is positive real if and only if the following generalized algebraic Riccati equation has a solution $X \in \mathbf{R}^{p\times p}$ with $M^T E_e^T X M > 0$

$$X^T A_e + A_e^T X + \gamma^{-2}(C_e - B_e^T X)^T (C_e - B_e^T X) = 0 \tag{8a}$$
$$E_e^T X = X^T E_e \tag{8b}$$

where $p := n + m$, $M := [I_n \ 0_{n\times m}]^T$ and $\gamma^2 := 2\alpha$.

Proof: (*Sufficiency*) It follows obviously from $M^T E_e^T XM > 0$ and (8b) that a solution X satisfying (8) takes the following form

$$X = \begin{bmatrix} X_{11} & 0_{n\times m} \\ X_{21} & X_{22} \end{bmatrix}, \quad X_{11} = X_{11}^T > 0 \tag{9}$$

Then (8a) becomes

$$X_{11}A + A^T X_{11} + L^T L = 0 \tag{10a}$$
$$X_{21} + L_2^T L = 0 \tag{10b}$$
$$X_{22}^T + X_{22} + L_2^T L_2 = 0 \tag{10c}$$

where $L := \gamma^{-1}(C - B^T X_{11} - B_2^T X_{21})$, $L_2 := \gamma^{-1}(I - B_2^T X_{22})$. By (10b) we obtain $(\gamma I - B_2^T L_2^T)L = C - B^T X_{11}$. Since $W := \gamma I - L_2 B_2$ satisfies $W^T W = D + D^T$ (see Appendix A), there exists a triple of $P := X_{11} > 0$, L and W satisfying (3).

(*Necessity*) Since $G(s)$ is positive real, there exist $P > 0$, L and W satisfying (3). Without loss of generality W is a square matrix (See Appendix B), we can define $L_2 := (\gamma I - W)B_2^{-1}$ and choose $X_{11} := P > 0$, $X_{21} := -L_2^T L$ and $X_{22} := B_2^{-T}(I - \gamma L_2)$. By $W^T L = C - B^T X_{11}$ and $X_{21} = -B_2^{-T}(\gamma I - W^T)L$, we obtain $\gamma L = C - B^T X_{11} - B_2^T X_{21}$. Therefore we can see that L and L_2 can be written by $L = \gamma^{-1}(C - B^T X_{11} - B_2^T X_{21})$ and $L_2 = \gamma^{-1}(I - B_2^T X_{22})$, respectively. With these matrices X_{11}, X_{21}, X_{22} and L_2, it is easy to show that (3) implies (8a), and hence the X of the form (9) associated with these X_{11}, X_{21} and X_{22} satisfies (8) and $M^T E_e^T XM > 0$. ♠

Although the GARE (8) in one variable X is replaced from the PRLE (3) in three variables P, L and W in Theorem 3, it is only the change of an algebraic relation and nothing changes essentially. However, since the GARE (8) reduces to generalized eigenvalue problems, it could be solved like algebraic Riccati equations.

3.2 Positive Real Lemma for stabilizable and Observable systems

In this subsection, we show that $P > 0$ (L, W) satisfying (3) even if (A, B) is stabilizable and (C, A) is controllable.

Below, we consider a system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ such that $(\bar{A}, \bar{B})$ is stabilizable and $(\bar{C}, \bar{A})$ is observable. Without loss of generality, the system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ can be represented as in the following Kalman canonical decomposition [15] using a minimal realization system (A, B, C, D) with a controllable pair (A, B).

$$\bar{G}(s) := \left[\begin{array}{c|c} \bar{A} - sI & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] := \left[\begin{array}{cc|c} A - sI & A_{un} & B \\ 0 & A_u - sI & 0 \\ \hline C & C_u & D \end{array}\right] \tag{11}$$

where $A_u \in \mathbf{R}^{l\times l}$ and $\sigma(A_u) \subset \mathbf{C}_-$, $A_{un} \in \mathbf{R}^{n\times l}$ and $C_u \in \mathbf{R}^{m\times l}$.

In the following two theorems, we first state the case of $\sigma(\bar{A}) \subset \mathbf{C}_-$, and then the case of $\sigma(\bar{A}) \subset \mathbf{C}_- \cup \mathbf{\Omega}$.

Theorem 4. Assume that $\sigma(\bar{A}) \subset \mathbf{C}_-$, $(\bar{A}, \bar{B})$ is stabilizable, $(\bar{C}, \bar{A})$ is observable and $|\bar{G}(s) + \bar{G}^T(-s)| \not\equiv 0$. Then $\bar{G}(s)$ is positive real if and only if there exist $\bar{P} > 0$, $\bar{L}$ and $\bar{W}$ satisfying the following equations.

$$\bar{P}\bar{A} + \bar{A}^T \bar{P} + \bar{L}^T \bar{L} = 0 \tag{12a}$$
$$\bar{C} - \bar{B}^T \bar{P} = \bar{W}^T \bar{L} \tag{12b}$$
$$\bar{D} + \bar{D}^T = \bar{W}^T \bar{W} \tag{12c}$$

Proof: (*Sufficiency*) Since there exist $\bar{P}$, $\bar{L}$ and $\bar{W}$ satisfying (12), $\bar{G}(s) + \bar{G}^T(\bar{s})$ becomes as follows:

$$
\begin{aligned}
&\bar{G}(s) + \bar{G}^T(\bar{s}) \\
&= \begin{bmatrix} (\bar{s}I - \bar{A})^{-1} \\ I \end{bmatrix}^T \begin{bmatrix} 0 & \bar{C}^T \\ \bar{C} & \bar{D} + \bar{D}^T \end{bmatrix} \begin{bmatrix} (sI - \bar{A})^{-1} \\ I \end{bmatrix} \\
&\geq \begin{bmatrix} (\bar{s}I - \bar{A})^{-1} \\ I \end{bmatrix}^T \begin{bmatrix} \bar{P}\bar{A} + \bar{A}^T\bar{P} & \bar{P}\bar{B} \\ \bar{B}^T\bar{P} & 0 \end{bmatrix} \begin{bmatrix} (sI - \bar{A})^{-1} \\ I \end{bmatrix} \\
&= (s + \bar{s})\bar{B}^T(\bar{s}I - \bar{A}^T)^{-1}\bar{P}(sI - \bar{A})^{-1}\bar{B} \qquad (13)
\end{aligned}
$$

Since $s+\bar{s} > 0$ for ${}^\forall\mathrm{Re}(s) > 0$ and $\bar{P} > 0$, $\bar{G}(s)+\bar{G}^T(\bar{s}) \geq 0$ for ${}^\forall\mathrm{Re}(s) > 0$.

(*Necessity*) *Case 1: Having no controllable part in* $(\bar{A}, \bar{B})$. By the assumptions, $\bar{B} = 0$, $|\bar{D}+\bar{D}^T| \neq 0$ and $\sigma(\bar{A}) \subset \mathbf{C}_-$ in Case 1. Also $\bar{D} + \bar{D}^T > 0$ due to the positive realness of $\bar{G}(s)$. Here, $\bar{W} := (\bar{D}+\bar{D}^T)^{1/2}$ and $\bar{L} := \bar{W}^{-T}\bar{C}$ satisfy (12b) and (12c), respectively. Since $\sigma(\bar{A}) \subset \mathbf{C}_-$ and $(\bar{L}, \bar{A})$ is observable, there always exists $\bar{P} > 0$ satisfying (12a). Hence there are $\bar{P} > 0$, $\bar{L}$ and $\bar{W}$ satisfying (12).

Case 2: Having a controllable part in $(\bar{A}, \bar{B})$.
As in the case with (7), let's consider the following descriptor form.

$$
\bar{G}(s) = \left[\begin{array}{cc|c} \bar{A} - sI & 0 & \bar{B} \\ 0 & I & \alpha I - \bar{D} \\ \hline \bar{C} & I & \alpha I \end{array}\right] =: \left[\begin{array}{c|c} \bar{A}_e - s\bar{E}_e & \bar{B}_e \\ \hline \bar{C}_e & \bar{D}_e \end{array}\right] \quad (14)
$$

where α is a positive scalar such that $\bar{B}_2 := \alpha I - \bar{D} = B_2$ is nonsingular. By Theorem 3, it is enough to show that there exists a solution $\bar{X} \in \mathbf{R}^{(p+l)\times(p+l)}$ with $\bar{M}^T\bar{E}_e\bar{X}\bar{M} > 0$ satisfying the following GARE.

$$
\bar{X}^T\bar{A}_e + \bar{A}_e^T\bar{X} + \gamma^{-2}(\bar{C}_e - \bar{B}_e^T\bar{X})^T(\bar{C}_e - \bar{B}_e^T\bar{X}) = 0 \qquad (15a)
$$

$$
\bar{E}_e^T\bar{X} = \bar{X}^T\bar{E}_e \qquad (15b)
$$

where $\bar{M} := \begin{bmatrix} I_{n+l} & 0_{(n+l)\times m} \end{bmatrix}^T$. $\bar{X}$ has the following form due to (15b).

$$
\begin{aligned}
\bar{X} &=: \begin{bmatrix} \bar{X}_{11} & 0_{(n+l)\times m} \\ \bar{X}_{21} & \bar{X}_{22} \end{bmatrix}, \quad \bar{X}_{11} = \bar{X}_{11}^T \qquad (16) \\
&=: \begin{bmatrix} \begin{bmatrix} X_{11} & X_\alpha \\ X_\alpha^T & X_\beta \end{bmatrix} & 0_{(n+l)\times m} \\ \begin{bmatrix} X_{21} & X_u \end{bmatrix} & X_{22} \end{bmatrix}
\end{aligned}
$$

Here, let $\Gamma_{\bar{X}}$ be the left hand side of (15a). Then

$$
T_a^T\bar{X}T_a = \begin{bmatrix} \begin{bmatrix} X_{11} & 0_{n\times m} \\ X_{21} & X_{22} \end{bmatrix} & \begin{bmatrix} X_\alpha \\ X_u \end{bmatrix} \\ \begin{bmatrix} X_\alpha^T & 0_{l\times m} \end{bmatrix} & X_\beta \end{bmatrix} =: \begin{bmatrix} X & X_\gamma \\ X_\gamma^T E_e & X_\beta \end{bmatrix}
$$

$$
T_a^T\bar{A}T_a = \begin{bmatrix} A_e & \bar{A}_{un} \\ 0_{l\times p} & A_u \end{bmatrix} \quad \left(\bar{A}_{un} := \begin{bmatrix} A_{un} \\ 0 \end{bmatrix}\right)
$$

$$
T_a^{-1}\bar{B} = \begin{bmatrix} B \\ B_2 \\ 0 \end{bmatrix}, \quad \bar{C}T_a = [C \;\; I \;\; C_u]
$$

where

$$
T_a := \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_m & 0 \end{bmatrix}
$$

Here $\Gamma_{\bar{X}}$ can be transformed as follows:

$$
\tilde{\Gamma}_{\bar{X}} := T_a^T\Gamma_{\bar{X}}T_a = \begin{bmatrix} \Gamma_X & \tilde{\Gamma}_{\bar{X}}(1,2) \\ \tilde{\Gamma}_{\bar{X}}^T(1,2) & \tilde{\Gamma}_{\bar{X}}(2,2) \end{bmatrix} \qquad (17)
$$

where $\tilde{\Gamma}_{\bar{X}}(1,2) := (\hat{A}_e^T + \gamma^{-2}X^TB_eB_e^T)X_\gamma + X^T\hat{A}_{12} + E_e^TX_\gamma A_u + \gamma^{-2}C_e^TC_u$, $\tilde{\Gamma}_{\bar{X}}(2,2) := X_\gamma^T\hat{A}_{12} + \hat{A}_{12}^TX_\gamma + \gamma^{-2}X_\gamma^TB_eB_e^TX_\gamma + X_\beta A_u + A_u^TX_\beta + \gamma^{-2}C_u^TC_u$ and $\hat{A}_e := A_e - \gamma^{-2}B_eC_e$.

With the above preliminaries, we first consider Γ_X. Since the minimal realization part $G(s) = C(sI - A)^{-1}B + D$ of $\bar{G}(s)$ is obviously positive real, there exists a solution X with $M^TE_e^TXM > 0$ to the GARE (8) by Theorem 3, which indicates $\Gamma_X = 0$.

Next we show an existence of X_γ satisfying $\tilde{\Gamma}_{\bar{X}}(1,2) = 0$. The pair $(E_e, \hat{A}_e + \gamma^{-2}B_eB_e^TX)$ can be transformed into the following Weierstrass form.

$$
S(\hat{A}_e + \gamma^{-2}B_eB_e^TX)T = \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix}, \; SE_eT = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \quad (18)
$$

where $\sigma(\Lambda) \subset \mathbf{C}_- \cup \mathbf{\Omega}$ and N is a nilpotent matrix. Note that there exists X such that $\sigma_f(sE_e - \hat{A}_e - \gamma^{-2}B_eB_e^TX) = \sigma(\Lambda) \subset \mathbf{C}_- \cup \mathbf{\Omega}$ in (8) (See Appendix B). Multiplying S from the left hand side of $\tilde{\Gamma}_{\bar{X}}(1,2) = 0$ yields the following equation.

$$
\begin{bmatrix} \Lambda^T\bar{X}_{\gamma1} + \bar{X}_{\gamma1}A_u + \mathrm{Const} \\ \bar{X}_{\gamma2} + N\bar{X}_{\gamma2}A_u + \mathrm{Const} \end{bmatrix} = 0 \qquad (19)
$$

where $T^{-1}\bar{X}_\gamma =: \begin{bmatrix} \bar{X}_{\gamma1}^T & \bar{X}_{\gamma2}^T \end{bmatrix}^T$. The above two Sylvester equations are feasible with respect to $\bar{X}_{\gamma1}$ and $\bar{X}_{\gamma2}$ due to $\sigma(\Lambda) \subset \mathbf{C}_- \cup \mathbf{\Omega}$, $\sigma(A_u) \subset \mathbf{C}_-$ and $\sigma(N) = 0$, respectively. Therefore, there always exists X_γ satisfying $\tilde{\Gamma}_{\bar{X}}(1,2) = 0$.

Next we show an existence of X_β satisfying $\tilde{\Gamma}_{\bar{X}}(2,2) = 0$. With substituting X_γ for $\tilde{\Gamma}_{\bar{X}}(2,2)$, $\tilde{\Gamma}_{\bar{X}}(2,2)$ becomes Lyapunov equation with respect to X_β. Since this Lyapunov equation is feasible due to $\sigma(A_u) \subset \mathbf{C}_-$, there always exists X_β. Therefore, there always exists $\bar{X}$ satisfying the GARE (15) if $\bar{G}(s)$ is positive real.

Finally, we show that $\bar{P} := \bar{X}_{11} = M^T\bar{E}_e^T\bar{X}M > 0$, $\bar{L} := \gamma^{-1}(\bar{C} - \bar{B}^T\bar{X}_{11} - \bar{B}_2^T\bar{X}_{21})$ and $\bar{W} := \gamma^{-1}(I - \bar{B}_2^T\bar{X}_{22})$ satisfy (12). Now, since $\sigma(\bar{A}) \subset \mathbf{C}_-$ and $\bar{L}^T\bar{L} \geq 0$, we have $\bar{P} \geq 0$. By Lemma 9 (See Appendix C), a solution $\bar{X}$ satisfying (15) is nonsingular since $(\bar{C}, \bar{A})$ is observable. Hence, by $|\bar{X}| = |\bar{X}_{11}||\bar{X}_{22}| \neq 0$, we obtain $\bar{X}_{11} = \bar{P} > 0$. ♠

Remark 1. The regularity of $\bar{G}(s) + \bar{G}^T(-s)$ is required as a basic condition to treat a generalized eigenvector problem of the Hamiltonian matrix pencil with respect to the GARE (8). See Reference [13] in detail to solve the GARE by using the generalized eigenvector problem.

Theorem 5. Assume that $\sigma(\bar{A}) \subset \mathbf{C}_- \cup \mathbf{\Omega}$, $(\bar{A}, \bar{B})$ is stabilizable and $(\bar{C}, \bar{A})$ is observable. Moreover, assume that $|\bar{G}(s) + \bar{G}^T(-s)| \not\equiv 0$ if $\sigma(\bar{A}) \not\subset \mathbf{\Omega}$. Then $\bar{G}(s)$ is positive real if and only if there exist $\bar{P} > 0$, $\bar{L}$ and $\bar{W}$ satisfying (12).

Proof: (*Sufficiency*) The proof is the same as that of Theorem 4.
(*Necessity*) The proofs in the cases of $\sigma(\bar{A}) \subset \mathbf{\Omega}$ and $\sigma(\bar{A}) \subset \mathbf{C}_-$ are obvious by Lemma 1 and Theorem 4, respectively. In the case of $\sigma(\bar{A}) \subset \mathbf{C}_- \cup \mathbf{\Omega}$, $\bar{G}(s)$ can be represented as follows:

$$\bar{G}(s) = \left[\begin{array}{c|c} \bar{A}-sI & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] =: \left[\begin{array}{cc|c} \bar{A}_0 - sI & 0 & \bar{B}_0 \\ 0 & \bar{A}_- - sI & \bar{B}_- \\ \hline \bar{C}_0 & \bar{C}_- & \bar{D}_- \end{array}\right] \quad (20)$$

where $\sigma(\bar{A}_0) \subset \mathbf{\Omega}$, $(\bar{A}_0, \bar{B}_0)$ is controllable, $(\bar{C}_0, \bar{A}_0)$ is observable, and $\sigma(\bar{A}_-) \subset \mathbf{C}_-$, $(\bar{A}_-, \bar{B}_-)$ is stabilizable, $(\bar{C}_-, \bar{A}_-)$ is observable. Now, since $\sigma(\bar{A}_0) \subset \mathbf{\Omega}$, $\sigma(\bar{A}_-) \subset \mathbf{C}_-$ and $\bar{G}(s)$ is positive real, both $\bar{G}_0(s) := C_0(sI - A_0)^{-1}B_0$ and $\bar{G}_-(s) := C_-(sI - A_-)^{-1}B_- + D_-$ are positive real. When $\bar{G}_0(s)$ is positive real, there exist $\bar{A}_0$ with $\bar{A}_0 + \bar{A}_0^T = 0$, and $\bar{B}_0, \bar{C}_0$ with $\bar{B}_0^T = \bar{C}_0$ [9]. Therefore, since $|\bar{G}(s) + \bar{G}^T(-s)| = |\bar{G}_-(s) + \bar{G}_-^T(-s)| \not\equiv 0$ by $\bar{G}_0(s) + \bar{G}_0^T(-s) = \bar{C}_0(sI - \bar{A}_0)^{-1}(\bar{A}_0 + \bar{A}_0^T)(sI + \bar{A}_0^T)^{-1}\bar{C}_0^T \equiv 0$, there exist $\bar{P}_- > 0$, $\bar{L}_-$, $\bar{W}_-$ satisfying Theorem 4 for $\bar{G}_-(s)$. Here, let $\bar{P}$, $\bar{L}$ and $\bar{W}$ be as follows:

$$\bar{P} := \begin{bmatrix} I & 0 \\ 0 & \bar{P}_- \end{bmatrix}, \ \bar{L} := \begin{bmatrix} 0 & \bar{L}_- \end{bmatrix}, \ \bar{W} := \bar{W}_- \quad (21)$$

Then $\bar{P} > 0$ and $\bar{L}$ as well as $\bar{W}$ satisfy (12). ♠

Remark 2. In Theorem 5, note that we have $|\bar{G}(s) + \bar{G}^T(-s)| \equiv 0$ when $\bar{G}(s)$ is positive real, $\sigma(\bar{A}_0) \neq \phi$, $\sigma(\bar{A}_-) \neq \phi$, $\bar{B}_- = 0$ and $|\bar{D}_- + \bar{D}_-^T| = 0$. In this case, there do not always exist solutions satisfying (12). In particular, when $\bar{D}_- + \bar{D}_-^T = 0$, there exist no solutions $\bar{P} > 0$, $\bar{L}$ and $\bar{W}$ satisfying (12), where there exists $\bar{P} \geq 0$ if $\bar{C}_- = 0$.

3.3 Strictly Positive Real Lemma for Stabilizable and Observable Systems

In this subsection, we state the strictly positive real lemma for stabilizable and observable systems (KYP Lemma for stabilizable and observable systems).

Theorem 6. Assume that $\sigma(\bar{A}) \subset \mathbf{C}_-$, $(\bar{A}, \bar{B})$ is stabilizable, $(\bar{C}, \bar{A})$ is observable and $|\bar{G}(s) + \bar{G}^T(-s)| \not\equiv 0$ in (11). Then $\bar{G}(s)$ is strictly positive real if and only if there exist matrices $\bar{P} > 0$, $\bar{L}$, $\bar{W}$ and a scalar $\varepsilon > 0$ satisfying the following equations.

$$\bar{P}\bar{A} + \bar{A}^T\bar{P} + \bar{L}^T\bar{L} + \varepsilon\bar{P} = 0 \quad (22a)$$

$$\bar{C} - \bar{B}^T\bar{P} = \bar{W}^T\bar{L} \quad (22b)$$

$$\bar{D} + \bar{D}^T = \bar{W}^T\bar{W} \quad (22c)$$

Proof: (*Sufficiency*) Obviously $\sigma(\bar{A} + (\varepsilon/2)I) \subset \mathbf{C}_- \cup \mathbf{\Omega}$ since (22a) can be written equivalently as follows:

$$\bar{P}(\bar{A} + (\varepsilon/2)I) + (\bar{A} + (\varepsilon/2)I)^T\bar{P} + \bar{L}^T\bar{L} = 0 \quad (23)$$

By Theorem 5, (23), (22b) and (22c) indicate that $\bar{C}(sI - (\bar{A} + (\varepsilon/2)I))^{-1}\bar{B} + \bar{D} = \bar{G}(s - \varepsilon/2)$ is positive real.

(*Necessity*) $\bar{G}(s - \varepsilon/2) = \bar{C}(sI - (\bar{A} + (\varepsilon/2)I))^{-1}\bar{B} + \bar{D}$ is positive real and $\sigma(\bar{A} + (\varepsilon/2)I) \subset \mathbf{C}_-$ for some sufficient small scalar $\varepsilon > 0$ since $\bar{G}(s)$ is strictly positive real. Therefore, by Theorem 4 there exist matrices $\bar{P} > 0$, $\bar{L}$ and $\bar{W}$ satisfying (23), (22b) and (22c), which indicates that $\bar{P} > 0$, $\bar{L}$, $\bar{W}$ and $\varepsilon > 0$ satisfy (22). ♠

Remark 3. Theorem 6 gives a general result without the constraint that the set of controllable modes and the set of uncontrollable modes do not intersect in Reference [3].

3.4 The case with Controllable and detectable Systems

In this subsection, we state the positive real and strictly positive real lemmas for controllable and detectable systems. Let us consider $\bar{G}(s) := \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$ such that $(\bar{A}, \bar{B})$ is controllable and $(\bar{C}, \bar{A})$ is detectable. Then we obtain two results as the dual cases of the lemmas for stabilizable and observable systems.

Theorem 7. Assume that $\sigma(\bar{A}) \subset \mathbf{C}_- \cup \Omega$, $(\bar{A}, \bar{B})$ is controllable and $(\bar{C}, \bar{A})$ is detectable. Moreover, assume that $|\bar{G}(s) + \bar{G}^T(-s)| \not\equiv 0$ if $\sigma(\bar{A}) \not\subset \mathbf{\Omega}$. Then $\bar{G}(s)$ is positive real if and only if there exist $\bar{P} > 0$, $\bar{L}$ and $\bar{W}$ satisfying (12).

Proof: (*Sufficiency*) The proof is the same as that of Theorem 4.

(*Necessity*) The transfer matrix $\bar{G}^T(s)$ is also positive real when $\bar{G}(s)$ is positive real. Since $(\bar{A}^T, \bar{C}^T)$ is stabilizable and $(\bar{B}^T, \bar{A}^T)$ is observable, by Theorem 5 there exist $\bar{Q} > 0$, $\bar{L}_q$ and $\bar{W}_q$ satisfying the following equations.

$$\bar{Q}\bar{A}^T + \bar{A}\bar{Q} + \bar{L}_q^T\bar{L}_q = 0$$

$$\bar{B}^T - \bar{C}\bar{Q} = \bar{W}_q^T\bar{L}_q$$

$$\bar{D}^T + \bar{D} = \bar{W}_q^T\bar{W}_q$$

By defining $\bar{P} := \bar{Q}^{-1}$, $\bar{L} := -\bar{L}_q\bar{Q}^{-1}$ and $\bar{W} := \bar{W}_q$, we obtain $\bar{P} > 0$, $\bar{L}$ and $\bar{W}$ satisfying (12). ♠

Theorem 8. Assume that $\sigma(\bar{A}) \subset \mathbf{C}_-$, $(\bar{A}, \bar{B})$ is controllable, $(\bar{C}, \bar{A})$ is detectable and $|\bar{G}(s) + \bar{G}^T(-s)| \not\equiv 0$. Then $\bar{G}(s)$ is strictly positive real if and only if there exist matrices $\bar{P} > 0$, $\bar{L}$, $\bar{W}$ and a scalar $\varepsilon > 0$ satisfying (22).

Proof: The proof is similar to Theorem 7. ♠

4. NUMERICAL EXAMPLE

Let us consider $\bar{G}(s)$ as follows:

$$\bar{G}(s) = \left[\begin{array}{c|c} \bar{A}-sI & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{cc|c} -2-s & 1 & 1 \\ 0 & -2-s & 0 \\ \hline 1 & 1 & 0 \end{array}\right] = \frac{1}{s+2}$$

where $\sigma(\bar{A}) \subset \mathbf{C}_-$, $(\bar{A}, \bar{B})$ is stabilizable, $(\bar{C}, \bar{A})$ is observable. Then $\bar{G}(s)$ is obviously positive real and the following $\bar{P}$, $\bar{L}$ and $\bar{W}$ satisfy (12).

$$\bar{P} = \begin{bmatrix} 1 & 1 \\ 1 & \frac{17}{16} \end{bmatrix} > 0, \quad \bar{L} = \pm\begin{bmatrix} 2 & \frac{3}{2} \end{bmatrix}, \quad \bar{W} = 0$$

By defining $\bar{V}(s)$ such that $\bar{G}(s) + \bar{G}^T(-s) = \bar{V}^T(-s)\bar{V}(s)$, we obtain

$$\bar{V}(s) := \left[\begin{array}{c|c} \bar{A}-sI & \bar{B} \\ \hline \bar{L} & \bar{W} \end{array}\right] = \frac{\pm 2}{s+2}$$

Next, consider the dual system $\bar{G}^T(s)$ of the above $\bar{G}(s)$. Then $\bar{G}^T(s)$ is positive real and the following $\bar{P}$, $\bar{L}$ and $\bar{W}$ satisfy (12) for $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ replaced by $(\bar{A}^T, \bar{B}^T, \bar{C}^T, \bar{D}^T)$.

$$\bar{P} = \begin{bmatrix} 1 & 1 \\ 1 & \frac{17}{16} \end{bmatrix}^{-1} = \begin{bmatrix} 17 & -16 \\ -16 & 16 \end{bmatrix} > 0$$

$$\bar{L} = -\left(\pm\begin{bmatrix} 2 & \frac{3}{2} \end{bmatrix}\right)\begin{bmatrix} 1 & 1 \\ 1 & \frac{17}{16} \end{bmatrix}^{-1} = \mp\begin{bmatrix} 10 & -8 \end{bmatrix}$$

$$\bar{W} = 0$$

which were calculated by using a method in the proof Theorem 8. By defining $\bar{V}_c(s)$ such that $\bar{G}^T(s) + \bar{G}(-s) = \bar{V}_c^T(-s)\bar{V}_c(s)$, we obtain

$$\bar{V}_c(s) := \left[\begin{array}{c|c} \bar{A}^T - sI & \bar{C}^T \\ \hline \bar{L} & \bar{W} \end{array}\right] = \frac{\pm 2(s-2)}{(s+2)^2}$$

Note that $\bar{V}_c(s)$ is minimal realization, $\bar{V}_c(s)$ has no unobservable mode and $\bar{V}_c(s)$ is a transfer function multiplying an all-pass transfer function $(2-s)/(2+s)$ for $\bar{V}(s)$.

5. CONCLUSION

In this paper, we have stated the positive real lemma and the strictly positive real lemma (the KYP lemma) for non-minimal realization systems. First we have shown the positive real lemma for stabilizable and observable systems under only the constraint with respect to the regularity of the systems, by using the generalized algebraic Riccati equation (GARE). Moreover we have shown that there always exist the solutions $\bar{P}$ with positive definite, $\bar{L}$ and $\bar{W}$. Next we have similarly shown the KYP lemma for stabilizable and observable systems under only the above constraint. Finally, as their dual problems, we have shown that the positive real and strictly positive real lemmas for controllable and detectable systems have positive definite $\bar{P}$, respectively.

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Appendix A. THE COMPLEMENT OF THEOREM 3

Using (10c), $B_2 = \alpha I - D$ and $\gamma L_2 = I - B_2^T X_{22}$, we obtain

$$\begin{aligned} W^T W &= \gamma^2 I - \gamma B_2^T L_2^T - \gamma L_2 B_2 - B_2^T (X_{22}^T + X_{22}) B_2 \\ &= 2\alpha I - \{B_2^T(\gamma L_2^T + X_{22}^T L_2) + (\gamma L_2 + B_2^T X_{22}) B_2\} \\ &= D + D^T \end{aligned}$$

Appendix B. PROOF OF $\sigma(\Lambda) \subset \mathrm{C}_- \cup \Omega$

When $G(s)$ is positive real, $G(s) + G^T(-s)$ can be written as

$$G(s) + G^T(-s) = V^T(-s)V(s) = \Pi^T(-s)\Pi(s)$$

Here, a square transfer matrix $\Pi(s)$ is defined by

$$\Pi(s) := \left[\begin{array}{c|c} A - sI & B \\ \hline -Z_{21} & I_m - Z_{22} \end{array}\right]$$

where Z_{21} and Z_{22} are the elements of Z defined by (B.1). In Reference [13], it is known that there always exists Z satisfying the following equations for $V(s)$. (Note that it is allowable that $V(s)$ is a $q \times m (q \geq m)$ transfer matrix [13].)

$$Z = \begin{bmatrix} Z_{11} & 0_{n\times m} \\ Z_{21} & Z_{22} \end{bmatrix}, \quad Z_{11} = Z_{11}^T \geq 0 \tag{B.1}$$

$$\sigma_f(sE_e - A_a + B_a B_a^T Z) \subset \mathrm{C}_- \cup \Omega$$

$$A_a := \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix}, \quad B_a := \begin{bmatrix} 0 \\ -I_m \end{bmatrix}$$

$$Z_{11}A + A^T Z_{11} + L^T L = Z_{21}^T Z_{21} \tag{B.2a}$$

$$B^T Z_{11} + W^T L = -(I - Z_{22})^T Z_{21} \tag{B.2b}$$

$$W^T W = (I - Z_{22})^T (I - Z_{22}) \tag{B.2c}$$

Let $\hat{P}$ be $\hat{P} := P - Z_{11}$. By substituting (B.2) for (3), we obtain

$$\hat{P}A + A^T \hat{P} + Z_{21}^T Z_{21} = 0$$

$$C - B^T \hat{P} = -(I - Z_{22})^T Z_{21}$$

$$W^T W = (I - Z_{22})^T (I - Z_{22})$$

Therefore, by $\sigma_f(sE_e - \hat{A}_e - \gamma^{-2} B_e B_e^T X) = \sigma_f(sE_e - A_a + B_a B_a^T Z)$, Theorem 3 and Lemma 9, we obtain a solution X with $M^T E_e^T X M > 0$ to the GARE (8) such that $\sigma_f(sE_e - \hat{A}_e - \gamma^{-2} B_e B_e^T X) = \sigma(\Lambda) \subset \mathrm{C}_- \cup \Omega$. ♠

Appendix C. ON THE NONSINGULARITY OF SOLUTIONS OF GARE

Lemma 9. Any solution $\bar{X}$ satisfying the GARE (15) is nonsingular if $(\bar{C}, \bar{A})$ is observable.

Proof: We assume $|\bar{X}| = 0$. Then there exists a basis matrix V with colum full rank of $\mathrm{Ker}(\bar{X})$. Multiplying V^T from the left hand side and V from the right hand side of (15a) yields $\bar{C}_eV = 0$. Multiplying V from the right hand side of (15a) and (15b) also yields $\bar{X}^T\bar{A}_eV = 0$ and $\bar{X}^T\bar{E}_eV = 0$, respectively. Here, we show that a matrix $\bar{E}_eV$ is of colum full rank. If $\bar{E}_eV$ is not of colum full rank, there exists a vector $y \neq 0$ such that $\bar{E}_eVy = 0$. Now although $\begin{bmatrix}\bar{E}_e^T & \bar{C}_e^T\end{bmatrix}^T Vy = 0$ by $\bar{C}_eV = 0$, we obtain $y = 0$ from the fact that $\begin{bmatrix}\bar{E}_e^T & \bar{C}_e^T\end{bmatrix}^T$ is of colum full rank. Therefore, $\bar{E}_eV$ is of colum full rank. Next, since $\dim\{\mathrm{Ker}(\bar{X}^T)\} = \dim\{\mathrm{Ker}(\bar{X})\}$ by the fact that $\bar{X}$ is a square matrix, we can represent $\bar{E}_eV\Theta = \bar{A}_eV$ by using the full rankness of $\bar{E}_eV$. Let λ and u be an arbitrary pair (λ, u) such that $\Theta u = \lambda u$ and $u \neq 0$. Then multiplying u from the right hand side of $\bar{E}_eV\Theta = \bar{A}_eV$ yields $(\lambda\bar{E}_e - \bar{A}_e)Vu = 0$. By $\bar{C}_eV = 0$, we obtain the following equation.

$$\begin{bmatrix}\lambda\bar{E}_e - \bar{A}_e \\ \bar{C}_e\end{bmatrix} Vu = 0, \quad Vu \neq 0$$

Since $(\bar{C}, \bar{A})$ is observable and V is of colum full rank, we obtain $u = 0$, which contradicts $u \neq 0$. Therefore, $\bar{X}$ is nonsingular. ♠