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Design of ISS-Lyapunov Functions for Discrete-Time Linear Uncertain Systems

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Abstract: In this work we consider robust control of discrete-time linear systems affected by time-varying additive disturbance inputs. We present a linear matrix inequality (LMI) based design technique that takes into account in an explicit manner, by means of a Minkowski function, the shape of the set in which the disturbances are bounded. This technique allows one to obtain tight bounds on the performance of the closed-loop system.

Keywords: Robust control; Robust controller synthesis; LMI optimization based controller synthesis.

1. INTRODUCTION

One practically relevant problem in control theory is the robust regulation towards a desired equilibrium of linear discrete-time systems affected by time-varying additive disturbance inputs. The input-to-state (ISS) stability notion, originally proposed in Sontag [1989, 1990], provides a natural framework that has been successfully employed in the stability analysis and control synthesis of a wide class of systems subject to disturbances.

ISS theory has been applied to discrete-time systems in Jiang and Wang [2001]. In this work, the notions of ISS stability and ISS-Lyapunov functions for discrete-time systems were introduced. However, there are few results in the literature that provide design methods to obtain control laws that guarantee that the closed-loop system is ISS with respect to a given disturbance. For linear systems subject to additive disturbances, it is possible to use H_∞ design techniques, see for example Kaminer et al. [1993], Chen and Scherer [2006], Limebeer et al. [1989], Lie and I. [1996], Doyle et al. [1989], to obtain controllers that guarantee that the closed-loop systems is ISS. This approach is appropriate for disturbances bounded in an ellipsoidal set. However, in many practical control applications the disturbances are bounded inside a polyhedral set (for example when lower and upper bounds on the disturbances are provided) to which this approach does not fit. One could overapproximate the polyhedron by an ellipsoidal set, but clearly this leads to conservative results.

In this work we consider robust control of discrete-time linear systems affected by time-varying additive disturbance inputs. We present a linear matrix inequality (LMI) based design technique that computes ISS gains and ISS stabilizing controllers using different norms to measure disturbance. In this way, the shape of the set in which the disturbances are bounded can be accounted for. The technique is based on defining a contractive constraint (that guarantees that the closed-loop system is ISS) that depends on the Minkowski functional, see Luenberger [1969], Blanchini [1994], of the set in which the disturbance is bounded. This method automatically provides better approximation of ultimate bounds given fixed polyhedron disturbance sets.

2. PRELIMINARY DEFINITIONS AND ASSUMPTIONS

A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

We now review the ISS framework for discrete-time autonomous systems introduced in Jiang and Wang [2001]. Consider the discrete-time autonomous perturbed system described by the following difference equation

$$x^+ = G(x, v), \tag{1}$$

where $x \in R^{n_x}$ is the state, $x^+ \in R^{n_x}$ is the one step predicted state vector, $v \in \mathcal{V} \subset R^{n_v}$ is an unknown time-varying disturbance input and $G : R^{n_x} \times R^{n_v}$ is an arbitrary, possibly discontinuous, function. In what follows we assume that $\mathcal{V}$ is a bounded set.

Definition 1. System (1) is said to be ISS with respect to v if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(0)$ and any bounded disturbance $v(k)$, the solution $x(k)$ exists for all $k \geq 0$ and satisfies

$$\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma\left(\sup_{0 \leq l \leq k-1} \|v(l)\|\right).$$

Definition 2. A function $V : R^{n_x} \to R_{\geq 0}$ is called an ISS-Lyapunov function for system (1) if there exist class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and a class $\mathcal{K}$ function σ such that

$$\begin{aligned} \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \\ V(G(x,v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|). \end{aligned} \tag{2}$$

for all $x \in R^{n_x}$ and $d \in R^{n_v}$.

Lemma 1. (Jiang and Wang [2001], Lazar [2006]). If system (1) admits an ISS-Lyapunov function, then it is ISS.

We denote by $\partial\mathcal{P}$ the boundary of a set $\mathcal{P}$. For any real $\lambda \geq 0$, the set $\lambda\mathcal{P}$ is defined as $\{x = \lambda y : y \in \mathcal{P}\}$. Let us introduce the following definition, see Blanchini [1994].

Definition 3. A function $\Psi : R^n \to R$ is said to be a gauge function if

- $\Psi(x + y) \leq \Psi(x) + \Psi(y)$ for all $x, y \in R^n$,
- $\Psi(x) \geq 0$,
- $\Psi(x) = 0 \Rightarrow x = 0$,
- $\Psi(\lambda x) = \lambda\Psi(x)$ for all $\lambda > 0$.

A gauge function is 0-symmetric if and only if it is a norm. In this work, we consider gauge functions in order to deal with possibly nonsymmetric sets. We define the ball of radius ξ with respect to Ψ as $\mathcal{B}_\xi = \{x \in R^{n_x} : \Psi(x) \leq \xi\}$.

The Minkowski functional Φ (see Luenberger [1969]) of a closed set $\mathcal{P}$ that contains the origin in its interior is defined as follows:

$$\Phi_{\mathcal{P}}(x) = \begin{cases} 0 \text{ if } x = 0 \\ \xi > 0, \text{ where } x \in \partial(\xi\mathcal{P}) \text{ otherwise.} \end{cases}$$

The Minkowski functional Φ is a gauge function and $\Phi_{\mathcal{P}}(x)$ is the value by which $\partial\mathcal{P}$ has to be scaled in order to include x. This implies that (due to the closedness of $\mathcal{P}$) $x \in \mathcal{P} \Leftrightarrow \Phi(x) \leq 1$.

3. ROBUST FEEDBACK DESIGN

Consider the following discrete-time linear system subject to bounded disturbances defined by the following difference equation

$$x^+ = Ax + Bu + Ev \tag{3}$$

where $x \in R^{n_x}$ is the state, $x^+ \in R^{n_x}$ is the one step predicted state vector, $u \in R^{n_u}$ is the input vector and $v \in \mathcal{V} \subset R^{n_v}$ is the disturbance input which is bounded inside a closed polyhedron $\mathcal{V}$ that contains the origin in its interior.

The main objective of this work is to design a feedback law $u = -Kx$ for system (3) such that the closed-loop system is guaranteed to be ISS with respect to the disturbance input v. To this end, given matrices $Q > 0, R > 0$, consider a linear feedback gain K, a positive definite matrix P and a positive constant γ such that the following inequality is satisfied:

$$V(x^+) - V(x) \leq -L(x, -Kx) + \gamma d(v)^2 \tag{4}$$

for all $x \in R^{n_x}$ and $v \in R^{n_v}$, where

$$\begin{aligned} V(x) &= x^T P x, \\ L(x,u) &= x^T Q x + u^T R u, \end{aligned}$$

and $d(v) = \Phi_{\mathcal{V}}(v)$ is the Minkowski functional of $\mathcal{V}$. Using the Minkowski functional to define this ISS constraint allows one to take explicitly into account the shape of $\mathcal{V}$ in the design technique. Inequality (4) guarantees that system (3) in closed-loop with the feedback control $u = -Kx$ is ISS with respect to v. This is proved in the following lemma.

Lemma 2. If $P > 0, K$ satisfy (4) for a positive constant γ then $V(x) = x^T P x$ is an ISS-Lyapunov function of system (3) in closed-loop with $u = -Kx$.

Proof. If d is a norm, i.e., $\mathcal{V}$ is zero-symmetric, it is easy to see that there exist class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and a class $\mathcal{K}$ function σ such that (2) is satisfied.

If d is not a norm, consider the function $\hat{d}(v) := \max\{d(v), d(-v)\}$. This function satisfies $d(v) \leq \hat{d}(v)$ for all $v \in R^{n_v}$. This implies that

$$V(x^+) - V(x) \leq -L(x, -Kx) + \gamma\hat{d}(v)^2.$$

Next we prove that $\hat{d}(v)$ is a norm. This implies that $\hat{d}(v)$ satisfies the following properties

- $\hat{d}(x + y) \leq \hat{d}(x) + \hat{d}(y)$ for all $x, y \in R^n$,
- $\hat{d}(x) \geq 0$,
- $\hat{d}(x) = 0 \to x = 0$,
- $\hat{d}(\lambda x) = |\lambda|\hat{d}(x)$ for all $\lambda \in R$.

The first property follows from:

$$\begin{aligned} \hat{d}(x+y) &= \max\{d(x+y), d(-x-y)\} \\ &\leq \max\{d(x) + d(y), d(-x) + d(-y)\} \\ &\leq \max\{d(x), d(-x)\} + \max\{d(y), d(-y)\} \\ &= \hat{d}(x) + \hat{d}(y). \end{aligned}$$

The second and third of the above properties hold by definition. Finally, taking into account that $\lambda \in R$, the fourth property holds as follows:

$$\begin{aligned} \hat{d}(\lambda x) &= \max\{d(\lambda x), d(-\lambda x)\} \\ &= \max\{|\lambda| d(x), |\lambda| d(-x)\} \\ &= |\lambda| \max\{d(x), d(-x)\} \\ &= |\lambda|\hat{d}(x). \end{aligned}$$

This implies that there exist class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and a class $\mathcal{K}$ function σ such that (2) is satisfied. □

Next, a LMI design technique to obtain matrices K, P and positive constant γ such that (4) is satisfied for a given system is presented. This technique takes into account the shape of the polyhedron in which the disturbance is bounded. This is possible because the ISS inequality (4) depends upon the Minkowski functional of the set $\mathcal{V}$ and hence, it depends on the shape of the set $\mathcal{V}$. This is the main contribution of this paper and it is presented in the following theorem.

Theorem 3. Consider system (3). If matrices there exist matrices W, Y and a constant γ such that the following inequality

$$\begin{bmatrix} \gamma & 0 & v_i^T E^T & 0 & 0 \\ * & W & WA^T - Y^T B^T & WQ^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ * & * & W & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} \geq 0, \tag{5}$$

is satisfied for all vertices v_i of $\mathcal{V}$, then (4) is satisfied for $P = W^{-1}$, $K = YW^{-1}$ and γ.

Proof. Taking into account the definition of V and L, inequality (4) is equivalent to

$$(Ax + Bu + Ev)^T P(Ax + Bu + Ev) - x^T Px \leq -x^T Qx - u^T Ru + \gamma d(v)^2,\ \forall x \in R^{n_x}, v \in R^{n_v}.$$

Substituting $u = -Kx$ we obtain

$$\Gamma(x, v) + v^T E^T PEv - \gamma d(v)^2 \leq 0,\ \forall x \in R^{n_x}, v \in R^{n_v},$$

where $\Gamma(x, v)$ is a quadratic function of x and v, of the form

$$\Gamma(x, v) = x^T((A - BK)^T P(A - BK) + Q + K^T RK - P)x + 2v^T E^T P(A - BK)x.$$

Inequality (4) must be satisfied for all x, so it must hold for the maximum value of $\Gamma(x, u)$ for any given v. In order to evaluate the maximum value for a given v, from now on we assume [1] that

$$S = (A - BK)^T P(A - BK) + Q + K^T RK - P < 0.$$

This assumption implies that for a fixed $v \in R^{n_v}$, $\Gamma(x, v)$ is a concave function of x and that the maximum for v is attained at the point $x^*(v)$ such that $x^*(v) \in \{x \in R^{n_x} | \frac{\partial \Gamma(x,v)}{\partial x} = 0\}$. Operating we obtain that

$$x^*(v) = -S^{-1}(A - BK)^T PEv$$

and

$$\Gamma(x^*(v), v) = -v^T E^T P(A - BK)S^{-1}(A - BK)^T PEv.$$

Therefore inequality (4) holds for all $x \in R^{n_x}$ and $v \in R^{n_v}$ if

$$-v^T E^T P(A - BK)S^{-1}(A - BK)^T PEv + v^T E^T PEv - \gamma d(v)^2 \leq 0,\ \forall v \in R^{n_v}.$$

This inequality is satisfied for $v = 0$. If $v \neq 0$ we divide the inequality by $d(v)^2$, which yields

$$-\frac{v}{d(v)}^T E^T P(A - BK)S^{-1}(A - BK)^T PE\frac{v}{d(v)} + \frac{v}{d(v)}^T E^T PE\frac{v}{d(v)} - \gamma \leq 0,\ \forall v \in R^{n_v}.$$

Taking into account that d is a gauge function we obtain the following equivalent inequality

$$-z^T E^T P(A - BK)S^{-1}(A - BK)^T PEz + z^T E^T PEz - \gamma \leq 0, \forall z \text{ such that } d(z) = 1.$$

Because $S < 0$ the right-hand side is a convex function in z, this implies that inequality is satisfied if it holds for all the vertices of the set $\mathcal{B}_1$, where $\mathcal{B}_1 = \{z \in R^{n_v} | d(z) = 1\}$ is the unit ball of d. It follows (recall the definitions of d and $\mathcal{V}$) that inequality (4) is satisfied holds for all $x \in R^{n_x}$ and $v \in R^{n_v}$ if

$$\begin{aligned} &-v_i^T E^T P(A - BK)S^{-1}(A - BK)^T PEv_i \\ &+v_i^T E^T PEv_i - \gamma \leq 0 \end{aligned} \tag{6}$$

for all vertices v_i of $\mathcal{V}$.

In what follows, we are going to manipulate this inequality to obtain (5). Applying Schur complement, inequality (6) is equivalent to

$$\begin{bmatrix} \gamma - v_i^T E^T PEv_i & -v_i^T E^T P(A - BK) \\ * & -S \end{bmatrix} \geq 0.$$

[1] This assumption will be immediately satisfied by the matrices P, K obtained following the proposed design method.

Taking into account the definition of S we obtain

$$\begin{bmatrix} \gamma & 0 \\ * & P - Q - K^T RK \end{bmatrix} - \begin{bmatrix} v_i^T E^T \\ (A - BK)^T \end{bmatrix} P\,[Ev_i\ \ (A - BK)] \geq 0.$$

Applying Schur complements we obtain

$$\begin{bmatrix} \gamma & 0 & v_i^T E^T \\ * & P - Q - K^T RK & (A - BK)^T \\ * & * & P^{-1} \end{bmatrix} \geq 0.$$

Following the same approach and applying twice the Schur complement it follows that (6) is equivalent to

$$\begin{bmatrix} \gamma & 0 & v_i^T E^T & 0 & 0 \\ * & P & (A - BK)^T & Q^{\frac{1}{2}} & K^T R^{\frac{1}{2}} \\ * & * & P^{-1} & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} \geq 0.$$

By pre- and post-multiplying with the positive definite matrix:

$$diag(I, P^{-1}, I, I, I)$$

we obtain

$$\begin{bmatrix} \gamma & 0 & v_i^T E^T & 0 & 0 \\ * & P^{-1} & P^{-1}(A - BK)^T & P^{-1}Q^{\frac{1}{2}} & P^{-1}K^T R^{\frac{1}{2}} \\ * & * & P^{-1} & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} \geq 0.$$

By applying the changes of variables $P = W^{-1}$ and $K = YW^{-1}$, we obtain (5). □

Using Theorem 3 it is possible to obtain matrices K, P such that the positive constant γ is minimized for a given system solving the following LMI optimization problem

$$\min_{W,Y,\gamma} \gamma \text{ subject to } (5) \tag{7}$$

and letting $P = W^{-1}$ and $K = YW^{-1}$.

Remark 1. The LMI constraint (3) depends on the vertices of the set $\mathcal{V}$ which defines the gauge function d of inequality (4). However, this inequality holds for all $v \in R^{n_v}$. This implies that regardless of the set $\mathcal{V}$ used to design K, P and γ, the trajectories of the state of the closed-loop system can be bounded using (4) for any disturbance realization.

Remark 2. The constant γ characterizes the set Ω_γ in which function V is not guaranteed to decrease for all $v \in \mathcal{V}$; that is,

$$\Omega_\gamma = \{x : L(x, -Kx) \leq \gamma\}.$$

This implies that minimizing γ, we obtain a linear feedback K that minimizes in an indirect way the size of the set in which the system is ultimately bounded. Note that the set is not bounded in Ω_γ, but in a set that contains Ω_γ, see for example Alamo et al. [2005] for details on this reasoning.

Remark 3. The linear feedback $u = -Kx$ and the cost function $V(x) = x^T Px$ obtained applying Theorem 3 can be used as the terminal cost and the corresponding local feedback in the design of robust MPC controllers. See Lazar et al. [2008], Limon et al. [2006].

Remark 4. The results obtained can be also applied to systems affected, possibly simultaneously, by time-varying parametric disturbances and additive disturbance inputs, defined by the following difference equation

$$x^+ = A(w)x + B(w)u + E(w)v,$$

where w is an unknown time-varying parametric uncertainty bounded in $\mathcal{W}$, where $\mathcal{W}$ is a compact polyhedron, $A(w), B(w), E(w)$ are affine functions of $w \in \mathcal{W}$ and there exists a $w \in \mathcal{W}$ such that the matrix $B(w)$ has full-column rank. In this case, the LMI constraint must hold for the combination of all the vertices of $\mathcal{W}$ and the vertices that define the polytopic uncertainties. The resulting LMI constraint is the following

$$\begin{bmatrix} \gamma & 0 & v_i^T E(w_j)^T & 0 & 0 \\ * & W & WA(w_j)^T - Y^T B(w_j)^T & WQ^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ * & * & W & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} > 0,$$

for all $v_i \in \text{vert } \mathcal{V}$ and $w_j \in \text{vert } \mathcal{W}$.

Remark 5. The proposed design technique is related to the H_∞ control problem, see Kaminer et al. [1993], Chen and Scherer [2006], Limebeer et al. [1989], Lie and I. [1996], Doyle et al. [1989] and the references therein. In the H_∞ design technique, the matrices P, K and the positive constant γ are obtained such that

$$V(x^+) - V(x) \leq -L(x, -Kx) + \gamma d(v)^2$$

for all v where $d(v)$ is the Minkowski functional of the set $v^T v \leq 1$, or equivalently, the 2-norm of the vector v. Note that for this set, the proposed approach cannot be applied because an ellipsoid is not defined by a finite set of vertices. This inequality leads to the following LMI, see for example Kaminer et al. [1993], Chen and Scherer [2006], Limebeer et al. [1989], Lie and I. [1996], Doyle et al. [1989],

$$\begin{bmatrix} \gamma I & 0 & E^T & 0 & 0 \\ * & W & WA^T - Y^T B^T & WQ^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ * & * & W & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} > 0. \qquad (8)$$

In this line of results, the disturbance is assumed to be bounded in a set defined by the unit-ball of the 2-norm. This assumption may lead to conservative estimates when the disturbance is bounded in a polyhedral set. The results presented in this paper help in this direction as they provide techniques better suited to polyhedral sets.

Remark 6. The results obtained can be applied to optimize not only ISS gains, but also induced norms (such as H_∞ in the ellipsoidal case). If $L(x, u) = x^T C^T C x$, where C is a matrix that defines an output $z = Cx$ of the system, it can be proved using a passivity type reasoning that the following inequality holds

$$\frac{\sum_{k=0}^{\infty} z(k)^T z(k)}{\sum_{k=0}^{\infty} d(v(k))} \leq \gamma$$

for all $\{d(k)\}$ where γ is the ISS gain obtained solving the proposed LMI optimization problem and $z(k)$ is the output trajectory of the system in closed-loop with the obtained linear feedback $u = Kx$ from from initial state zero for a disturbance trajectory $\{d(k)\}$.

4. EXAMPLE

Consider the discrete-time linear system (3) subject to bounded additive disturbances defined by the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

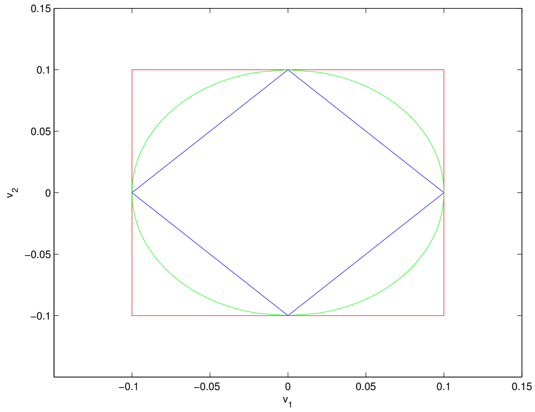

Fig. 1. Sets $0.1\mathcal{V}_1$ (blue), $0.1\mathcal{V}_\infty$ (red) and $0.1\mathcal{V}_2$ (green).

In this section we are going to obtain three different feedback laws and the corresponding ISS-Lyapunov functions. Each feedback law assumes that the disturbance is bounded in a closed set with a different shape.

First, we solve Problem (7) for $Q = I, R = 10$ and $\mathcal{V}_1 = \{v : \|v\|_1 \leq 1\}$. In this case, $d(v) = \|v\|_1$. The vertices of $\mathcal{V}_1$ are the following:

$$v_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, v_{13} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_{14} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

and the corresponding matrices are

$$P_1 = \begin{bmatrix} 4.2445 & 6.3364 \\ 6.3364 & 20.4440 \end{bmatrix},$$

$$K_1 = [0.3096 \; 1.3088],$$

$$\gamma_1 = 20.4643.$$

Next we solve Problem (7) for $Q = I, R = 10$ and $\mathcal{V}_\infty = \{v : \|v\|_\infty \leq 1\}$. In this case, $d(v) = \|v\|_\infty$. The vertices of $\mathcal{V}_\infty$ are the following:

$$v_{\infty 1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_{\infty 2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_{\infty 3} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_{\infty 4} = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

and the corresponding matrices are

$$P_\infty = \begin{bmatrix} 4.9827 & 8.1513 \\ 8.1513 & 23.3409 \end{bmatrix},$$

$$K_\infty = [0.4166 \; 1.4160],$$

$$\gamma_\infty = 44.6690.$$

Finally, we obtain the H_∞ feedback gain and corresponding ISS-Lyapunov function. In this case the gauge function is defined as the Minkowski functional of the set $\mathcal{V}_2$ defined as

$$\mathcal{V}_2 = \{v : \|v\|_2 \leq 1\}.$$

We solve Problem (7) for $Q = I, R = 10$ substituting the LMI inequality (4) by the H_∞ LMI (8), see Remark 5, and we obtain the following results

$$P_2 = \begin{bmatrix} 4.9710 & 7.7439 \\ 7.7439 & 22.5426 \end{bmatrix},$$

$$K_2 = [0.3771 \; 1.3757],$$

$$\gamma_2 = 25.5104.$$

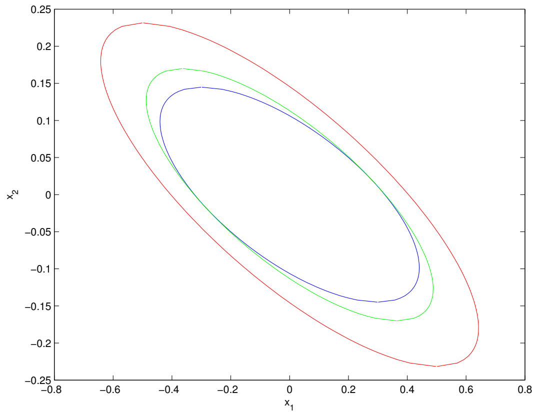

Fig. 2. Sets $L(x, -K_1 x) \leq \gamma_1 0.1^2$ (blue), $L(x, -K_\infty x) \leq \gamma_\infty 0.1^2$ (red) and $L(x, -K_2 x) \leq \gamma_2 0.1^2$ (green).

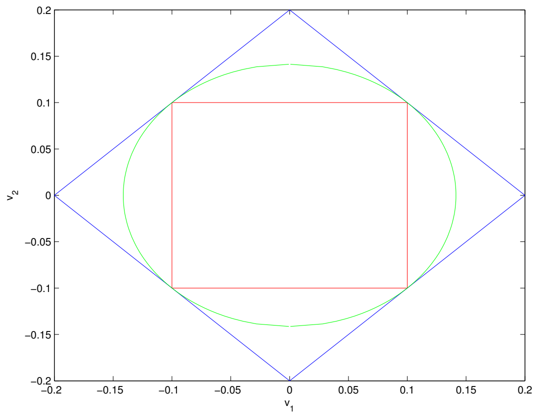

Fig. 3. Sets $0.2\mathcal{V}_1$ (blue), $0.1\mathcal{V}_\infty$ (red) and $0.1\sqrt{2}\mathcal{V}_2$ (green).

We have obtained three different feedback laws and the corresponding ISS-Lyapunov functions. Each of these feedback laws guarantee that (4) is satisfied for a given gauge function, namely,

$$V_1(x^+) - V_1(x) \leq -L(x, -K_1 x) + \gamma_1 d_1(v)^2,$$
$$V_\infty(x^+) - V_\infty(x) \leq -L(x, -K_\infty x) + \gamma_\infty d_\infty(v)^2,$$
$$V_2(x^+) - V_2(x) \leq -L(x, -K_2 x) + \gamma_2 d_2(v)^2.$$

Using these functions we can obtain bounds on the trajectories of the closed-loop system for a given closed set $\mathcal{V}$. In this example, we are going to consider two different cases.

Case 1: Assume that $\mathcal{V}$ is defined by $\{v : \|v\|_1 \leq 0.1\}$. In this case, the following inequalities hold

$$d_1(v) \leq 0.1, \ \forall v \in \mathcal{V},$$
$$d_\infty(v) \leq 0.1, \ \forall v \in \mathcal{V},$$
$$d_2(v) \leq 0.1, \ \forall v \in \mathcal{V}.$$

Although the value of the three gauge functions are equal, in Figure 1 it is shown that $0.1\mathcal{V}_1 \subseteq 0.1\mathcal{V}_2 \subseteq 0.1\mathcal{V}_\infty$. The set $\mathcal{V}$ is better approximated by $\mathcal{V}_1$. This implies that the feedback law $u = -K_1 x$ provides better bounds on the trajectories of the closed-loop system. If $u = -K_1 x$ is applied it is guaranteed that the function $V_1(x) = x^T P_1 x$ decreases for all initial states outside the ellipsoid $L(x, -K_1 x) \leq \gamma_1 0.1^2$. If $u = -K_\infty x$ is applied it is guaranteed that the function $V_\infty(x) = x^T P_\infty x$ decreases for all initial states outside the ellipsoid $L(x, -K_2 x) \leq \gamma_\infty 0.1^2$. If $u = -K_2 x$ is applied it is guaranteed that the function $V_2(x) = x^T P_2 x$ decreases for all initial states outside the ellipsoid $L(x, -K_2 x) \leq \gamma_2 0.1^2$. Figure 2 shows that the ellipsoid corresponding to K_1 is smaller than the one corresponding to K_∞ and K_2.

Case 2: Assume that $\mathcal{V}$ is defined by $\{v : \|v\|_\infty \leq 0.1\}$. In this case, the following inequalities hold

$$d_1(v) \leq 0.2, \ \forall v \in \mathcal{V},$$
$$d_\infty(v) \leq 0.1, \ \forall v \in \mathcal{V},$$
$$d_2(v) \leq 0.1\sqrt{2}, \ \forall v \in \mathcal{V}.$$

In Figure 3 it is shown that $0.1\mathcal{V}_\infty \subseteq 0.1\sqrt{2}\mathcal{V}_2 \subseteq 0.2\mathcal{V}_1$. The set $\mathcal{V}$ is better approximated by $\mathcal{V}_\infty$. This implies that the feedback law $u = -K_\infty x$ provides better bounds on the trajectories of the closed-loop system. If $u = -K_1 x$ is applied it is guaranteed that the function $V_1(x) = x^T P_1 x$ decreases for all initial states outside the ellipsoid $L(x, -K_1 x) \leq \gamma_1 0.2^2$. If $u = -K_\infty x$ is applied it is guaranteed that the function $V_\infty(x) = x^T P_\infty x$ decreases for all initial states outside the ellipsoid $L(x, -K_2 x) \leq \gamma_\infty 0.1^2$. If $u = -K_2 x$ is applied it is guaranteed that the function $V_2(x) = x^T P_2 x$ decreases for all initial states outside the ellipsoid $L(x, -K_2 x) \leq \gamma_2 (0.1\sqrt{2})^2$. Figure 4 shows that the ellipsoid corresponding to K_∞ is smaller than the one corresponding to K_1 and K_2.

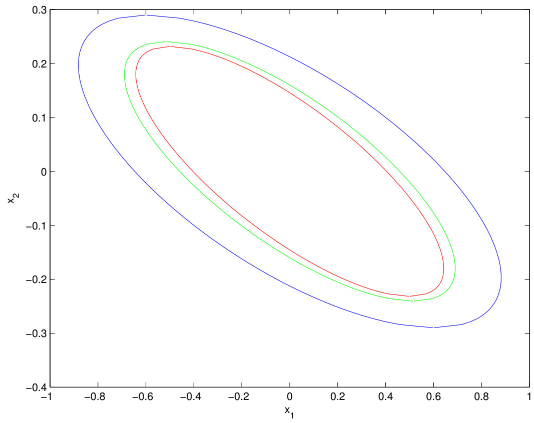

Fig. 4. Sets $L(x, -K_1 x) \leq \gamma_1 0.2^2$ (blue), $L(x, -K_\infty x) \leq \gamma_\infty 0.1^2$ (red) and $L(x, -K_2 x) \leq \gamma_2 (0.1\sqrt{2})^2$ (green).

This example demonstrates that using an appropriate description of the disturbance to design the linear feedback allows us to obtain better bounds on the trajectories of the closed-loop system.

5. CONCLUSIONS

In this paper a novel approach to the design of feedback laws and ISS-Lyapunov functions for discrete-time linear systems subject to bounded disturbance inputs bounded has been presented. The LMI-based proposed technique computes ISS gains and ISS stabilizing controllers using different norms to measure disturbance. This method automatically provides better approximation of ultimate bounds given fixed polyhedron disturbance sets. This leads generally to less conservative results if instead one would use a method based on an overapproximation of the disturbance set with an ellipsoidal set and rely on one of the existing approaches. The proposed method can be applied to obtain the terminal cost and the corresponding local controller needed to design robust min-max MPC controllers.

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