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Neural Network Robot Control with Noisy Learning

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Abstract: Neural network based control of a serial-link robotic manipulator is considered subject to a signal dependent noise (SDN) model corrupting the training signal. A radial basis function (RBF) network is utilized in the feedforward control to approximate the unknown inverse dynamics. The weights are adaptively adjusted according to a gradient descent plus a regulation term (Narendra's e-modification). A typical quadratic stochastic Lyapunov function is constructed which shows under certain noise models it is not necessary to employ quartic Lyapunov functions as is typically carried out in stochastic adaptive backstepping designs. Bounds on the feedback gains, and learning rate parameters are derived that guarantee the origin of the closed loop system is semi-globally, uniformly bounded in expectation (SGUBE).

1. INTRODUCTION

In this paper, we consider the neural network based control of a robotic manipulator subject to signal dependent noise (SDN) corrupting the training signal. Neural network based control of robotic manipulators has been considered in the case of deterministic bounded disturbances in the plant (Lewis et al. [1999]). The case of stochastic disturbances in the plant was considered in Psillakis [2002]. There, they employed an adaptive backstepping design utilizing a quartic Lyapunov function to prove that the mean square error is semi-global uniformly ultimately bounded. In this note, we consider the signal dependent noise model, where, roughly speaking, the variance of the noise is proportional to the mean of the signal being corrupted. This noise model has been considered in digital picture processing (Hirakawa and Parks [2005]) and in neuro-motor control (Harris and Wolpert [1998]).

2. BACKGROUND

In the following, we present the relevant material on the plant dynamics, neural network approximation properties, the SDN model, and stochastic differential equations (SDE's).

Plant Dynamics and Feedforward Control: We take the plant to be a serial link manipulator. The dynamics are given as follows: *Plant Dynamics*

$$M(q)\ddot{q} + V(q,\dot{q}) + F(\dot{q}) + G(q) + \tau_d = \tau \qquad (1)$$

where $M(q) \in R^{n \times n}$ is the inertia matrix, $V(q,\dot{q}) \in R^{nx1}$ is the corriolis/centripetal matrix, $G(q) \in R^{nx1}$ is the gravity matrix, and $\tau_d \in R^{nx1}$ represents unknown disturbances. $q \in R^{nx1}$ is the joint angle state vector, and $\tau \in R^{nx1}$ is the net torque applied at each joint. We have the following standard assumptions Lewis et al. [1999].

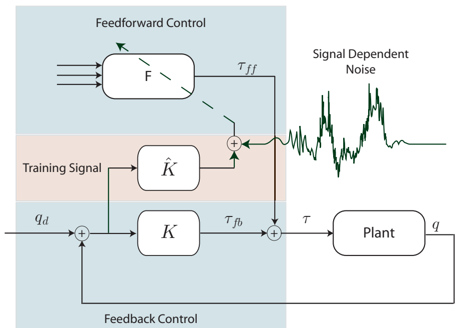

Fig. 1. General Neural Network based Feedforward Control with SDN

Assumption 1. M(q) is symmetric, positive definite and bounded.

Assumption 2. The corrolis/centripetal terms can be written as $V_m(q,\dot{q})\dot{q}$ such that $\dot{M} - 2V_m(q,\dot{q})$ is skew-symmetric.

Assumption 3. The gravity and disturbance terms are bounded.

We will neglect friction and disturbance terms in the analysis. The desired trajectory is denoted by $\boldsymbol{q_d} = \left[q_d^T \, \dot{q}_d^T \, \ddot{q}_d^T\right]^T$. The control input to the plant is denoted by $\tau = \tau_{fb} + \tau_{ff}$. The feedback control is given by $\tau_{fb} = K_p\tilde{q} + K_v\dot{\tilde{q}}$ where $\tilde{q} := q_d - q$, and K_s for $s = p, v$ denotes a positive definite symmetric feedback gain matrix. We will use the following notation for the lower and upper bounds of key terms:

$$\begin{aligned}
\kappa_M &\stackrel{\mathsf{d}}{=} \inf_{q\in\mathbb{R}^n} \sigma_{min}(M(q)) & \omega_M &\stackrel{\mathsf{d}}{=} \sup_{q\in\mathbb{R}^n} \|M(q)\| \\
\kappa_v &\stackrel{\mathsf{d}}{=} \sigma_{min}(K_v) & \omega_v &\stackrel{\mathsf{d}}{=} \|K_v\| \\
\kappa_p &\stackrel{\mathsf{d}}{=} \sigma_{min}(K_p) & \omega_p &\stackrel{\mathsf{d}}{=} \|K_p\| \\
\hat\kappa_v &\stackrel{\mathsf{d}}{=} \sigma_{min}(\hat K_v) & \hat\omega_v &\stackrel{\mathsf{d}}{=} \|\hat K_v\| \\
\hat\kappa_p &\stackrel{\mathsf{d}}{=} \sigma_{min}(\hat K_p) & \hat\omega_p &\stackrel{\mathsf{d}}{=} \|\hat K_p\|
\end{aligned} \tag{2}$$

where σ_{min} denotes the minimum singular value. The matrices, $\hat K_p$ and $\hat K_v$ are defined below in (4).

Neural Network Approximation:

The feedforward command, τ_{ff}, is the output of a radial basis function (RBF) network. In this control scheme, the inputs consist only of desired trajectory states, $\boldsymbol{q_d}$. Let $\boldsymbol{q_d} \in \mathcal{K}$ where $\mathcal{K}$ is compact. Let the desired feedforward command, $\tau^*_{ff} \in \mathrm{C}(\mathcal{K}, \mathbb{R}^n)$. It can be shown (see Liao et al. [2003]) that for any $\epsilon_B > 0$, there exists a number $N \in Z^+$, a matrix of weights, $W^* \in \mathbb{R}^{N\times n}$, and a basis function vector, $\phi \in \mathrm{C}^\infty(\mathcal{K}, \mathbb{R}^N)$, such that $\epsilon(\boldsymbol{q}_d) := \tau^*_{ff}(\boldsymbol{q}_d) - W^{*T}\phi(\boldsymbol{q}_d)$ is bounded uniformly from above by ϵ_B for all $\boldsymbol{q}_d \in \mathcal{K}$. The feedforward command that is implemented is given by

$$\begin{aligned}
\tau_{ff} = \hat W^T \phi(q_d) &= \left[W^* - \tilde W\right]\phi(q_d) \\
&= \tau^*_{ff} - \epsilon(q_d) - \tilde W^T\phi(q_d)
\end{aligned} \tag{3}$$

where $\hat W \in \mathbb{R}^{N\times n}$ is a matrix of estimated parameters, and $\tilde W \in \mathbb{R}^{N\times n}$ denotes the matrix of parameter errors defined by $\tilde W \stackrel{\mathsf{d}}{=} W^* - W$.

The Training Signal and Signal Dependent Noise: The training signal of the network is given by

$$\hat\tau_{fb} \stackrel{\mathsf{d}}{=} \hat K_p \tilde q + \hat K_v \dot{\tilde q} \in \mathbb{R}^{n\times 1} \tag{4}$$

where $\hat K_p$, $\hat K_v \in \mathbb{R}^{n\times n}$ are positive definite, symmetric matrices. We assume that each component of the training signal, $\hat\tau_{fb_i}$ is corrupted by SDN. In other words

$$\hat\tau_{fb_i} \to \hat\tau_{fb_i}(1 + \sigma_i \dot B_i) \quad i \in [1, n]$$

where σ_i represents the noise intensity of the ith component of the Brownian motion process. Brownian motion will be considered in detail in the next section. In the main theorem presented below, we will use as the learning rule, a gradient descent plus e-modification proposed by Narendra and Annaswamy [1987]. This is given by

$$\dot{\hat W} = -\gamma\phi(q_d)\hat\tau_{fb} + h\gamma\|y\|^p \hat W$$

where $\|y\| = \sqrt{\|\tilde q\|^2 + \|\dot{\tilde q}\|^2}$, $h > 0$, and $\gamma > 0$. When signal dependent noise on the training signal is considered the above equation becomes

$$\dot{\hat W} = -\gamma\phi(q_d)\hat\tau_{fb} + h\gamma\|y\|^p \hat W - \gamma\phi(q_d)(\Lambda\dot B)^T \tag{5}$$

where

$$\Lambda = \mathrm{diag}\{\sigma_i \hat\tau_{fb_i}\} \in \mathbb{R}^{n\times n}$$

and $B = [B_1 B_2 \dots B_m]^T$ denotes an m-dimensional Brownian motion. In the above equation, we use the term, $\dot B$ to denote the effect of noise on the learning dynamics. This term, however, does not mean the time derivative of B, as Brownian motion is not differentiable on $[0, \infty)$ almost surely.

Relevant Stochastic Calculus and SGUBE

Here we define the relevant stochastic terminology, and the term semi-global, uniform boundedness in expectation (SGUBE). In the previous subsection, we briefly referred to a Brownian motion process which corrupted the components of the training signal. In this section, we make this notion precise. We also state Ito's Lemma which is critical to the stability analysis of SDE's. We refer the reader to Mao [1997] for further details. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space. Let B_t be an m-dimensional Brownian motion vector adapted to the filtration, $\{\mathcal{F}_t\}$. An $\bar n$-dimensional Ito process, $x(t, \omega)$, is a continuous, $\{\mathcal{F}_t\}$-adapted stochastic process that is the solution of the stochastic differential equation denoted by

$$dx = F(x(t,\omega), t)dt + H(x(t,\omega), t)dB \tag{6}$$

or, by 'dividing' by dt,

$$\dot x(t,\omega) = F(x(t,\omega), t) + H(x(t,\omega), t)\dot B \tag{7}$$

The above two equations have no mathematical meaning. They are simply short hand notation to mean that $x(t, \omega)$ satisfies the equality

$$x(t,\omega) = x(0,\omega) + \int_0^t F(x(s,\omega), s)ds + \int_0^t H(x(s,\omega), s)dB$$

for all $t \in [0, T]$. For example, the stochastic learning dynamics in (5) are written in the shorthand notation as in (7). In the following, we will suppress the dependence of x on $\omega \in \Omega$ and denote $x(t, \omega)$ as simply $x(t)$. It is to be understood that for each fixed t, x is a random variable. Let $\mathcal{L}^p(X; Y)$ denote the collection of stochastic processes $f : X \times \Omega \to Y$ such that

$$\int_0^T \|f(s)\|_Y^p ds < \infty \quad a.s \ \ \forall T > 0$$

We now state the Ito formula, which is the stochastic version of the chain rule from ordinary calculus.

Theorem 1. (Ito's Lemma-*adapted from Mao [1997]*). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space, and $x(t, \omega)$ be a $\bar n$-dimensional Ito process satisfying

$$x(t,\omega) - x(0,\omega) = \int_0^t F(x,s)ds + \int_0^t H(x,s)dB \tag{8}$$

where $F \in \mathcal{L}^1(\mathbb{R}^{\bar n} \times \mathbb{R}^+ \times \Omega; \mathbb{R}^{\bar n})$, $H \in \mathcal{L}^2(\mathbb{R}^{\bar n} \times \mathbb{R}^+ \times \Omega; \mathbb{R}^{\bar n\times m})$, and $B : \mathbb{R}^+ \times \Omega \to \mathbb{R}^m$ is an m dimensional Brownian motion. Let $V \in \mathrm{C}^{2,1}(\mathbb{R}^{\bar n} \times \mathbb{R}^+; \mathbb{R})$. Then,

$$V(x(t), t) - V(x(0), 0) = \int_0^t \bar F ds + \int_0^t \bar H dB \quad \text{a.s}$$

where $\bar F = V_t + V_x F + \frac{1}{2}\mathrm{Tr}\left(H^T V_{xx} H\right)$ and $\bar H = V_x H$. The notation, V_x denotes the gradient of V, and V_{xx} denotes the Hessian of V.

Note the additional term given by

$$\frac{1}{2}\mathrm{Tr}\left(H^T V_{xx} H\right) \tag{9}$$

This is the Ito *anomaly* and is the reason why stability analysis of stochastic systems is very different than its deterministic counterpart. The Ito *anomaly* is intimately related to the fact that 1) Brownian motion is almost surely nowhere differentiable with respect to time, and 2)

Brownian motion has infinite first order variation. We do not go into further details here.

Suppose that the dynamics can be written as

$$\dot{x} = F(x,t) + H(x,t)\dot{B} \tag{10}$$

where

$$x = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(d)} \end{bmatrix} \qquad F(x,t) = \begin{bmatrix} F_1(x,t) \\ \vdots \\ F_d(x,t) \end{bmatrix}$$

$$H(x,t) = \begin{bmatrix} H_1(x,t) \\ \vdots \\ H_d(x,t) \end{bmatrix} \qquad B(t) = \begin{bmatrix} B_1(t) \\ \vdots \\ B_m(t) \end{bmatrix}$$

and the dimensions of the components are $x^{(i)} \in \mathbb{R}^{n_i}$, $F_i \in \mathbb{R}^{n_i}$, $H_i \in \mathbb{R}^{n_i \times m}$ and $\sum n_i = \bar{n}$. Then, it can be shown that the Ito Anomaly can be written as

$$\begin{aligned} \mathrm{Tr}\left(V_{xx}HH^T\right) &= \mathrm{Tr}\left(\sum_{j=1}^{d}\sum_{i=1}^{d} V_{x^{(j)}x^{(i)}} H_i H_j^T\right) \\ &= \sum_{i,j=1}^{d} \mathrm{Tr}\left(V_{x^{(j)}x^{(i)}} H_i H_j^T\right) \end{aligned} \tag{11}$$

where

$$V_{x^{(j)}x^{(i)}} := \begin{bmatrix} \frac{\partial}{\partial x_1^{(j)}}\frac{\partial V}{\partial x_1^{(i)}} & \cdots & \frac{\partial}{\partial x_1^{(j)}}\frac{\partial V}{\partial x_{n_i}^{(i)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_{n_j}^{(j)}}\frac{\partial V}{\partial x_1^{(i)}} & \cdots & \frac{\partial}{\partial x_{n_j}^{(j)}}\frac{\partial V}{\partial x_{n_i}^{(i)}} \end{bmatrix} \in \mathbb{R}^{n_j \times n_i}$$

Theorem 2. (GUBE). Let F and H in (10) satisfy the local Lipschitz and linear growth condition described in Mao [1997] (page 51). Let $V \in \mathrm{C}^{2,1}(\mathbb{R}^{\bar{n}} \times \mathbb{R}^+; \mathbb{R})$ be positive definite and decrescent. That is, there exists class K functions, μ_1 and μ_2 such that $\mu_1(\|x\|) \leq V(x,t) \leq \mu_2(\|x\|)$ for all $(x,t) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^+$. Denote the operator L acting on V by

$$LV := V_t + V_x F + \frac{1}{2}\mathrm{Tr}\left(H^T V_{xx} H\right) \tag{12}$$

Then, if for some $R_0 > 0$, we have $LV \leq 0$ for all $(x,t) : \|x\| \geq R_0$ and $t \geq t_0$, the expected value,

$$E(\|x\|) \leq \mu_1^{-1}(\mu_2(R_0))$$

Proof: Due to page limitations, we only provide a sketch of the proof as it follows very closely to the proof of the deterministic version first proposed by Yoshizawa [1960]. Taking the expected value of LV results in an inequality that is similar to the deterministic case, only that we have an expectation on both the right and left hand sides. Appealing to the convexity of class K functions and Jensens inequality, we get a bound on the expected value, $E\|x\|$. The main differences in the proof involve conditioning the expectation on the values of a set of stopping times (see Mao [1997] for a definition of stopping times) $\tau_1, \tau_2, \ldots$. These are random numbers defined to be the ordered time values (between the initial time t_0 and the current time t) at which the trajectory $x(t)$ crosses the sphere $\|x\| = R_0$; there may be no such times, or a finite number, or an infinity of them. Since the norm, $\|x\|$ is bounded inside the sphere (by definition), and since LV is negative semi-definite outside of it, we are able to prove that the conditional expectation of $\|x(t)\|$ is bounded, and that this bound is uniform in t and in the values of the stopping times. The condition is then removed, leading to a proof that $E\|x(t)\|$ is bounded uniformly in time, with bound described above. □

Closed Loop Dynamics: In this subsection, we determine the closed loop dynamics of the error system. Let $x^{(1)} = \tilde{q}$ and $x^{(2)} = \dot{\tilde{q}}$. Vectorizing the weight matrix, we let $x^{(3)} = [\tilde{W}_{[1,:]}\ \tilde{W}_{[2,:]} \ldots \tilde{W}_{[N,:]}]^T$ where $\tilde{W}_{[i,:]}$ denotes the ith row of matrix $\tilde{W}$. With this, the closed loop system is given by

$$\begin{aligned} \dot{x}^{(1)} &= F_1(x,t) + H_1(x,t)\dot{B} \\ \dot{x}^{(2)} &= F_2(x,t) + H_2(x,t)\dot{B} \\ \dot{x}^{(3)} &= F_3(x,t) + H_3(x,t)\dot{B} \end{aligned} \tag{13}$$

where

$$F_1(x,t) = x^{(2)}$$

$$F_2(x,t) = \ddot{q}_d - M^{-1}(x^{(1)})\left(\tau - V_m x^{(2)} - G(x^{(1)})\right)$$

$$F_3(x,t) = -\gamma \begin{bmatrix} \phi_1 \tau_{fb} \\ \phi_2 \tau_{fb} \\ \vdots \\ \phi_N \tau_{fb} \end{bmatrix} + h\gamma\|y\|^p \begin{bmatrix} (W^*_{[1,:]} - \tilde{W}_{[1,:]})^T \\ (W^*_{[2,:]} - \tilde{W}_{[2,:]})^T \\ \vdots \\ (W^*_{[N,:]} - \tilde{W}_{[N,:]})^T \end{bmatrix}$$

$$H_3(x,t) = -\gamma \begin{bmatrix} \phi_1 \Lambda \\ \phi_2 \Lambda \\ \vdots \\ \phi_N \Lambda \end{bmatrix}$$

and $H_1 = H_2 = 0$.

3. MAIN RESULTS

We require an estimate of the maximum component of the noise intensity vector: σ, as well as the norm of the RBF network. We define the constants σ_B and ϕ_B such that

$$\max_{i \in [1,n]} \{|\sigma_i|\} \leq \sigma_B \tag{14}$$

and

$$\phi_B := \sup_{q_d \in \mathcal{K}} \|\phi(q_d)\| \tag{15}$$

Since the basis functions and the compact set $\mathcal{K}$ are known in advance, ϕ_B is computed exactly, whereas the constant σ_B is an estimate of the upper bound of the infinity norm of the noise intensity vector. For notational convenience, we define the constant

$$b_0 := \sigma_B^2 \phi_B^2 \tag{16}$$

Theorem 3. (Main Stability Theorem: SGUBE). Consider the closed loop system depicted in Fig. (1) and described by the SDE in (13). Let c_γ and $\hat{\omega}_v$ be arbitrary positive real numbers. Then, if $\kappa_p \geq \bar{K}_{p_{lb}}(c_\gamma, b_0)$, $\hat{\omega}_p = \hat{\omega}_p^*$, $\kappa_v \geq \bar{K}_{v_{lb}}(\kappa_p, \hat{\omega}_p^*, c_\gamma, b_0)$, and $\gamma \leq \gamma^* = \frac{c_\gamma}{\max\{\hat{\omega}_p^*, \hat{\omega}_v\}}$, the closed loop system in semi-globally, uniformly bounded in expectation. The terms, $\bar{K}_{p_{lb}}$, $\bar{K}_{v_{lb}}$, and $\hat{\omega}_p^* = \hat{\beta}_p \hat{\kappa}_p^*$ are defined (35), (34) and (36), respectively.

Proof: Consider the following Lyapunov function candidate:

$$V(\tilde{q}, \dot{\tilde{q}}, \tilde{W}) = V_0 + \frac{1}{2}\mathrm{Tr}\left[\tilde{W}^T\Gamma^{-1}\tilde{W}\right]$$

where $V_0 = \frac{1}{2}\dot{\tilde{q}}^T\hat{K}_v M(q)\dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T\hat{K}_v K_p\tilde{q} + \frac{1}{2}\tilde{q}^T\hat{K}_p K_v\tilde{q} + \tilde{q}^T\hat{K}_p M(q)\dot{\tilde{q}}$. We will take $\gamma I = \Gamma$. We first show that V_0 is positive definite and decrescent. We lower bound V_0 as

$$\begin{aligned} V_0\left(\tilde{q}(t), \dot{\tilde{q}}(t)\right) &\geq \frac{1}{2}\hat{\kappa}_v\kappa_M\|\dot{\tilde{q}}\|^2 + \frac{1}{2}\hat{\kappa}_v\kappa_p\|\tilde{q}\|^2 + \frac{1}{2}\hat{\kappa}_p\kappa_v\|\tilde{q}\|^2 \\ &\quad -\hat{\omega}_p\omega_M\|\dot{\tilde{q}}\|\|\tilde{q}\| \\ &\geq \frac{1}{2}\hat{\kappa}_v\kappa_M\|\dot{\tilde{q}}\|^2 + \frac{1}{2}\hat{\kappa}_v\kappa_p\|\tilde{q}\|^2 + \frac{1}{2}\hat{\kappa}_p\kappa_v\|\tilde{q}\|^2 \\ &\quad -\frac{1}{2}\hat{\omega}_p\omega_M\nu_1^2\|\tilde{q}\|^2 - \frac{1}{2}\hat{\omega}_p\omega_M\frac{\|\dot{\tilde{q}}\|^2}{\nu_1^2} \end{aligned}$$

where the last term follows from completion of squares and holds for any non-zero ν_1. It follows that

$$V_0 \geq c_1\|\tilde{q}\|^2 + c_2\|\dot{\tilde{q}}\|^2 \tag{17}$$

where

$$\begin{aligned} c_1 &= \frac{1}{2}(\hat{\kappa}_v\kappa_p + \hat{\kappa}_p\kappa_v - \hat{\omega}_p\omega_M\nu_1^2) \text{ and} \\ c_2 &= \frac{1}{2}(\hat{\kappa}_v\kappa_M - \hat{\omega}_p\omega_M\nu_1^{-2}) \end{aligned} \tag{18}$$

To guarantee that c_1 and c_2 are positive, we find that

$$0 < \hat{\kappa}_p < \frac{\kappa_v\kappa_M\hat{\kappa}_v + \sqrt{(\hat{\kappa}_v\kappa_v\kappa_M)^2 + 4\hat{\beta}_p\omega_M^2\hat{\kappa}_v^2\kappa_M\kappa_p}}{2\hat{\beta}_p^2\omega_M^2} \tag{19}$$

Note that this bound is independent of the feedback gains, K_p and K_v, as well as the training matrix, $\hat{K}_v$. Hence, we have converted the condition on the positive defin

$$V_0 \leq \frac{1}{2}\hat{\omega}_v\omega_M\|\dot{\tilde{q}}\|^2 + \frac{1}{2}\hat{\omega}_v\omega_p\|\tilde{q}\|^2 + \frac{1}{2}\hat{\omega}_p\omega_v\|\tilde{q}\|^2 + \hat{\omega}_p\omega_M\|\tilde{q}\|\|\dot{\tilde{q}}\|$$

which yields

$$V_0 \leq c_3\|\tilde{q}\|^2 + c_4\|\dot{\tilde{q}}\|^2 \tag{20}$$

where

$$\begin{aligned} c_3 &= \frac{\hat{\omega}_v\omega_p + \hat{\omega}_p\omega_v + \hat{\omega}_p\omega_M\nu_2^2}{2} \text{ and} \\ c_4 &= \frac{\hat{\omega}_v\omega_M + \hat{\omega}_p\omega_M\nu_2^{-2}}{2} \end{aligned} \tag{21}$$

where ν_2 is any non-zero real number. Note that in this case, we do not require any additional conditions on ν_2 since c_3 and c_4 are always positive. Combining (17) and (20) we have

$$\min(c_1, c_2, \frac{\gamma^{-1}}{2})\|x\|^2 \leq V \leq \max(c_3, c_4, \frac{\gamma^{-1}}{2})\|x\|^2 \tag{22}$$

where $x = [x^{(1)^T} \ x^{(2)^T} \ x^{(3)^T}]^T$. Hence, V is positive definite and decrescent.

We now consider the first term on the right hand side of the LV equation (12) given by

$$\left[\frac{\partial V}{\partial x}\right]^T F = \sum_{i=1}^{d}\left[\frac{\partial V}{\partial x^{(i)}}\right]^T F_i(x)$$

It can be shown that

$$\begin{aligned} &\left[\frac{\partial V}{\partial x}\right]^T F \\ &= (\hat{K}_v\dot{\tilde{q}} + \hat{K}_p\tilde{q})^T\left(M(q)\ddot{q}_d + V_m\dot{q}_d + G - \tau_{ff}\right) \\ &+\frac{1}{2}\dot{\tilde{q}}^T\hat{K}_v\left(\dot{M} - 2V_m\right)\dot{\tilde{q}} - \dot{\tilde{q}}^T\hat{K}_v K_v\dot{\tilde{q}} - \tilde{q}^T\hat{K}_p K_p\tilde{q} \\ &+\dot{\tilde{q}}^T\hat{K}_p M\dot{\tilde{q}} + \tilde{q}^T\hat{K}_p V_m^T\dot{\tilde{q}} \\ &+\mathrm{Tr}\left[\tilde{W}^T\gamma^{-1}\left(-\gamma\phi(q_d)\hat{\tau}_{fb} + h\gamma\|y\|^p\hat{W}\right)\right] \end{aligned}$$

□

where we have used the fact that $K_p\hat{K}_v = \hat{K}_v K_p$ and that $\dot{M} = V_m + V_m^T$. Substituting the feedforward control given in (3), we get

$$\begin{aligned} \left[\frac{\partial V}{\partial x}\right]^T F &= \overbrace{(\hat{K}_v\dot{\tilde{q}} + \hat{K}_p\tilde{q})^T\left(M(q)\ddot{q}_d + V_m\dot{q}_d + G - \tau_{ff}^*\right)}^{①} \\ &+ \overbrace{(\hat{K}_v\dot{\tilde{q}} + \hat{K}_p\tilde{q})^T\epsilon(q_d)}^{②} + \overbrace{\frac{1}{2}\dot{\tilde{q}}^T\hat{K}_v(V_m^T - V_m)\dot{\tilde{q}}}^{③} \\ &+ \underbrace{\tilde{q}^T\hat{K}_p V_m^T\dot{\tilde{q}}}_{④}\underbrace{-\tilde{q}^T\hat{K}_p K_p\tilde{q} - \dot{\tilde{q}}^T\hat{K}_v K_v\dot{\tilde{q}} + \dot{\tilde{q}}^T\hat{K}_p M\dot{\tilde{q}}}_{⑤} \\ &+ \underbrace{h\gamma\|y\|^p\mathrm{Tr}\left\{\tilde{W}^T\hat{W}\right\}}_{⑥} \end{aligned}$$

Term ① is bounded by:

$$\begin{aligned} &\|\hat{\tau}_{fb}\|\|M(q)\ddot{q} + V_m\dot{q}_d + G - \tau_{ff}^*\| \\ &= \|\hat{\tau}_{fb}\|\|(M - M_d)\ddot{q}_d + (V_m - V_{m_d})\dot{q}_d + (G - G_d)\| \\ &\leq \|\hat{\tau}_{fb}\|\left(\|M - M_d\|\|\ddot{q}_d\| + \|V_m - V_{m_d}\|\|\dot{q}_d\| + \|G - G_d\|\right) \\ &\leq \hat{\omega}_p\left(\alpha_2\|\tilde{q}\|^2 + \alpha_3\|\tilde{q}\|\|\dot{\tilde{q}}\|\right) + \hat{\omega}_v\left(\alpha_2\|\tilde{q}\|\|\dot{\tilde{q}}\| + \alpha_3\|\dot{\tilde{q}}\|^2\right) \end{aligned}$$

Term ② is bounded by:

$$\left|(\hat{K}_v\dot{\tilde{q}} + \hat{K}_p\tilde{q})^T\epsilon(q_d)\right| \leq \hat{\omega}_p\epsilon_B\|\tilde{q}\| + \hat{\omega}_v\epsilon_B\|\dot{\tilde{q}}\| \tag{23}$$

Term ③ is bounded by:

$$\left|\frac{1}{2}\dot{\tilde{q}}^T\hat{K}_v(V_m^T - V_m)\dot{\tilde{q}}\right| \leq \hat{\omega}_v\left(\eta_1\|\dot{\tilde{q}}\|^2 + \eta_2\|\dot{\tilde{q}}\|^3\right) \tag{24}$$

where $\eta_1 = C_v C_{q_d}$, and $\eta_2 = C_v$. Note, that if $\hat{K}_v = \hat{\kappa}_v I$, then the matrix $\hat{K}_v(V_m^T - V_m)$ is skew-symmetric, and hence we take $\eta_1 = \eta_2 = 0$.

Term ④ is bounded by:

$$\begin{aligned} \left|\tilde{q}^T\hat{K}_p V_m^T\dot{\tilde{q}}\right| &\leq \hat{\omega}_p\|\tilde{q}\|\|\dot{\tilde{q}}\|\left(C_v\|\dot{\tilde{q}}\| + C_v\|\dot{q}_d\|\right) \\ &= \hat{\omega}_p\left(C_v\|\tilde{q}\|\|\dot{\tilde{q}}\|^2 + C_v\|\dot{q}_d\|\|\tilde{q}\|\|\dot{\tilde{q}}\|\right) \\ &= \hat{\omega}_p\left(\xi_2\|\tilde{q}\|\|\dot{\tilde{q}}\|^2 + \xi_1\|\tilde{q}\|\|\dot{\tilde{q}}\|\right) \end{aligned} \tag{25}$$

where $\xi_1 = C_v\|\dot{q}_d\|$, and $\xi_2 = C_v$. The bound used on V_m is derived in the appendix.

Term ⑤ is bounded by:

$$\begin{aligned} &-\tilde{q}^T\hat{K}_p K_p\tilde{q} - \dot{\tilde{q}}^T\hat{K}_v K_v\dot{\tilde{q}} + \dot{\tilde{q}}^T\hat{K}_p M\dot{\tilde{q}} \\ &\leq -\hat{\kappa}_p\kappa_p\|\tilde{q}\|^2 - \hat{\kappa}_v\kappa_v\|\dot{\tilde{q}}\|^2 + \hat{\omega}_p\omega_M\|\dot{\tilde{q}}\|^2 \end{aligned}$$

Term ⑥ is bounded by:

$$h\|y\|^p\left(\left\|\tilde{W}\right\|_{\mathrm{F}}\|W^*\|_{\mathrm{F}} - \left\|\tilde{W}\right\|_{\mathrm{F}}^2\right)$$

Combining the above bounds, we have

$$\left[\frac{\partial V}{\partial x}\right]^T F \leq -\|\tilde{q}\|^2 \left(\hat{\kappa}_p\kappa_p - \hat{\omega}_p\alpha_2\right) \\ -\|\dot{\tilde{q}}\|^2 \left(\hat{\kappa}_v\kappa_v - \hat{\omega}_p\omega_M - \hat{\omega}_v\alpha_3 - \hat{\omega}_v\eta_1\right) \\ +\|\tilde{q}\|\|\dot{\tilde{q}}\| \left(\hat{\omega}_p\alpha_3 + \hat{\omega}_v\alpha_2 - \hat{\omega}_p\xi_1\right) \\ +h\|y\|^p \left(\left\|\tilde{W}\right\|_{\mathrm{F}} \|W^*\|_{\mathrm{F}} - \left\|\tilde{W}\right\|_{\mathrm{F}}^2\right) \\ +\mathcal{W}(\|\tilde{q}\|, \|\dot{\tilde{q}}\|, \hat{K}_p, \hat{K}_v) \tag{26}$$

where

$$\mathcal{W} = \hat{\omega}_p \left(\|\tilde{q}\|\|\epsilon_B + \|\dot{\tilde{q}}\|^2\|\tilde{q}\|\xi_2\right) + \hat{\omega}_v \left(\|\tilde{q}\|\|\epsilon_B + \|\dot{\tilde{q}}\|^3\eta_2\right) \tag{27}$$

Ito Term: We now compute (11) along the trajectories of the closed loop system given in (13). It can be shown that

$$\frac{1}{2}\mathrm{Tr}\left(V_{xx}GG^T\right) \leq \frac{\gamma b_0}{2}\hat{\omega}_v^2\|\dot{\tilde{q}}\|^2 + \gamma b_0\hat{\omega}_v\hat{\omega}_p\|\tilde{q}\|\|\dot{\tilde{q}}\| \\ +\frac{\gamma b_0}{2}\hat{\omega}_p^2\|\tilde{q}\|^2 \tag{28}$$

Using (26) and (28) we have the bound

$$LV \leq -\begin{bmatrix}\|\tilde{q}\| \\ \|\dot{\tilde{q}}\|\end{bmatrix}^T A \begin{bmatrix}\|\tilde{q}\| \\ \|\dot{\tilde{q}}\|\end{bmatrix} + \mathcal{W}(\|\tilde{q}\|, \|\dot{\tilde{q}}\|, K_p, K_v) \\ +h\|y\|^p \left(\left\|\tilde{W}\right\|_{\mathrm{F}} \|W^*\|_{\mathrm{F}} - \left\|\tilde{W}\right\|_{\mathrm{F}}^2\right) \tag{29}$$

where

$$A = \begin{bmatrix}\kappa_p\hat{\kappa}_p - f_0 & -f_1 \\ -f_1 & \kappa_v\hat{\kappa}_v - f_2\end{bmatrix}$$

$f_0 = \hat{\omega}_p\alpha_2 - \frac{\gamma b_0}{2}\hat{\omega}_p^2$, $f_1 = \frac{\hat{\omega}_p(\alpha_3+\xi_1)+\hat{\omega}_v\alpha_2+\gamma b_0\hat{\omega}_v\hat{\omega}_p}{2}$ and $f_2 = \hat{\omega}_p\omega_M - \hat{\omega}_v(\alpha_3+\eta_1) - \frac{\gamma b_0}{2}\hat{\omega}_v^2$. The matrix A is positive definite if and only if we require $b^2 < ad$ and $d + a > 0$ where $a = \kappa_p\hat{\kappa}_p - f_0$, $b = -f_1$ and $d = \kappa_v\hat{\kappa}_v - f_2$. The first condition ($b^2 < ad$) yields the inequality

$$\frac{\left(\hat{\omega}_p(\alpha_3 + \xi_1) + \hat{\omega}_v\alpha_2 + \gamma b_0\hat{\omega}_v\hat{\omega}_p\right)^2}{4} \\ < \left(\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{\gamma b_0}{2}\hat{\omega}_p^2\right) \cdot \\ \left(\kappa_v\hat{\kappa}_v - \hat{\omega}_p\omega_M - \hat{\omega}_v(\alpha_3 + \eta_1) - \frac{\gamma b_0}{2}\hat{\omega}_v^2\right) \tag{30}$$

The second condition ($d + a > 0$) yields

$$\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{\gamma b_0}{2}\hat{\omega}_p^2 \\ +\kappa_v\hat{\kappa}_v - \hat{\omega}_p\omega_M - \hat{\omega}_v(\alpha_3 + \eta_1) - \frac{\gamma b_0}{2}\hat{\omega}_v^2 > 0 \tag{31}$$

Suppose that we select gains such that (30) is satisfied. Then, it follows that the left hand side of (31) is bounded from below by

$$\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{\gamma b_0}{2}\hat{\omega}_p^2 \\ +\kappa_v\hat{\kappa}_v - \hat{\omega}_p\omega_M - \hat{\omega}_v(\alpha_3 + \eta_1) - \frac{\gamma b_0}{2}\hat{\omega}_v^2 \\ > \kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{\gamma b_0}{2}\hat{\omega}_p^2 \\ +\frac{\left(\hat{\omega}_p(\alpha_3 + \xi_1) + \hat{\omega}_v\alpha_2 + \gamma b_0\hat{\omega}_v\hat{\omega}_p\right)^2}{4\left(\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{\gamma b_0}{2}\hat{\omega}_p^2\right)}$$

The right hand side is positive if we require that

$$\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{\gamma b_0}{2}\hat{\omega}_p^2$$

or, equivalently

$$\kappa_p > \hat{\kappa}_p\left(\hat{\omega}_p\alpha_2 + \frac{\gamma b_0}{2}\hat{\omega}_p^2\right) =: K_{p_{lb}} \tag{32}$$

For (30) to be satisfied we find

$$\kappa_v > \frac{1}{\hat{\kappa}_v}\left[\frac{\left(\hat{\omega}_p(\alpha_3 + \xi_1) + \hat{\omega}_v\alpha_2 + \gamma b_0\hat{\omega}_v\hat{\omega}_p\right)^2}{4\left(\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{\gamma b_0}{2}\hat{\omega}_p^2\right)} + \hat{\omega}_p\omega_M \right. \\ \left. +\hat{\omega}_v(\alpha_3 + \eta_1) + \frac{\gamma b_0}{2}\hat{\omega}_v^2\right] =: K_{v_{lb}} \tag{33}$$

For any c_γ, choose $\gamma \leq \gamma^* = \frac{c_\gamma}{\max\{\hat{\omega}_p, \hat{\omega}_v\}}$. Then, it follows that

$$\gamma\hat{\omega}_v\hat{\omega}_p \leq c_\gamma\hat{\omega}_p \qquad \gamma\hat{\omega}_p^2 \leq c_\gamma\hat{\omega}_p \qquad \gamma\hat{\omega}_v^2 \leq c_\gamma\hat{\omega}_v$$

Using the above inequalities, it follows that

$$K_{v_{lb}} \leq \frac{1}{\hat{\kappa}_v}\left[\frac{\left(\hat{\omega}_p(\alpha_3 + \xi_1) + \hat{\omega}_v\alpha_2 + c_\gamma b_0\hat{\omega}_p\right)^2}{4\left(\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{c_\gamma b_0\hat{\omega}_p}{2}\right)} + \hat{\omega}_p\omega_M \right. \\ \left. +\hat{\omega}_v(\alpha_3 + \eta_1) + \frac{c_\gamma b_0\hat{\omega}_v}{2}\hat{\omega}_v^2\right] =: \bar{K}_{v_{lb}} \tag{34}$$

Choosing $\kappa_v > \bar{K}_{v_{lb}}$ implies $\kappa_v > K_{v_{lb}}$. Note also also that $\gamma\hat{\omega}_p \leq c_\gamma$. Using this in (32), we have

$$K_{p_{lb}} \leq \bar{K}_{p_{lb}} := \hat{\beta}_p\left(\alpha_2 + \frac{b_0 c_\gamma}{2}\right) \tag{35}$$

Choosing $\kappa_p > \bar{K}_{p_{lb}}$ implies $\kappa_p > K_{p_{lb}}$. Observe that the lower bound $\bar{K}_{p_{lb}}$ is independent of $\hat{\omega}_p$ and $\hat{\omega}_v$. The term, $\hat{\beta}_p$ is regarded as a fixed quantity. On the other hand, $\bar{K}_{v_{lb}}$ is a function of $\hat{\kappa}_p$, and $\hat{\kappa}_v$.

Optimizing $\bar{K}_{v_{lb}}$: If we regard $\hat{\kappa}_v$ as fixed, we can optimize $\bar{K}_{v_{lb}}$ with respect to $\hat{\kappa}_p$. Setting $\frac{\partial\bar{K}_{v_{lb}}}{\partial\hat{\kappa}_p} = 0$ we get

$$8\left(\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{c_\gamma b_0\hat{\omega}_p}{2}\right)\left(\hat{\omega}_p g_1 + \hat{\omega}_v\alpha_2\right)\hat{\beta}_p g_1 \\ -4\left(\hat{\omega}_p g_1 + \hat{\omega}_v\alpha_2\right)^2\left(\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{c_\gamma b_0\hat{\omega}_p}{2}\right) \\ +16\hat{\beta}_p\omega_M\left(\kappa_p\hat{\kappa}_p - \hat{\omega}_p\alpha_2 - \frac{c_\gamma b_0\hat{\omega}_p}{2}\right)^2 = 0$$

Solving for $\hat{\kappa}_p$ in the above equations, and denoting the solution as $\hat{\kappa}_p^* = \underset{\hat{\kappa}_p}{\mathrm{argmin}}\,\bar{K}_{v_{lb}}(\hat{\kappa}_p)$, we find that

$$\hat{\kappa}_p^* = \frac{\hat{\omega}_v\alpha_2}{\sqrt{\hat{\beta}_p^2 g_1^2 + 4\hat{\beta}_p\omega_M(\kappa_p - \hat{\beta}_p g_2)}} \tag{36}$$

where $g_1 = \alpha_3 + \xi_1 + c_\gamma b_0$ and $g_2 = \alpha_2 + \frac{c_\gamma b_0}{2}$. Selecting $\hat{\kappa}_p = \hat{\kappa}_p^*$ results in the minimum lower bound on κ_v. When selecting $\hat{\kappa}_p = \hat{\kappa}_p^*$, it can be shown that the lower bound is given by

$$\bar{K}_{v_{lb}}(\hat{\kappa}_p^*) = \hat{\beta}_v\left[\frac{(h\hat{\beta}_p(\alpha_3 + \xi_1) + \alpha_2 + c_\gamma b_0 h\hat{\beta}_p)^2}{4h\left(1 - \hat{\beta}_p\alpha_2 - \frac{c_\gamma b_0\hat{\beta}_p}{2}\right)} + h\hat{\beta}_p\omega_M \right. \\ \left. +\alpha_3 + \eta_1 + \frac{c_\gamma b_0}{2}\right]$$

where $h = \frac{\alpha_2}{\sqrt{\hat{\beta}_p^2 g_1^2 + 4\hat{\beta}_p \omega_M (\kappa_p - \hat{\beta}_p g_2)}}$. The important point here is to note that with $\hat{\kappa}_p = \hat{\kappa}_p^*$, the lower bound $\bar{K}_{v_{lb}}$ is independent of $\hat{\kappa}_v$. Thus, it follows that both the lower bounds on the minimum singular values of the feedback gain matrices are independent of the learning rate parameter, $\hat{\kappa}_v$.

Bound on the Learning Rate: We now examine the bound on the learning rate given by $\gamma^* = \frac{c_\gamma}{\max\{\hat{\omega}_p, \hat{\omega}_v\}}$. Previously, we argued that $\hat{\omega}_p = \hat{\omega}_p^* = \hat{\beta}_p \hat{\kappa}_p^*$ so that the lower bound on κ_v is minimized. Hence, we have the following condition

$$\gamma^* = \begin{cases} \frac{c_\gamma}{\hat{\omega}_p^*} & \text{if } \frac{\hat{\beta}_p \alpha_2}{\sqrt{\hat{\beta}_p^2 g_1^2 + 4\hat{\beta}_p \omega_M (\kappa_p - \hat{\beta}_p g_2)}} \geq 1 \\ \frac{c_\gamma}{\hat{\omega}_v} & \text{if } \frac{\hat{\beta}_p \alpha_2}{\sqrt{\hat{\beta}_p^2 g_1^2 + 4\hat{\beta}_p \omega_M (\kappa_p - \hat{\beta}_p g_2)}} < 1 \end{cases} \quad (37)$$

Uniform Boundedness in Expectation: With the feedback gains selected such that $\kappa_p \geq \bar{K}_{p_{lb}}$ and $\kappa_v \geq \bar{K}_{v_{lb}}$, and learning rate $\gamma \leq \gamma^*$, we are guaranteed that the matrix A given in (29) is positive definite. Hence, we can now bound LV as follows

$$\begin{aligned} LV \leq & -\lambda_{min}(A)\|y\|^2 + \epsilon_B \|y\|(\hat{\omega}_p^* + \hat{\omega}_v) \\ & + \|y\|^3 \left(\frac{\hat{\omega}_p^* \xi_2}{2} + \hat{\omega}_v \eta_2 \right) \\ & - h\|y\|^3 \left(\left\|\tilde{W}\right\|_F^2 - \left\|\tilde{W}\right\|_F \|W^*\|_F \right) \\ = & -\lambda_{min}(A)\|y\|^2 + \epsilon_B \|y\|(\hat{\omega}_p^* + \hat{\omega}_v) \\ & + \|y\|^3 \left(\frac{\hat{\omega}_p^* \xi_2}{2} + \hat{\omega}_v \eta_2 \right) \\ & - h\|y\|^3 \left[\left(\left\|\tilde{W}\right\|_F - \frac{\|W^*\|_F}{2} \right)^2 - \frac{\|W^*\|_F^2}{4} \right] \\ = & -\|y\| \Bigg(\lambda_{min}(A)\|y\| - \epsilon_B(\hat{\omega}_p^* + \hat{\omega}_v) + \\ & \|y\|^2 \left[h \left(\left(\left\|\tilde{W}\right\|_F - \frac{\|W^*\|_F}{2} \right)^2 - \frac{\|W^*\|_F^2}{4} \right) - \right. \\ & \left. \left(\frac{\hat{\omega}_p^* \xi_2}{2} + \hat{\omega}_v \eta_2 \right) \right] \Bigg) \end{aligned}$$

It follows that $LV \leq 0$ for all $x : \|x\| \geq R_0$ where $R_0 = \sqrt{R_y^2 + R_W^2}$ where

$$R_y = \frac{\epsilon_B(\hat{\omega}_p + \hat{\omega}_v)}{\lambda_{min}(A)}$$

and

$$R_W = \frac{\|W^*\|_F}{2} + \sqrt{\frac{1}{h}\left(\frac{\hat{\omega}_p \xi_2}{2} + \hat{\omega}_v \eta_2 \right) + \frac{\|W^*\|_F^2}{4}}$$

Hence, the closed loop system is semi-globally, uniformly bounded in expectation. □

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Appendix A. BOUNDING CONSTANTS OF THEOREM 1

In this section, we define the bounding constants used in Theorem 1. The bounds presented here are similar to that in Kelly and Salgado [1994].

$$\begin{aligned} &\xi_1 = C_V \|\dot{q}_d\| && \alpha_2 = C_M \|\ddot{q}_d\| + C_{V_q} \|\dot{q}_d\|^2 + C_G \\ &\alpha_3 = C_{V_{\dot{q}}} \|\dot{q}_d\| && C_{V_{\dot{q}}} = \max_j \sup_q \sum_{i=1}^n \|\omega_{ij}(q)\| \\ &\eta_1 = C_V && C_V = \max_j \sup_q \sum_{i=1}^n \|\omega_{ij}(q)\| \\ &\omega_{ij}(q) = [V_i(q)]_{[:,j]} && C_M = \max_j \sum_{i=1}^n \sup_q \left\| \frac{\partial m_{ij}(q)}{\partial q} \right\| \\ & && C_{V_q} = \max_j \sup_q \sum_{i=1}^n \sqrt{\sum_{k=1}^n \left\| \frac{\partial \omega_{ij}}{\partial q_k} \right\|^2} \\ & && C_G = \max_j \sup_q \sum_i \left| \left[\frac{\partial G}{\partial q} \right]_{ij} \right| \end{aligned} \quad (A.1)$$

where m_{ij} denotes that ith row and jth column of the inertia matrix, $M(q)$, and $[V_i(q)]_{[:,j]}$ denotes the jth column of the matrix $V_i(q)$ defined by

$$V_m(q, \dot{q}) = \begin{bmatrix} \dot{q}^T V_1(q) \\ \dot{q}^T V_2(q) \\ \vdots \\ \dot{q}^T V_n(q) \end{bmatrix}$$