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On Robustness of Discrete-Time LTI Systems with Varying Time Delays ⋆

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Abstract: This manuscript concerns robust stability analysis of discrete-time LTI systems with varying time delays. The stability problem is treated in the Integral Quadratic Constraint (IQC) framework. The novelty and main contribution of the manuscript is the integral quadratic constraint characterization of the discrete-time time-varying delay operator. The characterization enables the IQC analysis to be applied for studying robustness property in the presence of time-varying delays.

Keywords: Time-varying delay, Robust stability, Integral Quadratic Constraint

1. INTRODUCTION

Consider the following discrete-time linear time delay system

$$x[k+1] = Ax[k] + A_d(x[k-\tau[k]] + f) \tag{1}$$

where $\tau[k]$ is an unknown time-varying parameter which satisfies

$$\begin{aligned} &\bullet\ \ \tau[k] \in \{0, 1, \cdots, \mathcal{T}\};\\ &\bullet\ \ |\delta[k]| := |\tau[k+1] - \tau[k]| \in \{0, 1, \cdots, d\}, d \le \mathcal{T}. \end{aligned} \tag{2}$$

$x \in \mathbf{R}^n$ is the signal of interest, f is a finite-energy disturbance, A and $A_d \in \mathbf{R}^{n\times n}$ are constant matrices. The initial condition $x[\theta]$ is a function defined on $[-\mathcal{T}, 0]$. In this manuscript, delay-dependent conditions for robust stability of system (1) is to be developed. More specifically, given a pair of scalars $(\mathcal{T}, d)$, the main objective of this manuscript is to derive conditions under which the delay system (1) is stabile for all $\tau[k]$ that satisfy condition (2).

It is well-known that the manifestation of time delays in a system can lead to performance degradation and even destabilization of the system. As such, robustness in the presence of time delay has been a long standing research field in the systems theory and control community. Recently, robustness with respect to time-varying delays gains substantial attention due to their relevance to practical applications such as regulation of internet traffic and control over networked communication channels (Misra et al. [2002], Low et al. [2002]), real-time implementation of control systems (Cervin et al. [2003]), and control of fuel injection systems (Cho and Hedrick [1989]). Most existing results for checking the stability of systems with time-varying delays were developed in time domain based on the Lyapunov stability theorem – in which certain form of Lyapunov function candidates are used to derive stability conditions (Li and de Souza [1997], Cao et al. [1998], Song et al. [1999], Gu and Han [2000], Kim [2001], Mehdi et al. [2002], Fridman and Shaked [2002, 2003], Kharitonov and Niculescu [2003], Wu et al. [2004], Gao et al. [2004], Suplin et al. [2006], Gao and Chen [2007], Kao and Ranzter [2007]). The form of Lyapunov functions is often tied to the formulation of systems under consideration. As such, it could be difficult to generalize the result to other similar but slightly different systems because the generalization involves modification of the form of the Lyapunov function, which might not be easy to come up with.

It is also worthwhile mentioning that most of results regarding robustness against time-varying delays are concerned with the continuous time case. To the best of the author's knowledge, little effort has been made for stability analysis of linear discrete-time systems with time-varying delays. The problem was considered in Song et al. [1999], Gao et al. [2004] and Gao and Chen [2007], where the Lyapunov function approach was taken to derive stability conditions. In all these papers, the robust stability problem was studied for the case where only the knowledge on the bounds of the length of time delay were taken into account. The information about variation of the delay parameter was not considered.

In contrast to the Lyapunov approach, we will tackle the robust stability problem via a frequency-domain approach called Integral Quadratic Constraint (IQC) Analysis(Megretski and Rantzer [1997]). The main step of applying the IQC analysis to time-varying delay systems is to acquire integral quadratic constraint characterization of the time-varying delay operator. With such characterization, stability conditions can be straightforwardly obtained following the IQC stability theorem. The advantage of IQC analysis lies in its flexibility. Under this framework, it is easy to deal with systems where multiple time-varying delays, parametric uncertainties, un-modelled dynamics, and/or nonlinear elements such as saturation, relay, hysteresis appear simultaneously. Another distinct feature of the work presented in this manuscript is that the variation of the time delay parameter is taken into account. Our work here demonstrates how this information could be used in deriving less conservative criteria for checking robust stability under the presence of time-varying delay.

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Notation: Symbol I_n is used to denote n-dimensional identity matrix. The subscript n is dropped when the dimension is evident from the text. Given a matrix M, the transposition and the conjugate transposition are denoted by M' and M^*, respectively. The notation $M > 0$ ("$\geq$","$<$", and "$\leq$", respectively) is used to denote positive definiteness (positive semi-definiteness, negative definiteness, and negative semi-definiteness, respectively). Symbol l_2^m denotes the space of $\mathbf{R}^m$-valued, square summable functions defined on time interval $(-\infty, \infty)$, and l_{2e}^m denotes the extension of the space l_2^m, which consists of functions whose time truncation lies in l_2^m. Notation $\mathbf{Rl}_\infty^{l\times m}$ is used to denote the space of proper rational transfer matrices (of dimension $l\times m$) with no poles on the unit circle, while $\mathbf{Rh}_\infty^{l\times m}$ denotes the subspace of $\mathbf{Rl}_\infty^{l\times m}$ consisting of functions which have no poles outside the open unit disk. Every $H \in \mathbf{Rl}_\infty^{l\times m}$ defines a convolution operator on l_2: let $h(t)$ be the inverse Laplace transform of H. Then for any $u \in l_2$,

$$(Hu)[k] := \sum_{l=-\infty}^{\infty} h[k-l]u[l].$$

Given a signal f in the l_2 space, we use $\|f\|_{l_2}$ to denote the l_2 norm of f. Given a bounded operator G on the l_2 space, we use $\|G\|_{l_2}$ to denote the l_2 induced norm of G.

Let $\mathcal{D}_\tau$ denote the time-delay operator $\mathcal{D}_\tau(v) := v[k-\tau[k]]$, and $\mathcal{S}_\tau$ be the "delay-difference" operator $(I - \mathcal{D}_\tau)$; i.e., $\mathcal{S}_\tau(v) := v[k] - v[k-\tau[k]]$. To simplify the notation, in the rest of the paper we will suppress the time dependency on $\tau[k]$ and $\delta[k] := \tau[k+1] - \tau[k]$, and simply write them as τ and δ.

2. STABILITY ANALYSIS VIA INTEGRAL QUADRATIC CONSTRAINTS

In this section, the Integral Quadratic Constraint (IQC) analysis, which is needed for the main result of this paper, is briefly introduced. The system under consideration is

$$v = Gw + e, \quad w = \Delta(v) \tag{3}$$

where $G \in \mathbf{Rh}_\infty^{l\times m}$ and Δ is a bounded and causal operator on $l_{2e}^{m\times l}$. *Well-posedness* and *stability* of such a system are defined similar to that of Megretski and Rantzer [1997].

The feedback system (3) is said to be *well-posed* if the map $(v, w) \mapsto e$ has a causal inverse on l_{2e}. That is, for any $e \in l_{2e}$, there exists a solution $(v, w) \in l_{2e}$ which depends causally on e. If, in addition, there exists a positive constant C such that

$$\sum_{k=-\infty}^{T} \|v[k]\|^2 + \|w[k]\|^2 \leq C \sum_{k=-\infty}^{T} \|e[k]\|^2, \quad \forall\, T \geq 0$$

then the system is said to be *stable*.

Let Π be a bounded self-adjoint operator on l_2 space. Then Π defines a quadratic form on l_2

$$\sigma_\Pi(v,w) := \left\langle \begin{bmatrix} v \\ w \end{bmatrix}, \Pi \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle = \sum_{k=-\infty}^{\infty} \begin{bmatrix} v[k] \\ w[k] \end{bmatrix}' \left(\Pi \begin{bmatrix} v \\ w \end{bmatrix}\right)[k]$$
$$= \int_{-\pi}^{\pi} \begin{bmatrix} \hat{v}(e^{j\omega}) \\ \hat{w}(e^{j\omega}) \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} \hat{v}(e^{j\omega}) \\ \hat{w}(e^{j\omega}) \end{bmatrix} d\omega$$

where $\hat{v}$ and $\hat{w}$ are Fourier transforms of v and w, respectively. The operator Π is referred to as the multiplier of the quadratic form σ_Π. The multiplier Π is often block partitioned into the form

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$$

where the dimensions of Π_{ij} are consistent with those of v and w.

Given an operator $\mathcal{H}$ and a quadratic form $\sigma_\Pi(v,w)$ defined on l_2 space, we said that $\mathcal{H}$ satisfies the integral quadratic constraint defined by σ_Π, or more often "$\mathcal{H}$ satisfies IQC defined by Π" to emphasize the multiplier involved, if $\sigma_\Pi(v, \mathcal{H}(v)) \geq 0$ for all $v \in l_2$.

Stability of (3) can be determined by the following theorem.

Theorem 1. Let $G \in \mathbf{Rh}_\infty^{l\times m}$ and let Δ be a bounded causal operator. Suppose

(i) for every $\rho \in [0, 1]$, the interconnection of G and $\rho\Delta$ is well-posed;
(ii) for every $\rho \in [0, 1]$, the IQC defined by Π is satisfied by $\rho\Delta$;
(iii) there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(e^{j\omega}) \\ I \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} G(e^{j\omega}) \\ I \end{bmatrix} \leq -\epsilon I, \ \forall\, \omega \in [-\pi, \pi]. \tag{4}$$

Then the feedback interconnection of G and Δ is stable.

Proof. The stability theorem presented above is completely analog to that of Megretski and Rantzer [1997], where the continuous-time systems were considered. The proof for the discrete-time case is identical to that of the continuous-time case.

Condition (ii) in Theorem 1 can be eased if $\Pi_{11} \geq 0$ and $\Pi_{22} \leq 0$. In this case, Δ satisfies IQCs defined by Π implies that $\rho\Delta$ satisfies IQCs defined by Π for all $\rho \in [0, 1]$.

Assume that the overall Δ is diagonally structured by n components, Δ_i, $i = 1, \cdots, n$; i.e., $\Delta = \mathrm{diag}(\Delta_1, \cdots, \Delta_n)$. Suppose that each Δ_i satisfies IQC defined by Π_i, respectively. Then an IQC for Δ can be easily defined by assembling Π_i appropriately. Furthermore, if Δ satisfies IQCs defined by Π_i, $i = 1, \cdots, n$, then the conic combination $x_1\Pi_1 + \cdots + x_n\Pi_n$, $x_i \geq 0$ also defines an IQC for Δ. Hence, if Δ satisfies IQCs defined by Π_i, $i = 1, \cdots, n$, a sufficient condition for stability is the existence of $x_1, \cdots, x_n \geq 0$ such that (4) holds for $\Pi := x_1\Pi_1 + \cdots + x_n\Pi_n$.

Condition (4) is a frequency dependent, infinite dimensional Linear Matrix Inequality (LMI). Suppose that $\Pi \in \mathbf{Rl}_\infty$. Then this matrix inequality can be converted into a frequency independent finite dimensional LMI using the Kalman-Yakubovich- Popov (KYP) Lemma. More details will be given in Section 4.

3. INTEGRAL QUADRATIC CONSTRAINTS FOR $\mathcal{D}_\tau$ AND $\mathcal{S}_\tau$

In this section, conically parameterized integral quadratic constraints for operators $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$ are derived, which are essential for robustness analysis of system (1) under the IQC framework. We first present a few technical lemmas which will be used for deriving IQCs. To facilitate the development of the paper, let us consider the following sets of discrete-time sequences

$$\Upsilon_1 := \{s[k] : s[k] \in \{0, 1, \cdots, \mathcal{T}\},\ \forall k\}$$
$$\Upsilon_2 := \{s[k] : s[k] \in \{0, 1, \cdots, \mathcal{T}\},\ |s[k+1] - s[k]| \in \{0, 1, \cdots, d\},\ \forall k\}.$$

Lemma 2. Consider the time-varying delay operator $\mathcal{D}_\tau$ where the delay parameter τ could be any sequence from Υ_1. Then the following characterization holds

$$\sup_{\tau[k]\in\Upsilon_1} \|\mathcal{D}_\tau\|_{l_2} = \sqrt{\mathcal{T}+1}.$$

Suppose now τ belongs to Υ_2 (i.e., the variation of τ is restricted). Then

$$\sup_{\tau[k]\in\Upsilon_2} \|\mathcal{D}_\tau\|_{l_2} = \sqrt{\mathcal{T}+1}.$$

as long as $d \geq 1$.

Proof. Let $w[k] := (\mathcal{D}_\tau v)[k] := v[k-\tau]$. Since $\tau \in \{0, 1, \cdots, \mathcal{T}\}$, we know $w[k] \in \{v[k], v[k-1], \cdots, v[k-\mathcal{T}]\}$ and therefore

$$w[k]^2 \leq \sum_{j=0}^{\mathcal{T}} v[k-j]^2,$$

which in turn implies

$$\|w\|_{l_2}^2 := \sum_{k=-\infty}^{\infty} w[k]^2 \leq \sum_{k=-\infty}^{\infty}\sum_{j=0}^{\mathcal{T}} v[k-j]^2 = \sum_{j=0}^{\mathcal{T}}\sum_{k=-\infty}^{\infty} v[k-j]^2 = (\mathcal{T}+1)\|v\|_{l_2}^2$$

This proves $\sup_{\tau[k]\in\Upsilon_1} \|\mathcal{D}_\tau\|_{l_2} \leq \sqrt{\mathcal{T}+1}$. To see that $\sup_{\tau(t)\in\Upsilon_1} \|\mathcal{D}_\tau\|_{l_2}$ is equal to $\sqrt{\mathcal{T}+1}$, let us consider the following signals

$$v[k] = \begin{cases} \tilde{v}[l] & \text{if } k = l(\mathcal{T}+1) \text{ for some } l \in \mathcal{Z} \\ 0 & \text{otherwise} \end{cases},$$
$$\tau[k] = k \bmod (\mathcal{T}+1)$$

where $\tilde{v}$ is some l_2 signal. Then one can easily verify that

$$w[k] := v[k-\tau[k]] = \tilde{v}\left[\left\lfloor \frac{k}{\mathcal{T}+1} \right\rfloor\right]$$

and thus $\|w\|_{l_2}^2 = (\mathcal{T}+1)\|\tilde{v}\|_{l_2}^2 = (\mathcal{T}+1)\|v\|_{l_2}^2$.

To see that $\sup_{\tau[k]\in\Upsilon_2} \|\mathcal{D}_\tau\|_{l_2} \leq \sqrt{\mathcal{T}+1}$, let

$$v[k] = \begin{cases} \tilde{v}[l] & \text{if } k = 2l\mathcal{T} \text{ for some } l \in \mathcal{Z} \\ 0 & \text{otherwise} \end{cases},$$
$$\tau[k] = \begin{cases} (k \bmod 2\mathcal{T}), & \text{if } (k \bmod 2\mathcal{T}) \leq \mathcal{T} \\ 2\mathcal{T} - (k \bmod 2\mathcal{T}), & \text{if } (k \bmod 2\mathcal{T}) > \mathcal{T} \end{cases}$$

where $\tilde{v}$ is some l_2 signal. Then one can again verify that $\|w\|_{l_2}^2 = (\mathcal{T}+1)\|\tilde{v}\|_{l_2}^2 = (\mathcal{T}+1)\|v\|_{l_2}^2$. This concludes the proof.

Remark 3. Lemma 2 shows that, as long as τ is allowed to vary, the worse case l_2-gain of $\mathcal{D}_\tau$ depends only on *the length of the time-delay.* The information on the variation of τ given in the form of $|\tau[k+1] - \tau[k]| \in \{0, 1, \cdots, d\}, d \leq \mathcal{T},\ \forall\ k$, provides no help for improving the l_2-gain of $\mathcal{D}_\tau$. This is in contrast to the continuous-time case where the $\mathcal{L}_2$-gain of $\mathcal{D}_\tau$ depends only on *the variation of the time-delay* but not the length of the time-delay (see Kao and Ranzter [2007]).

Lemma 4. Consider the "delay-difference" operator $\mathcal{S}_\tau$. The following characterization holds for $\mathcal{S}_\tau$: for any l_2 signal v,

$$\|\mathcal{S}_\tau v\|_{l_2}^2 \leq \sum_{k=-\infty}^{\infty}\sum_{i=1}^{\mathcal{T}} (v[k] - v[k-i])^2. \tag{5}$$

Proof. Let $w[k] := (\mathcal{S}_\tau v)[k] := v[k] - v[k-\tau]$. Since $\tau \in \{0, 1, \cdots, \mathcal{T}\}$, we know $w[k] \in \{v[k] - v[k-1], \cdots, v[k] - v[k-\mathcal{T}]\}$ and therefore

$$w[k]^2 \leq \sum_{i=1}^{\mathcal{T}} (v[k] - v[k-i])^2.$$

This concludes the proof.

Lemmas 2 and 4 give rise to the following integral quadratic constraints for $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$.

Proposition 5. Suppose $\tau[k] \in \{0, 1, \cdots, \mathcal{T}\}$. Then the operator $\mathcal{D}_\tau$ satisfies any integral quadratic constraint defined by

$$\Pi_1 = \begin{bmatrix} (\mathcal{T}+1)X_1 & 0 \\ 0 & -X_1 \end{bmatrix} \tag{6}$$

where $X_1 = X_1' \geq 0$ is any positive semi-definite matrix.

Proof. Proposition 5 follows Lemma 2 and that, given a positive semi-definite matrix X_1, $(X_1^{\frac{1}{2}}\mathcal{D}_\tau)v = (\mathcal{D}_\tau X_1^{\frac{1}{2}})v$.

Proposition 6. Suppose $\tau[k] \in \{0, 1, \cdots, \mathcal{T}\}$. Then the operator $\mathcal{S}_\tau$ satisfies any integral quadratic constraint defined by

$$\Pi_2 = \begin{bmatrix} |\phi(\mathrm{e}^{j\omega})|^2 X_2 & 0 \\ 0 & -X_2 \end{bmatrix} \tag{7}$$

where $\phi(z) \in \mathbf{Rl}$ satisfies

$$|\phi(\mathrm{e}^{j\omega})|^2 = \sum_{\kappa=1}^{\mathcal{T}} |1 - \mathrm{e}^{-j\kappa\omega}|^2 \tag{8}$$

and $X_2 = X_2' \geq 0$ is any positive semi-definite matrix.

Proof. Let $w := \mathcal{S}_\tau v$ and $v_\kappa := (1 - z^{-\kappa})v$, $\kappa = 1, \cdots, \mathcal{T}$, where $z^{-\kappa}$ denotes the κ-step delay operator. Lemma 4 implies

$$\|w\|_{l_2}^2 \leq \sum_{\kappa=1}^{\mathcal{T}} \|v_\kappa\|_{l_2}^2.$$

Therefore, we have

$$\int_{-\pi}^{\pi} \|\hat{w}(\mathrm{e}^{j\omega})\|^2 d\omega \leq \sum_{\kappa=1}^{\mathcal{T}} \int_{-\pi}^{\pi} \|\hat{v}_\kappa(\mathrm{e}^{j\omega})\|^2 d\omega$$

where $\hat{w}$ and $\hat{v}_\kappa$ denote the Fourier transforms of w and v_κ, respectively. Furthermore, let $\hat{v}$ be the Fourier transform of v. Then $\hat{v}_\kappa(\mathrm{e}^{j\omega})$ is equal to $(1 - \mathrm{e}^{-j\kappa\omega})\hat{v}(\mathrm{e}^{j\omega})$, and we have

$$\int_{-\pi}^{\pi} \|\hat{w}(\mathrm{e}^{j\omega})\|^2 d\omega \leq \int_{-\pi}^{\pi} \left(\sum_{\kappa=1}^{\mathcal{T}} |1 - \mathrm{e}^{-j\kappa\omega}|^2\right) \|\hat{v}(\mathrm{e}^{j\omega})\|^2 d\omega$$

This implies that v and w satisfy IQC defined by

$$\begin{bmatrix} |\phi(\mathrm{e}^{j\omega})|^2 I & 0 \\ 0 & -I \end{bmatrix}$$

Finally, the IQC defined by Π_2 follows that, given any $v \in l_2$ and positive semi-definite matrix X_2, $(X_2^{\frac{1}{2}}\mathcal{S}_\tau)v = (\mathcal{S}_\tau X_2^{\frac{1}{2}})v$. This concludes the proof.

The scaling matrices X_1 and X_2 of IQCs defined in (6) and (7) are constant (as opposed to frequency dependent) over all frequencies. Frequency dependent scalings, which usually provide better characterization of $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$, are allowed if certain "correction terms" are added to the IQCs. The key idea behind is the following swapping lemma.

Lemma 7. (Swapping lemma). Define $H(z) := C(zI - A)^{-1}B + D$, $H_l(z) := C(zI - A)^{-1}$, and $H_r(z) := (zI - A)^{-1}B$ to be proper rational transfer matrices from $\mathbf{Rl}_\infty^{n\times n}$. Furthermore, let T denote the operator $z \circ \mathcal{D}_\tau - \mathcal{D}_\tau \circ z$. Then

$$\mathcal{D}_\tau \circ H(z) = H(z) \circ \mathcal{D}_\tau + H_l(z) \circ T \circ H_r(z). \tag{9}$$

Proof. Let $y = \mathcal{D}_\tau \circ Hv$, and $z = H \circ \mathcal{D}_\tau(v)$. Then $y[k] = Cx_1[k-\tau] + Dv[k-\tau]$, where x_1 is the state of the system

$$x_1[k+1] = Ax_1[k] + Bv[k], \quad x_1[0] = 0,$$

and $z[k] = Cx_2[k] + Dv[k-\tau]$, where x_2 satisfies

$$x_2[k+1] = Ax_2[k] + Bv[k-\tau], \; x_2[0] = 0.$$

Let $x_3[k] := x_1[k - \tau[k]] - x_2[k]$ and $w := y - z = Cx_3$. We find that x_3 satisfies the difference equation

$$\begin{aligned} x_3[k+1] &= x_1[k+1-\tau[k+1]] - x_2[k+1] \\ &= x_1[k+1-\tau[k]] - x_2[k+1] + \\ &\quad (x_1[k+1-\tau[k+1]] - x_1[k+1-\tau[k]]) \\ &= A(x_1[k-\tau[k]] - x_2[k]) + (Tx_1)[k] \\ &= Ax_3[k] + (Tx_1)[k]. \end{aligned}$$

Hence, we have shown that

$$w := (\mathcal{D}_\tau \circ H - H \circ \mathcal{D}_\tau)v = H_l(Tx_1) = (H_l \circ T \circ H_r)v$$

This concludes the proof.

Remark 8. Using (9), the following equality can be readily verified

$$\mathcal{S}_\tau \circ H(z) = H(z) \circ \mathcal{S}_\tau - H_l(z) \circ T \circ H_r(z).$$

Notice that the operator T can also be expressed as $(I - \mathcal{D}_\delta) \circ z\mathcal{D}_\tau$, where $\delta[k] := \tau[k] - \tau[k+1]$. This identity and the swapping formulas lead to the following IQC characterization for $\mathcal{D}_\tau$.

Proposition 9. Let $H(z) := h(z) \cdot I_n$ where $h(z) \in \mathbf{Rl}_\infty^{1\times 1}$, and the corresponding $H_l(z)$ and $H_r(z)$ be defined similarly as those in Lemma 7. Suppose $\tau[k] \in \{0, 1, \cdots, \mathcal{T}\}$ and $|\delta[k]| \in \{0, 1, \cdots, d\}$, where $d \le \mathcal{T}$. Let $w := \mathcal{D}_\tau v$, $\hat{w} := z\mathcal{D}_\tau H_r v$, and $\tilde{w} := (I - \mathcal{D}_\delta)\hat{w}$. Then the following integral quadratic constraint holds

$$\left\langle \begin{bmatrix} v \\ \hat{w} \\ \tilde{w} \\ w \end{bmatrix}, \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & M_3 & M_{34} \\ 0 & 0 & M_{34}^* & M_4 \end{bmatrix} \begin{bmatrix} v \\ \hat{w} \\ \tilde{w} \\ w \end{bmatrix} \right\rangle \ge 0 \tag{10}$$

where

$$\begin{aligned} &M_1 := (\mathcal{T}+1)(H^* X_3 H + H_r^* X_4 H_r) \\ &M_2 := \psi^*\psi X_5 - X_4 \\ &M_3 := -X_5, \quad M_{34} := -H_l^* X_3 H, \quad M_4 := -H^* X_3 H \end{aligned}$$

$\psi(z) \in \mathbf{Rl}$ satisfies

$$|\psi(e^{j\omega})|^2 = \sum_{\kappa=1}^{d} 2|1 - e^{-j\kappa\omega}|^2$$

and $X_i = X_i'$, $i = 3, 4, 5$, are any positive semi-definite matrices.

Proof. Note that by the swapping lemma, $X_3^{\frac{1}{2}} Hw = X_3^{\frac{1}{2}} \mathcal{D}_\tau Hv - X_3^{\frac{1}{2}} H_l \tilde{w} = \mathcal{D}_\tau X_3^{\frac{1}{2}} Hv - X_3^{\frac{1}{2}} H_l \tilde{w}$. Therefore, we have

$$\begin{aligned} &2\langle X_3^{\frac{1}{2}} Hw, X_3^{\frac{1}{2}} Hw\rangle \\ &= 2\langle X_3^{\frac{1}{2}} Hw, \mathcal{D}_\tau X_3^{\frac{1}{2}} Hv\rangle - 2\langle X_3^{\frac{1}{2}} Hw, X_3^{\frac{1}{2}} H_l\tilde{w}\rangle \\ &\le \langle X_3^{\frac{1}{2}} Hw, X_3^{\frac{1}{2}} Hw\rangle + (\mathcal{T}+1)\langle X_3^{\frac{1}{2}} Hv, X_3^{\frac{1}{2}} Hv\rangle \\ &\quad - 2\langle X_3^{\frac{1}{2}} Hw, X_3^{\frac{1}{2}} H_l\tilde{w}\rangle. \end{aligned} \tag{11}$$

Furthermore, since $\hat{w} := z\mathcal{D}_\tau H_r v$ and $\tilde{w} := (I - \mathcal{D}_\delta)\hat{w}$, we have $X_4^{\frac{1}{2}}\hat{w} := X_4^{\frac{1}{2}} z\mathcal{D}_\tau H_r v = z\mathcal{D}_\tau X_4^{\frac{1}{2}} H_r v$ and $X_5^{\frac{1}{2}}\tilde{w} := X_5^{\frac{1}{2}}(I - \mathcal{D}_\delta)\hat{w} = (I - \mathcal{D}_\delta)X_5^{\frac{1}{2}}\hat{w}$; therefore,

$$\langle X_4^{\frac{1}{2}}\hat{w}, X_4^{\frac{1}{2}}\hat{w}\rangle \le (\mathcal{T}+1)\langle X_4^{\frac{1}{2}} H_r v, X_4^{\frac{1}{2}} H_r v\rangle \tag{12}$$

$$\langle X_5^{\frac{1}{2}}\tilde{w}, X_5^{\frac{1}{2}}\tilde{w}\rangle \le \langle \phi X_5^{\frac{1}{2}}\hat{w}, \phi X_5^{\frac{1}{2}}\hat{w}\rangle \tag{13}$$

IQCs (12) and (13) follows Propositions 5 and 6, respectively. Note that the form of ψ is slightly different from that of ϕ because $\delta[k]$ ranges between $-d$ to d instead of 0 to $\mathcal{T}$. Combining IQCs (11), (12), and (13), we arrive IQC (10). This concludes the proof.

An integral quadratic constraint which is very similar to (10) can be derived for $\mathcal{S}_\tau$.

Proposition 10. Let $H(z) := h(z) \cdot I_n$ where $h(z) \in \mathbf{Rl}_\infty^{1\times 1}$, and the corresponding $H_l(z)$ and $H_r(z)$ be defined similarly as those in Lemma 7. Suppose $\tau \in \{0, 1, \cdots, \mathcal{T}\}$ and $|\delta| \in \{0, 1, \cdots, d\}$, where $d \le \mathcal{T}$. Let $w := \mathcal{S}_\tau v$, $\hat{w} := z\mathcal{D}_\tau H_r v$, and $\tilde{w} := (I - \mathcal{D}_\delta)\hat{w}$. Then the following integral quadratic constraint holds

$$\left\langle \begin{bmatrix} v \\ \hat{w} \\ \tilde{w} \\ w \end{bmatrix}, \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & M_3 & M_{34} \\ 0 & 0 & M_{34}^* & M_4 \end{bmatrix} \begin{bmatrix} v \\ \hat{w} \\ \tilde{w} \\ w \end{bmatrix} \right\rangle \ge 0 \tag{14}$$

where

$$\begin{aligned} &M_1 := (\phi H)^* X_3 (\phi H) + (\mathcal{T}+1) H_r^* X_4 H_r \\ &M_2 := \psi^*\psi X_5 - X_4 \\ &M_3 := -X_5, \quad M_{34} := H_l^* X_3 H, \quad M_4 := -H^* X_3 H \end{aligned}$$

operator ϕ is defined in (8), operator ψ satisfies

$$|\psi(e^{j\omega})|^2 = \sum_{\kappa=1}^{d} 2|1 - e^{-j\kappa\omega}|^2,$$

and $X_i = X_i'$, $i = 3, 4, 5$, are any positive semi-definite matrices.

Proof. The proof of Proposition 10 is completely analog to that of Proposition 9.

4. STABILITY ANALYSIS OF DISCRETE-TIME LINEAR TIME-VARYING DELAY SYSTEMS

Consider now the linear time-varying delay system

$$x[k+1] = Ax[k] + A_d(x[k-\tau] + f) \tag{15}$$

where the delay parameter τ satisfies conditions

$$\begin{aligned} &\tau[k] \in \{0, 1, \cdots, \mathcal{T}\}, \\ &|\delta[k]| := |\tau[k+1] - \tau[k]| \in \{0, 1, \cdots, d\}, \quad d \le \mathcal{T} \end{aligned}$$

We assume that $A + A_d$ is stable (i.e., all eigenvalues of $A + A_d$ are strictly inside the unit circle), which is a necessary condition for stability.

The system can be modelled as the feedback interconnection

$$x = Gw + e, \quad w = \Delta(x) \tag{16}$$

where $G := -(zI-(A+A_d))^{-1}A_d$ is a linear time invariant stable system, $e = -Gf$, and $\Delta := \mathcal{S}_\tau$. With the integral quadratic constraints derived in Section 3, IQC analysis can be applied to the transformed system (16) to study l_2 stability of system (15). Note that any IQC for $\mathcal{D}_\tau$ immediately leads to an IQC for $\mathcal{S}_\tau$. For example, let $w = \mathcal{S}_\tau v := v - \mathcal{D}_\tau v$. That $\mathcal{D}_\tau$ satisfies IQC defined by Π_1 in (6) implies v and w satisfy IQC

$$\left\langle \begin{bmatrix} v \\ w \end{bmatrix}, \begin{bmatrix} \mathcal{T}X_1 & X_1 \\ X_1 & -X_1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle \geq 0.$$

Stability criteria derived via IQC analysis are naturally posed as semi-infinite optimization problems. Final dimensional formulation (in terms of linear matrix inequalities) can be derived using the Kalman-Yakubovich-Popov (KYP) lemma.

To further illustrate the idea, let us consider IQCs defined by Π_1 and Π_2 (equations (6) and (7)) for $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$. Then $\mathcal{S}_\tau$ satisfies IQC defined by

$$\Pi_{\text{comb}} := \begin{bmatrix} \mathcal{T}X_1 + |\phi(e^{j\omega})|^2 X_2 & X_1 \\ X_1 & -X_1 - X_2 \end{bmatrix}$$

With this IQC, Theorem 1 leads to the following stability criteria: the system is stable if there exist symmetric matrices $X_1 \geq 0$, $X_2 \geq 0$, and $\epsilon > 0$ such that

$$\begin{aligned} &G(e^{j\omega})^*(\mathcal{T}X_1 + |\phi(e^{j\omega})|^2 X_2)G(e^{j\omega}) + G(e^{j\omega})^* X_1 \\ &+ X_1 G(e^{j\omega}) - X_1 - X_2 \leq -\epsilon I, \quad \forall \omega \in [-\pi, \pi]. \end{aligned} \tag{17}$$

where $G(z) := -(zI-(A+A_d))^{-1}A_d$. Let $(A_\phi, B_\phi, C_\phi, D_\phi)$ be the minimum state space realization of $\phi(z) \cdot I_n$. Define

$$A_t = \begin{bmatrix} A + A_d & 0 \\ B_\phi & A_\phi \end{bmatrix}, \quad B_t = \begin{bmatrix} A_d \\ 0 \end{bmatrix}, \quad C_t = \begin{bmatrix} I_n & 0 \\ D_\phi & C_\phi \end{bmatrix}.$$

and

$$M_{11} = \begin{bmatrix} \mathcal{T}X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} -X_1 \\ 0 \end{bmatrix}, \quad M_{22} = -X_1 - X_2.$$

A final dimensional formulation of stability criterion (17) can be obtained by the KYP lemma: the system is stable if there exist symmetric matrices P, $X_1 \geq 0$, and $X_2 \geq 0$ such that

$$\begin{bmatrix} A_t'PA_t - P & A_t'PB_t \\ B_t'PA_t & B_t'PB_t \end{bmatrix} + \begin{bmatrix} C_t'M_{11}C_t & C_t'M_{12} \\ M_{12}'C_t & M_{22} \end{bmatrix} < 0.$$

The above mentioned stability criteria are obtained by utilizing IQC defined by Π_{comb} for $\mathcal{S}_\tau$. Other IQCs such as those stated in Propositions 9 and 10 can also be used to derived stability criteria. Take IQC defined in (14) for example. This IQC gives rise to the following stability condition: the system is stable if there exists $H(z) := h(z) \cdot I_n$, $h(z) \in \mathbf{Rl}_\infty^{1\times1}$, symmetric matrices $X_3 \geq 0$, $X_4 \geq 0$, $X_5 \geq 0$, and $\epsilon > 0$, such that

$$\begin{aligned} &M_2(e^{j\omega}) \leq -\epsilon I, \\ &\begin{bmatrix} M_3(e^{j\omega}) & M_{34}(e^{j\omega}) \\ M_{34}^*(e^{j\omega}) & G^*(e^{j\omega})M_1(e^{j\omega})G(e^{j\omega}) + M_4(e^{j\omega}) \end{bmatrix} \leq -\epsilon I, \end{aligned} \tag{18}$$

for all $\omega \in [-\pi, \pi]$. In the above (frequency-dependent) inequalities, $G(z) := -(zI - (A + A_d))^{-1}A_d$ and M_i, $i = 1, \cdots, 4$, and M_{34} are defined as in Proposition 10.

Note that, to apply the above stability criterion, the upper bound on the variation of time delay parameter (i.e., the bound on $|\tau[k+1] - \tau[k]|$) is required.

As a final remark, if $\tau[k+1] - \tau[k] \equiv 0$ for all k, stability condition (18) reduces to

$$\begin{aligned} &|\phi(e^{j\omega})|^2 G^*(e^{j\omega})H(e^{j\omega})^* X_3 H(e^{j\omega})G(e^{j\omega}) \\ &- H(e^{j\omega})^* X_3 H(e^{j\omega}) \leq -\epsilon I, \quad \forall\, \omega \in [-\pi, \pi]. \end{aligned}$$

This criterion is identical to that obtained by using IQC defined by

$$\Pi_3 := \begin{bmatrix} |\phi(e^{j\omega})|^2 H(e^{j\omega})^* X_3 H(e^{j\omega}) & 0 \\ 0 & -H(e^{j\omega})^* X_3 H(e^{j\omega}) \end{bmatrix}$$

for operator $\mathcal{S}_\tau$. Compared to the IQC defined by Π_2 (cf. equation (7)), we recover the frequency scaling in the IQC, which is only valid for *constant* (uncertain) delays.

5. CONCLUSIONS

Robustness analysis of discrete-time linear systems with varying time delays under the integral quadratic constraint framework is investigated. The delay parameter is assumed to be an unknown time-varying function for which the upper bounds on the length and the variation are given. The influence of the time-varying delay is modelled as an uncertainty in the system, for which integral quadratic constraint characterization is derived. IQC analysis is then applied to derive robust stability criteria. The advantage of this approach lies in its flexibility: the result obtained here can be easily generalized to analyze systems with multiple delays, parametric uncertainties, unmodelled dynamics, and/or various simple non-linearities.

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