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Stabilization of switched discrete-time systems with time-varying delay ⋆

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Abstract: A convex approach is proposed to deal with switched discrete-time systems with time-varying delays. It uses a parameter dependent Lyapunov-Krasovskii functional that allows to assure the robust stability or the robust stabilization of a switched system for arbitrary switching functions. The analysis and the design conditions are formulated as simple feasibility tests of linear matrix inequalities (LMIs). The presented conditions encompass previous results found in the literature, yielding less conservative and convex design methods. The design conditions can take into account the rate of variation of delay and deal with decentralized control. A design example is presented to illustrate the efficacy of the proposed LMI conditions, including some time-simulations.

Keywords: Time delay systems; time-varying systems; switched systems; robust control; Linear matrix inequalities.

1. INTRODUCTION

The class of switched systems encompasses systems in many fields, such as chemical process, transportation systems, communication systems, etc. In control systems, switching among different structures is an essential feature in some applications, for example in electrical power converters Montagner et al. [2004], sludge process Gómez Quintero et al. [2004] and network controlled systems Lin et al. [2003]. Switched discrete-time systems with state delay have been extensively investigated in the last years Johansson and Rantzer [1998], Liberzon and Morse [1999], Daafouz et al. [2001], Daafouz et al. [2002] but some open problems still deserve attention. In general, two main problem types have been investigated in the class of switched systems (see DeCarlo et al. [2000]): the first problem type considers how to design a controller that assures the stability of the entire system despite of arbitrary switching functions. The second one is related to the search of switching sequences that stabilize the system. In this paper, the first problem type is addressed in the context of switched discrete-time systems with time varying delay. Recently, an important number of studies has been published about switched delayed systems. See, for instance, Xie and Wang [2004], Phat [2005], Montagner et al. [2005], Yu et al. [2007], Du et al. [2007].

In Xie and Wang [2004], quadratic stability approach, i.e., Lyapunov-Krasovskii functionals with constant matrices, is used conjointly with system augmentation to design switched gains for discrete-time system with constant delay and an arbitrary switching function. Riccati-like inequalities yielding sufficient conditions for the robust stability and stabilizability of switched discrete-time systems with constant state delay are given in Phat [2005]. A Lyapunov-Krasovskii functional with a constant matrix dealing with delayed states and other depending on the switching function, that deals with the delay-free dynamic part, is presented yielding convex synthesis conditions for switched gains is investigated in Montagner et al. [2005]. In Yu et al. [2007], a parameter dependent Lyapunov-Krasovskii functional is used to propose some convex conditions to deal with switched discrete-time systems affect by constant delay, subject to actuator saturation and norm-bounded uncertainties presented at each operation mode. $\mathcal{H}_\infty$-filtering design conditions are proposed in Du et al. [2007] by means of a switched Lyapunov-Krasovskii functional. Similar approach has been taken before, in the context of stabilization of switched delay-free discrete-time systems in Daafouz et al. [2002]. Note, however, that none of the cited works can deal with time-varying delays.

Thus, this paper is focused on the problem of stability analysis and stabilization of arbitrary switched discrete-time systems with time-varying delay. Some convex conditions formulated as feasibility tests of linear matrix inequalities (LMIs) are proposed. Parameter dependent Lyapunov-Krasovskii functionals are used to assure the asymptotic stability of the entire system. It has been verified that the proposed conditions leads to less conservative results and encompass the quadratic stability approach. Some numerical examples, including time-simulation, are presented to illustrate the efficacy of the proposed conditions.

Notation: The notation used here is quite standard. $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{m\times n}$ is the set of $m \times n$ matrices with real entries and $\mathbb{N}$ ($\mathbb{N}^*$) is the set of natural numbers (excluded the zero). $\mathbf{I}_n$ and $\mathbf{0}$ denotes, respectively, the

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$n \times n$ identity matrix and the null matrix of appropriate dimensions. $M > \mathbf{0}$ ($M < \mathbf{0}$) means that matrix M is positive (negative) definite. M' is the transpose of M. The symbol $\star$ stands for symmetric blocks in the LMIs.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the following switched discrete-time system with delayed states

$$x_{k+1} = A(\alpha(k))x_k + A_d(\alpha(k))x_{k-d_k} + B(\alpha(k))u(\alpha(k)),$$
$$x_k = \phi(k), \quad k \in [-\bar{d},\ 0] \quad (1)$$

where k is the sampling time, $u_k = u(\alpha(k)) \in \mathbb{R}^p$ is the input control signal, $x_k \equiv x(k) \in \mathbb{R}^n$ is the state vector, $x_{k-d_k} \equiv x(k-d_k) \in \mathbb{R}^n$ is the delayed state vector, $d_k \equiv d(k)$ is the time-varying state delay limited by

$$\underline{d} \leq d_k \leq \bar{d} \quad (2)$$

with $(\underline{d}, \bar{d}) \in \mathbb{N}^* \times \mathbb{N}^*$ representing the possible variation band of the delay value. The uncertain parameter $\alpha(k)$ is directly related to the arbitrary switching function

$$\sigma_k \equiv \sigma(k) : \mathbb{N} \to \mathcal{I}, \quad \mathcal{I} = \{1, \ldots, \kappa\} \quad (3)$$

where $\mathcal{I}$ is the set of selectable subsystems and κ is the number of subsystems. Defines

$$\alpha_i(k) = \begin{cases} 1, \text{ for } i = \sigma_k \\ 0, \text{ otherwise} \end{cases} \quad (4)$$

Thus, the system matrices $[A(\alpha(k))|A_d(\alpha(k))|B(\alpha(k))] \equiv [A|A_d|B](\alpha(k)) \in \mathbb{R}^{n\times 2n+p}$ are switched matrices depending on the switching function (3) and can be written as

$$[A|A_d|B](\alpha(k)) = \sum_{i=1}^{\kappa} \alpha_i(k)[A|A_d|B]_i \quad (5)$$

Therefore, function σ_k can select one of the subsystems $[A|A_d|B]_i$, $i = 1, \ldots, \kappa$, at each instant k. Consider the following standard hypothesis w.r.t. switched system (1):

Hypothesis 1. The switching function is not known *a priori*, but it is available at each sample-time, k.

Hypothesis 2. All matrices of system (1) are switched simultaneously by (3).

Also, assume that:

Hypothesis 3. Both state vectors, x_k and x_{k-d_k}, are available for feedback.

Note that this last hypothesis can be achieved in physical systems by employing, for instance, a time-stamped in the measurements or in the estimated states Srinivasagupta et al. [2004]. An initial problem investigated in this paper is given below.

Problem 4. Determine if the autonomous system (1)-(2), i.e. with $u_k = \mathbf{0}$, is stable for any switching function (3).

In case of an unstable system, a state feedback control law given by

$$u(k) = K(\alpha(k))x_k + K_d(\alpha(k))x_{k-d_k} \quad (6)$$

where the switched gains $[K|K_d](\alpha(k)) \in \mathbb{R}^{p\times 2n}$, are considered to stabilize the respective subsystem i, $i = 1, \ldots, \kappa$, and assure stable transitions $\sigma_k \to \sigma_{k+1}$. In this case, it is stated the following stabilizing problem.

Problem 5. Find a pair of gains $[K|K_d](\alpha(k))$ such that the system (1)-(2) controlled by (6) is stable for arbitrary switching functions σ_k.

Thus, the stable switched closed-loop system eventually achieved by a solution of Problem 5 is written as

$$x_{k+1} = \tilde{A}(\alpha(k))x_k + \tilde{A}_d(\alpha(k))x_{k-d_k} \quad (7)$$

with

$$\tilde{A}(\alpha(k)) \equiv A(\alpha(k)) + B(\alpha(k))K(\alpha(k)) \quad (8)$$
$$\tilde{A}_d(\alpha(k)) \equiv A_d(\alpha(k)) + B(\alpha(k))K_d(\alpha(k)) \quad (9)$$

It is worth to mention that, if the delay d_k is not known, then it is enough to make $K_d(\alpha(k)) = \mathbf{0}$ in (6). On the other hand, if d_k is known, then the use of $K(\alpha(k))$ and $K_d(\alpha(k))$ may improve the performance of the closed-loop system (7).

To investigate problems 4 and 5 the following parameter dependent Lyapunov-Krasovskii candidate functional is employed:

$$V(\alpha(k), k) = \sum_{v=1}^{3} V_v(\alpha(k), k) \quad (10)$$

with

$$V_1(\alpha(k), k) = x_k' P(\alpha(k))x_k, \quad (11)$$
$$V_2(\alpha(k), k) = \sum_{j=k-d(k)}^{k-1} x_j' Q(\alpha(j))x_j, \quad (12)$$
$$V_3(\alpha(k), k) = \sum_{\ell=2-\bar{d}}^{1-\underline{d}} \sum_{j=k+\ell-1}^{k-1} x_j' Q(\alpha(j))x_j, \quad (13)$$

where $\alpha(k)$ is given in (4) and matrices $P(\alpha(k))$ and $Q(\alpha(k))$ can assume a different value at each instant k as a function of the switching function σ_k. Note that (13) is defined only for $\bar{d} > \underline{d}$.

3. MAIN RESULTS

Initially, convex conditions to solve Problem 4 are presented. Then, convex conditions for the design of state feedback gains $K(\alpha(k))$ and $K_d(\alpha(k))$ are given. In both cases, the Lyapunov-Krasovskii candidate functional is exploited to obtain less conservative conditions than those available in the literature. The following Lyapunov-Krasovskii candidate matrices are used:

$$P(\alpha(k)) = \sum_{i=1}^{\kappa} \alpha_i(k)P_i \quad (14)$$

$$Q(\alpha(k)) = \sum_{i=1}^{\kappa} \alpha_i(k)Q_i \quad (15)$$

Observe that these switched matrices are depend on σ_k by (4).

3.1 Robust Stability Analysis

Theorem 6. The switched time-varying delay system (1)-(5) with $u_k = \mathbf{0}$ is stable for arbitrary switching function σ_k, if there exist symmetric matrices $\mathbf{0} < P_i \in \mathbb{R}^{n\times n}$, $\mathbf{0} < Q_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, \kappa$, and a scalar $\beta = \bar{d} - \underline{d} +$

1, with $\underline{d}$ and $\bar{d}$ known, such that one of the following equivalent conditions is verified

a)

$$\Gamma(i,j,\ell) \equiv \begin{bmatrix} A_i' P_j A_i + \beta Q_i - P_i & A_i' P_j A_{di} \\ \star & A_{di}' P_j A_{di} - Q_\ell \end{bmatrix} < \mathbf{0}, \quad (i,j,\ell) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I} \tag{16}$$

b)

$$\Psi(i,j,\ell) \equiv \begin{bmatrix} -P_j & P_j A_i & P_j A_{di} \\ \star & \beta Q_i - P_i & \mathbf{0} \\ \star & \star & -Q_\ell \end{bmatrix} < \mathbf{0}, \quad (i,j,\ell) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I} \tag{17}$$

c) There exist matrices $F_i \in \mathbb{R}^{n\times n}$, $G_i \in \mathbb{R}^{n\times n}$ and $H_i \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,\kappa$ such that

$$\Omega(i,j,\ell) \equiv \begin{bmatrix} P_j + F_i' + F_i & G_i' - F_i A_i & H_i' - F_i A_{di} \\ \star & \beta Q_i - P_i - A_i' G_i' - G_i A_i & -A_i' H_i' - G_i A_{di} \\ \star & \star & -(Q_\ell + H_i A_{di} + A_{di}' H_i') \end{bmatrix} < \mathbf{0}, \quad (i,j,\ell) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I} \tag{18}$$

Besides this, (10)-(13) with (14)-(15) is a Lyapunov-Krasovskii functional for considered autonomous system.

Proof. The positivity of the functional (10) is clearly assured with the conditions of $P(\alpha(k)) = P(\alpha(k))' > \mathbf{0}$ and $Q(\alpha(k)) = Q(\alpha(k))' > \mathbf{0}$. For (10) be a Lyapunov-Krasovskii functional, besides its positivity, it is necessary to verify

$$\Delta V(\alpha(k), k) < 0, \ \forall \ [x_k' \ \ x_{k-d_k}']' \neq \mathbf{0} \tag{19}$$

From hereafter, the $\alpha(k)$ dependency is omitted in the argument of $V_v(k)$, $v = 1,\ldots,3$, for simplicity of the notation. To calculate (19), consider

$$\Delta V_1(k) = x_{k+1}' P(\alpha(k+1)) x_{k+1} - x_k' P(\alpha(k)) x_k \tag{20}$$

$$\Delta V_2(k) = x_k' Q(\alpha(k)) x_k - x_{k-d_k}' Q(\alpha(k-d_k)) x_{k-d_k} + \sum_{i=k+1-d(k+1)}^{k-1} x_i' Q(\alpha(i)) x_i - \sum_{i=k+1-d_k}^{k-1} x_i' Q(\alpha(i)) x_i \tag{21}$$

and

$$\Delta V_3(k) = (\bar{d} - \underline{d}) x_k' Q(\alpha(k)) x_k - \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i' Q(\alpha(i)) x_i \tag{22}$$

Observe that the third term in (21),

$$\Xi_k \equiv \sum_{i=k+1-d(k+1)}^{k-1} x_i' Q(\alpha(i)) x_i,$$

can be rewritten as

$$\begin{aligned} \Xi_k &= \sum_{i=k+1-\underline{d}}^{k-1} x_i' Q(\alpha(i)) x_i + \sum_{i=k+1-d(k+1)}^{k-\underline{d}} x_i' Q(\alpha(i)) x_i \\ &\leq \sum_{i=k+1-d_k}^{k-1} x_i' Q(\alpha(i)) x_i + \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i' Q(\alpha(i)) x_i \end{aligned} \tag{23}$$

Using (23) in (21), one gets

$$\Delta V_2(k) \leq x_k' Q(\alpha(k)) x_k - x_{k-d_k}' Q(\alpha(k-d_k)) x_{k-d_k} + \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i' Q(\alpha(i)) x_i \tag{24}$$

So, taking into account (20), (22) and (24) the following upper bound for (19) can be obtained

$$\begin{aligned} \Delta V(k) \leq\ & x_{k+1}' P(\alpha(k+1)) x_{k+1} \\ & + x_k' [\beta Q(\alpha(k)) - P(\alpha(k))] x_k \\ & - x_{k-d_k}' Q(\alpha(k-d_k)) x_{k-d_k} < 0 \end{aligned} \tag{25}$$

Considering (14)-(15) and (4), matrices $P(\alpha(k))$, $Q(\alpha(k))$, $P(\alpha(k+1))$, $Q(\alpha(k-d_k))$ are replaced by P_i, Q_i, P_j, Q_ℓ, respectively, with $i,j,\ell \in \mathcal{I}$. Replacing x_{k+1} in (25) by the right hand side of (1) with $u_k = \mathbf{0}$, one gets (16). The equivalence between (16) and (17) can be established as follows. First, note that (16) can be rewritten as

$$\Gamma(i,j,\ell) = \Pi_{ij}' P_j^{-1} \Pi_{ij} - \begin{bmatrix} P_i - \beta Q_i & \mathbf{0} \\ \mathbf{0} & Q_\ell \end{bmatrix} < \mathbf{0} \tag{26}$$

with $\Pi_{ij} = [P_j A_i \ P_j A_{di}]$, which by Schur complement is equivalent to (17). Therefore, the equivalence between $\Gamma(i,j,\ell) < \mathbf{0}$ and $\Psi(i,j,\ell) < \mathbf{0}$ has been stablished. So, if (17) is verified, then (18) is true for $F_i = F_i' = -P_j$, $G_i = H_i = \mathbf{0}$. On the other hand, if (18) is verified, then $\Gamma(i,j,\ell) = T_i' \Omega(i,j,\ell) T_i$ with

$$T_i = \left[\begin{array}{c} A_i \ A_{di} \\ \hline \mathbf{I}_{2n} \end{array}\right]$$

completing the proof.

Observe that, Theorem 6 can deal with system defined by $A(\alpha_k)$ and $A_d(\alpha(k))$ as well as with its dual given by $A(\alpha(k))'$ and $A_d(\alpha(k))'$, respectively. Also, note that Theorem 6 is a delay-independent condition encompassing the case where the delay is constant, i.e. $\beta = 1$ for $\underline{d} = \bar{d}$. However, the conditions presented here seems similar to those presented in Montagner et al. [2005], but in this last, besides the constant delay, only matrix $P(\alpha(k))$ depends on the switching function σ_k. Therefore, the conditions presented in Montagner et al. [2005] lead to more conservative results, in general.

Another relevant issue of Theorem 6 is that its LMI conditions encompass the results of a quadratic stability based approach, i.e., with constant and σ_k-independent matrices in the Lyapunov-Krasovskii candidate functional. Indeed, quadratic stability conditions can be recovered from the particular choice $P_i = P$ and $Q_i = Q$. Although this may seem to be a straightforward simplification, the resulting LMIs obtained from (17) and (18) lead to very different design conditions. Thus, only these two conditions are presented in the next corollary.

Corollary 7. The switched time-varying delay system (1)-(5) with $u(\alpha(k)) = \mathbf{0}$ is quadratically stable for any arbitrary switching function σ_k, if there exist symmetric matrices $\mathbf{0} < P \in \mathbb{R}^{n\times n}$, $\mathbf{0} < Q \in \mathbb{R}^{n\times n}$, and a scalar $\beta = \bar{d} - \underline{d} + 1$, with $\underline{d}$ and $\bar{d}$ known, such that one of the following equivalent conditions is verified

a)

$$\begin{bmatrix} -P & PA_i & PA_{di} \\ \star & \beta Q - P & \mathbf{0} \\ \star & \star & -Q \end{bmatrix} < \mathbf{0}, \quad i \in \mathcal{I} \tag{27}$$

b) There exist matrices $F_i \in \mathbb{R}^{n\times n}$, $G_i \in \mathbb{R}^{n\times n}$ and $H_i \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,\kappa$ such that

$$\begin{bmatrix} P + F_i' + F_i & G_i' - F_iA_i & H_i' - F_iA_{di} \\ \star & \beta Q - P - A_i'G_i' - G_iA_i & -A_i'H_i' - G_iA_{di} \\ \star & \star & -(Q + H_iA_{di} + A_{di}'H_i') \end{bmatrix} < \mathbf{0}, \quad i \in \mathcal{I} \quad (28)$$

Besides this, (10)-(13) with $P(\alpha(k)) = P$ and $Q(\alpha(k)) = Q$ is a Lyapunov-Krasovskii functional for considered autonomous system.

3.2 Robust Stabilization

Convex conditions are derived from Theorem 6 to design robust state feedback gains $K(\alpha(k))$ and $K_d(\alpha(k))$ for (6) providing a solution to Problem 5.

Theorem 8. If there exist symmetric matrices $\mathbf{0} < P_i \in \mathbb{R}^{n\times n}$, $\mathbf{0} < Q_i \in \mathbb{R}^{n\times n}$, matrices $F_i \in \mathbb{R}^{n\times n}$, $W_i \in \mathbb{R}^{n\times \ell}$ and $W_{di} \in \mathbb{R}^{n\times \ell}$, $i = 1,\ldots,\kappa$, and a scalar $\beta = \bar{d} - \underline{d} + 1$, with $\underline{d}$ and $\bar{d}$ known, such that

$$\Theta(i,j,\ell) \equiv \begin{bmatrix} P_j + F_i' + F_i & -W_iB_i' - F_iA_i' & -W_{di}B_i' - F_iA_{di}' \\ \star & \beta Q_i - P_i & \mathbf{0} \\ \star & \star & -Q_\ell \end{bmatrix} < \mathbf{0}, \quad (i,j,\ell) \in \mathcal{I}\times\mathcal{I}\times\mathcal{I} \quad (29)$$

then the switched system with time-varying delay (1) is robustly stabilizable by the control law (6) with

$$K_i = W_i'(F_i')^{-1} \quad \text{and} \quad K_{di} = W_{di}'(F_i')^{-1} \quad (30)$$

Besides this, (10)-(13) with $P(\alpha(k)) = P$ and $Q(\alpha(k)) = Q$ is a Lyapunov-Krasovskii functional for resulting switched closed-loop system (7).

Proof. The proof can be obtained by replacing $A(\alpha(k))$ and $A_d(\alpha(k))$ by $\tilde{A}_i'$ and $\tilde{A}_{di}'$ given in (8) and (9), respectively, imposing $G_i = H_i = \mathbf{0}$ and making the changing of variables $W_i = F_iK_i'$ and $W_{di} = F_iK_{di}'$.

Note that, in case where the delay value is not known, i.e., when x_{k-d_k} is not available for feedback, then condition (29) can be used with $W_{di} = \mathbf{0}$, $i = 1,\ldots,\kappa$. A quadratic stability condition can be derived from (29) by imposing $P_i = P = P'$. Thus, whenever the time-varying system (1) is quadratically stabilizable, it is also robustly stabilizable through condition of Theorem 8. The following two corollaries are presented without proofs. They state different quadratic conditions for the design of state feedback gains $K(\alpha(k))$ and $K_d(\alpha(k))$.

Corollary 9. If there exist symmetric matrices $\mathbf{0} < P \in \mathbb{R}^{n\times n}$, $\mathbf{0} < Q \in \mathbb{R}^{n\times n}$, matrices $F_i \in \mathbb{R}^{n\times n}$, $W_i \in \mathbb{R}^{n\times \ell}$ and $W_{di} \in \mathbb{R}^{n\times \ell}$, $i = 1,\ldots,\kappa$, and a scalar $\beta = \bar{d} - \underline{d} + 1$, with $\underline{d}$ and $\bar{d}$ known, such that

$$\begin{bmatrix} P + F_i' + F_i & -W_iB_i' - F_iA_i' & -W_{di}B_i' - F_iA_{di}' \\ \star & \beta Q - P & \mathbf{0} \\ \star & \star & -Q \end{bmatrix} < \mathbf{0}, \quad i \in \mathcal{I} \quad (31)$$

then the switched system with time-varying delay (1) is quadratically stabilizable by the control law (6) with (30). Besides this, (10)-(13) with $P(\alpha(k)) = P$ and $Q(\alpha(k)) = Q$ is a Lyapunov-Krasovskii functional for resulting switched closed-loop system (7).

Corollary 10. If there exist symmetric matrices $\mathbf{0} < P \in \mathbb{R}^{n\times n}$, $\mathbf{0} < Q \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,\kappa$, matrices $W \in \mathbb{R}^{n\times \ell}$ and $W_d \in \mathbb{R}^{n\times \ell}$ and a scalar $\beta = \bar{d} - \underline{d} + 1$, with $\underline{d}$ and $\bar{d}$ known, such that

$$\begin{bmatrix} -P & WB_i' + PA_i' & W_dB_i' + PA_{di}' \\ \star & \beta Q - P & \mathbf{0} \\ \star & \star & -Q \end{bmatrix} < \mathbf{0}, \ i \in \mathcal{I} \quad (32)$$

then the switched system with time-varying delay (1) is quadratically stabilizable by the control law (6) with

$$K(\alpha(k)) = K = W'(F')^{-1} \quad (33)$$

and

$$K_d(\alpha(k)) = K_d = W_d'(F')^{-1} \quad (34)$$

Besides this, (10)-(13) with $P(\alpha(k)) = P$ and $Q(\alpha(k)) = Q$ is a Lyapunov-Krasovskii functional for resulting switched closed-loop system (7).

An important remark is that Corollary 9 encompasses Corollary 10, since if (32) is verified, condition (31) is also verified with $F_i = -P$, $W_i = -W$ and $W_{di} = -W_d$, $i = 1,\ldots,\kappa$. Therefore, Corollary 9 is an important issue in the context of quadratic stability approach, since it allows to design parameter dependent state feedback gains. This is a more general condition than those with constant state feedback gains as in Corollary 10 and in [Montagner et al., 2005, Corollary 2].

Also observe that, decentralized control can be casted in conditions presented in Theorem 8 and Corollary 9 by imposing block-diagonal structure to matrices

$$F_i = F_{Di} = \text{block-diag}\{F_i^1,\ldots,F_i^\eta\},$$
$$W_i = W_{Di} = \text{block-diag}\{W_i^1,\ldots,W_i^\eta\},$$
$$W_{di} = W_{dDi} = \text{block-diag}\{W_{di}^1,\ldots,W_{di}^\eta\},$$

$i = 1,\ldots,\kappa$, where η denote the number of subsystems defined. In this case, it is possible to get robust block-diagonal state feedback gains $K_{Di} = W_{Di}'(F_{Di}')^{-1}$ and $K_{dDi} = W_{dDi}'(F_{Di}')^{-1}$, $i = 1,\ldots,\kappa$. In this case, the matrices of the Lyapunov-Krasovskii functional, $P(\alpha_k)$, $Q(\alpha_k)$, do not have any restrictions in their structures, which results in less conservative design conditions if no extra variable is employed. Observe that conditions of Corollary 10 could be used to deal with decentralized control, but in this case, the block-diagonal structure must be imposed to Z, Z_d and directly over matrices P and Q yielding, in general, more conservative results.

It is worth to mention that the combinations $(i,j,\ell) \in \mathcal{I}\times\mathcal{I}\times\mathcal{I}$ considered in theorems 6 and 8 can be simplified if the considered system does not take all possible combinations of transitions. However, for sake of space, this is not presented here.

Note that, the conditions proposed in Montagner et al. [2005] can be obtained as special cases of those presented here. For this, it is enough to choose $\underline{d} = \bar{d} = d$ and to impose $Q_i = S$, $i = 1,\ldots,\kappa$. Note that, in this case, the functional V_3 given in (13) is not defined and, thus, $V(\alpha(k),k) = V_1(\alpha(k),k) + V_2(\alpha(k),k)$ which is the same employed in Montagner et al. [2005]. Therefore the proposed conditions represent an improvement in the available tools for dealing with switched discrete-time systems with time-varying delay.

A final note on the design conditions is that LMIs in (29) and (31) can be used to design constant gains $K(\alpha(k)) = K$ and $K_d(\alpha(k)) = K_d$ by imposing $F_i = F$, $W_i = W$ and $W_{di} = W_d$, $i = 1, \ldots, \kappa$.

3.3 Numerical complexity

The numerical complexity of the conditions presented in this paper can be determined by the number of scalar variables, $\mathcal{K}$, and the number of rows, $\mathcal{R}$, involved in the optimization problems. In case of using *LMI Control Toolbox* Gahinet et al. [1995], the numerical complexity is $\mathcal{O}(\mathcal{K}^3\mathcal{R})$ and using the solver SeDuMi Sturm [1999] the numerical complexity is $\mathcal{O}(\mathcal{K}^2\mathcal{R}^{2.5} + \mathcal{R}^{3.5})$. Note that, nowadays efficient algorithms can solve the conditions presented here in polynomial time. The number of scalar variables and the number of LMI rows of the feasibility tests proposed in this paper are presented in Table 1.

Condition	$\mathcal{K}$	$\mathcal{R}$
Theorem 6.**a)**	$\kappa n(n+1)$	$2\kappa n(\kappa^2+1)$
Theorem 6.**b)**	$\kappa n(n+1)$	$3\kappa^3 n$
Theorem 6.**c)**	$\kappa n(3n^2+n+1)$	$\kappa n(3\kappa^2+2)$
Corollary 7.**a)**	$n(n+1)$	$3\kappa n$
Corollary 7.**b)**	$(3\kappa+1)n^2+n$	$n(3\kappa+2)$
Theorem 8	$\kappa n(2n+2p+1)$	$3\kappa^3 n$
Corollary 9	$n(2n+2\kappa p+1)$	$3\kappa n$
Corollary 10	$n(n+2p+1)$	$3\kappa n$

Table 1. Number of scalar variables ($\mathcal{K}$) and LMI rows ($\mathcal{R}$) for the proposed conditions

4. NUMERICAL EXAMPLE

In this section an example is given to illustrate the efficacy of the proposed LMI conditions. The example deals with the Problem 5 discussed in this paper.

Example 11. Consider the switched discrete-time system with time varying delay described by (1) where where $A(\sigma_k) = A_n + (-1)^{\sigma_k}\rho L'J$, $A_d(\sigma_k) = (0.225 + (-1)^{\sigma_k}0.025)A_n$ and

$$B(\sigma_k) = [0\ 1.5\ 0\ 1.5]' + (-1)^{\sigma_k}[0\ 0.5\ 0\ 0.5]'$$

with

$$A_n = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.03 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (35)$$

$L = [0,\ 0,\ 1,\ 0]'$, $J = [0.8,\ -0.5,\ 0,\ 1]$, $\sigma_k \in \{1,\ 2\}$, $\rho = 0.35$. This defines a switched system with 2 submodes. Note that, even for $\underline{d} = \bar{d} = 1$, conditions from Theorem 6 fail to identify this system as a stable one. Supposing $\underline{d} = 1$, the objective here is to search the maximum value of $\bar{d}$ such that the considered system is stabilizable. Since the delay is time-varying, conditions presented in Montagner et al. [2005], Phat [2005] and Yu et al. [2007] cannot be applied. Two alternatives are taken into account. Firstly, consider that only x_k is available for feedback, i.e., $K_d = \mathbf{0}$. In this case, Corollary 10 can yields a feasible solutions for $\underline{d} \le \bar{d} = 8$. For $\bar{d} = 8$, it yields

$$K = [\,-0.0537\ 0.1111\ -1.1188\ -0.4768\,]$$

On the other hand, Corollary 9 and Theorem 8 achieve feasible solutions for $\bar{d} = 15$ with the following gains

$$K_{\mathrm{C9},1} = [\,0.1218\ 0.0475\ -1.6331\ -0.4745\,]$$
$$K_{\mathrm{C9},2} = [\,-0.1488\ 0.1548\ -0.8166\ -0.4994\,]$$
$$K_{\mathrm{Th8},1} = [\,0.1215\ 0.0475\ -1.6326\ -0.4744\,]$$
$$K_{\mathrm{Th8},2} = [\,-0.1494\ 0.1551\ -0.8168\ -0.5002\,]$$

Now, consider that both x_k and x_{k-d_k} are available for feedback. In this case, Corollary 10 can be used to obtain constant feedback gains given by

$$K = [\,-0.0233\ 0.0772\ -1.0140\ -0.3503\,]$$

and

$$K_d = [\,-0.0341\ 0.0318\ -0.2130\ -0.1300\,]$$

for $1 \le d_k \le 21$. If Corollary 9 is used, then it can be verified feasible solutions for $1 \le d_k \le 333$, using K_{σ_k} and K_{d,σ_k}. Theorem 8 can reaches a little better result, achieving feasible solutions for $1 \le d_k \le 335$. In this case, with $\bar{d} = 335$, conditions of Theorem 8 lead to

$$K_1 = [\,-0.6129\ 0.3269\ -1.2873\ -1.1935\,] \quad (36)$$
$$K_2 = [\,-0.2199\ 0.1107\ -0.6450\ -0.4890\,] \quad (37)$$
$$K_{d1} = [\,-0.1291\ 0.0677\ -0.3228\ -0.2685\,] \quad (38)$$
$$K_{d2} = [\,-0.0518\ 0.0271\ -0.1291\ -0.1076\,] \quad (39)$$

Thus, it is clear that the use of switched gains leads to less conservative results than when constant gains are employed in the feedback control law. Also, it has been shown that conditions state in Theorem 8 can lead to less conservative results. This is achieved thanks to the switched Lyapunov-Krasovskii functional employed and to the extra matrices F_i, $i = 1, \ldots, \kappa$. This last condition is simulated and the results are presented in what follows. Random signals for $\sigma_k \in \{1,\ 2\}$ and for $1 \le d(k) \le 335$ have been generated as indicated in Figure 1. These signals

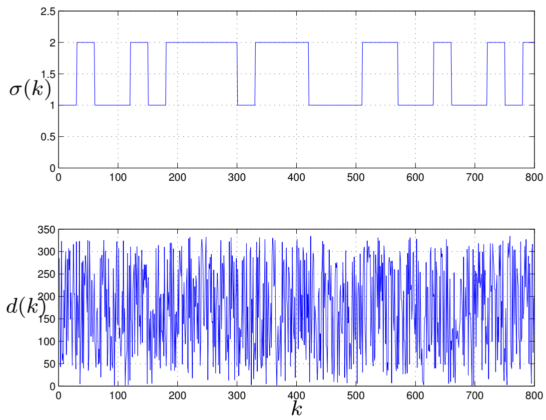

Fig. 1. The switched rule, $\sigma(k)$ and the varying delay, $d(k)$.

have been used in the system considered in this example, with gains given in (36)-(39). An initial condition $x(k) = [1,\ -1,\ 1, -1]$, $k \in [-335,\ 0]$, has been used. Thus, it is expected that the delayed states degenerate the overall system response, at least for the first $\bar{d} = 335$ samples.

Note that, this initial condition is harder than the ones usually found in the literature. The state behavior of the switched closed-loop system with time-varying delay is presented in Figure 2. Observe that the initial value of the states are not presented due to the scale choice. Clearly,

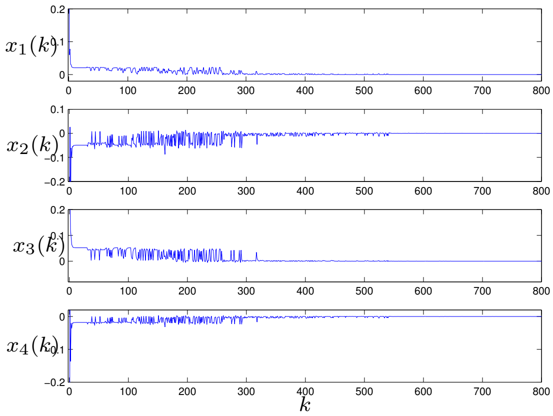

Fig. 2. The behaviors of the states $x_1(k)$ to $x_4(k)$, with $1 \leq d(k) \leq 335$ (see Fig. 1).

by the response behavior presented in Figure 2, the states are almost at the the equilibrium point after 300 samples. The control signal is presented in Figure 3. In the top of this figure, it is shown the control signal due to $K_{\sigma_k}x(k)$. In the bottom of this figure it is shown the control signal due to $K_{d\sigma_k}x(k-d(k))$. Observe that the total control signal applied to the system at each instant is given by $u(k) = u_1(k) + u_2(k) = K_{\sigma_k}x(k) + K_{d\sigma_k}x(k-d(k))$

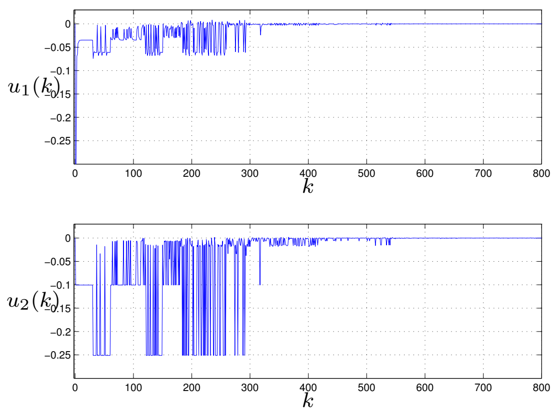

Fig. 3. Control signal $u(k) = u_1(k) + u_2(k)$, with $u_1(k) = K_i x(k)$ and $u_2(k) = K_{di}x(k-d(k))$, $i = \sigma_k \in \{1,\ 2\}$.

5. CONCLUSIONS

Convex conditions have been presented for both stability analysis and control design for switched discrete-time systems with time-varying delay. It has been used a Lyapunov-Krasovskii functional depending on an arbitrary switching function. The LMI conditions proposed here encompass quadratic stability based approach and other results in the literature. A design problem is developed, including time-simulation, to illustrate the efficacy of the proposed LMI conditions.

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