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Target problems under state contraints for anisotropic affine dynamics A numerical analysis based on viability theory

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Abstract: We propose an approximation procedure for the dual problems of viability and reachability from a robust control point of view. Namely, given a region of the state space, we want to know whether it is possible for one control to guarantee that whatever the actions of the other control, the state stays in this region, or that starting outside of the region it is possible to reach it. We consider anisotropic affine dynamics in the sense that the dynamics is described by the sum of an autonomous term and a linear term in each input. This particular form enables us to approximate the solution of the robust control problem using a sequence of simpler control problems. We demonstrate how this approach can alleviate the computational effort required for the solution of the original problem. Our formulation is based on viability theory and differential games.

1. INTRODUCTION

Let us consider a continuous time system with two inputs

$$\dot{x} = f_0(x) + f_1(x)u + f_2(x)v, \quad (u, v) \in U \times V \qquad (1)$$

with $x \in \mathbb{R}^n$, $U \subset \mathbb{R}^p$ the control input, and $V \subset \mathbb{R}^q$ the disturbance input. We are interested in numerical solution of the dual problems of viability and reachability from a robust control point of view. Namely, given a region of the state space, we want to know whether it is possible to find a control input which guarantees that whatever the disturbance input, the state stays in this region, or that starting outside of the region it is possible to reach it. We address the robustness aspect by considering the worst case approach in which the disturbance is an *intelligent* opponent of the control. This leads to the formulation of the robust control problem as a two-player differential game. In this paper, we consider the robust viability/reachability problems in the unified framework of viability theory (Aubin [1991], Cardaliaguet et al. [1999]) for the particular case when U and V are convex and compact sets which contain 0 in their interior. If for all x the norm of $f_0(x)$ is large with respect to the norm of $f_1(x)U + f_2(x)V$, then dynamics $\dot{x} = f_0(x)$ can be considered as an approximation of (1). We use this observation and approximate problems with only one input in order to speed up numerical solution of the robust control problem.

Questions of (robust) viability, invariance or reachability arise for numerous applications in engineering, biology or economics. Related concepts have been studied extensively in the literature (Aubin [1991], Cardaliaguet et al. [1999], Lygeros [2002], Mitchell et al. [2005], Blanchini [1999]). The interest for these questions has been renewed with the study of safety problems in the framework of hybrid systems (Aubin et al. [2002], Gao et al. [2006], Crück and Saint-Pierre [2004]). Ground transportation systems (Lygeros and Lynch [1998]) and air traffic management systems (Livadas et al. [2000], Tomlin et al. [2001]), for instance, have been considered from this perspective.

Direct characterization of the extremal sets enjoying (robust) viability, reachability or invariance properties is at the basis of viability theory. The development of computational tools follows closely the theoretical achievements and requires mild regularity assumptions on the dynamics, the target sets and the constraints (Cardaliaguet et al. [1999]). An alternative, indirect approach to reachability analysis relies on optimal control theory and characterizes the sets of interest as level sets of the value function of an appropriate optimal control problem. Under regularity assumptions (Lygeros [2002]) the value function is the viscosity solution to a first order partial differential equation (variant of the standard Hamilton-Jacobi equation), and reachability computation can take advantage of efficient algorithms developed for this class of equations (Mitchell et al. [2005]). The algorithms derived from both approaches rely on gridding of the state space. They perform well only if the dynamics can be accurately "projected" on this grid. The convergence properties of the algorithms in (Mitchell et al. [2005]) explicitly require isotropic dynamics; algorithms from viability theory are guaranteed to converge without this assumption to the price of heavier computation for good accuracy. Both approaches are subject to the *curse of dimensionality*, in the sense that memory and time required to perform computation grow exponentially with the dimension of the state space. Current implementations can deal with 3D or 4D problems, up to 5D with coarse gridding. Higher dimensions can be dealt with in the case of linear convex problem using ellipsoidal or polytopic techniques (see Blanchini [1999] and references therein). These techniques are powerful

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when there is only one input (control or disturbance). In the case when there are two antagonistic inputs, convexity can readily be lost (Raković et al. [2007]). In any case, the robust control problem is more computationally intensive than the single input problem.

In this paper, our purpose is to simplify the resolution of the robust control problem by using a sequence of approximate control problems. Our approach takes advantage of the unified framework of viability theory and differential games for formulating meaningful approximate problems. The main advantage is that this provides a (possibly rough) approximate solution to the game problem using relatively simple computation. The quality of this approximation can be estimated since we obtain an inner and an outer approximation of the game solution. Moreover, the result of this approximation can be used in the computation of the game solution, thus alleviating the overall computation effort. Another advantage is that the approximation provides insight on the effect of each player on the behavior of the system, which is not always consistant with off-hand intuition.

The material is organized as follows: The next section is devoted to the introduction of the concepts from viability theory and differential games which will be used for solving the robust control problems. Section 3 describes our approximation procedure. Some numerical considerations based on the viability kernel algorithm (Cardaliaguet et al. [1999]) are provided in Section 4. We illustrate our approach on an example in Section 5.

2. NOTATIONS AND DEFINITIONS

2.1 Preliminaries

We assume that the dynamics satisfy the following conditions:

Hypothesis 1.

- U and V are compact, convex and contain 0 in their interior;
- f_0, f_1 and f_2 are Lipschitz-continuous.

We denote by $\mathcal{U}$ (resp. $\mathcal{V}$) the set of Lebesgue measurable functions $u(\cdot) : [0, \infty) \to U$ (resp. $v(\cdot) : [0, \infty) \to V$). We denote by $x(\cdot : x_0, u(\cdot), v(\cdot))$ the trajectory of system (1) originating at x_0 with inputs $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$. Let us define

$$\forall x \in \mathbb{R}^n, \quad \begin{cases} F(x) := f_0(x) + f_1(x)U + f_2(x)V \\ F_1(x) := f_0(x) + f_1(x)U \\ F_2(x) := f_0(x) + f_2(x)V \end{cases} \tag{2}$$

Under Hypothesis 1, the differential inclusions

$$\dot{x} \in F(x), \quad \dot{x} \in F_1(x), \text{ and } \quad \dot{x} \in F_2(x),$$

are equivalent to the differential equations (1) and

$$\dot{x} = f_0(x) + f_1(x)u, \text{ and } \dot{x} \in f_0(x) + f_2(x)v$$

respectively. Solutions to the differential inclusions are absolutely continuous functions verifying the relevant inclusion for almost all time. We denote by $S_F(x_0)$, $S_{F_1}(x_0)$, $S_{F_2}(x_0)$ the set of such solutions originating at x_0.

2.2 Viability concepts

Let us consider the differential inclusion

$$\dot{x} \in F(x).$$

Let $E \subset \mathbb{R}^n$. We can define four basic problems:

Is E viable, in the sense that

$$\forall x_0 \in E, \quad \exists x(\cdot) \in S_F(x_0), \quad \forall t \geq 0,\ x(t) \in E\ ?$$

Is E invariant, in the sense that

$$\forall x_0 \in E, \quad \forall x(\cdot) \in S_F(x_0), \quad \forall t \geq 0,\ x(t) \in E\ ?$$

Is E reachable, in the sense that

$$\forall x_0 \in \mathbb{R}^n \setminus E, \quad \exists x(\cdot) \in S_F(x_0), \quad \exists T \geq 0,\ x(T) \in E\ ?$$

Is E an absorbing set, in the sense that

$$\forall x_0 \in \mathbb{R}^n \setminus E, \quad \forall x(\cdot) \in S_F(x_0), \quad \exists T \geq 0,\ x(T) \in E\ ?$$

Remark 1. Viability is sometimes referred to as *controlled invariance* (Blanchini [1999]).

In the sequel, we use the following notations:

Definition 1. Let $K \subset \mathbb{R}^n$ and $C \subset K$ be closed sets.

- A trajectory $x(\cdot)$ is *viable in K with target C* if
$$\forall t \leq \inf\{s : x(s) \in C\}, \quad x(t) \in K.$$
- $\mathrm{Viab}_F(K, C)$ denotes the *viability kernel of K with target C*. It is the set of initial conditions in K from which there exists a trajectory viable in K with target C.
- $\mathrm{Inv}_F(K, C)$ denotes the *invariance kernel of K with target C*. It is the set of initial conditions in K from which all trajectories are viable in K with target C.

Under Hypothesis 1, $\mathrm{Viab}_F(K, C)$ and $\mathrm{Inv}_F(K, C)$ are closed sets which are the largest viable (resp. invariant) closed subsets of K.

Viability and invariance kernels provide obviously direct solutions to viability and invariance problems. They also provide a solution for reachability and absorption problems, if $\mathrm{Viab}_F(\overline{K \setminus C}) = \emptyset$ (resp. $\mathrm{Inv}_F(\overline{K \setminus C}) = \emptyset$). If this is not the case, we can set

$$\Phi(\tau, x) = \{-1\} \times F(x).$$

Then (Cardaliaguet et al. [1999]):

- $\mathrm{Viab}_\Phi([0, +\infty) \times K, [0, +\infty) \times C)$ is the epigraph of the minimal time function for reaching the target (under constraints). Namely, it is the set of initial conditions (τ_0, x_0) such that
$$\exists x(\cdot) \in S_F(x_0),\ \exists T \leq \tau_0, \quad x(T) \in C$$
and the condition $\forall t \leq T,\ x(t) \in K$ is true.
- $\mathrm{Inv}_\Phi([0, +\infty) \times K, [0, +\infty) \times C)$ is the set of initial conditions (τ_0, x_0) such that
$$\forall x(\cdot) \in S_F(x_0),\ \exists T \leq \tau_0, \quad x(T) \in C$$
and the condition $\forall t \leq T,\ x(t) \in K$ is true.

Viability and invariance kernels can be computed using algorithms defined in (Cardaliaguet et al. [1999]) and outlined in Section 4 for sake of completeness.

2.3 Differential game concepts

We define a game in which the control (playing with u) tries to reach the target and/or respect the constraints. The disturbance (playing with v) has the opposite goal. It is well known since the early 60s (see for instance Varaiya [1967]) that the solution of the game depends on the information that each player can use to determine his/her actions. Unfortunately, the natural generalization of control problems which would allow both players to use state feedbacks does not lead to a *good* solution of the game in general (see Cardaliaguet [1994] for a discussion on *context of strategies*).

In order to generalize the notion of viability and invariance kernel, we introduce the notion of non-anticipative strategies (Cardaliaguet et al. [1999]):

Definition 2. (Non-anticipative strategy). A function $\alpha : \mathbb{R}^n \times \mathcal{V} \to \mathcal{U}$ is a non anticipative strategy if for all time $T > 0$, all initial condition $x_0 \in \mathbb{R}^n$ and for all pair of functions $v(\cdot)$ and $\tilde{v}(\cdot)$ of $\mathcal{V}$ which coincide almost everywhere on $[0,T)]$, $u(\cdot) = \alpha(x_0, v(\cdot))$ and $\tilde{u}(\cdot) = \alpha(x_0, \tilde{v}(\cdot))$ coincide almost everywhere on $[0,T]$.

In words, a player which uses a non-anticipative strategy can make decisions using the past history of the game, not only the current state. Let us mention that the notion of non-anticipative strategy encompasses the notion of feedback. Indeed, let us assume for instance that the control uses a feedback $\tilde{u} : \mathbb{R}^n \to U$, and let us consider the application

$$\alpha : (x_0, v(\cdot)) \longrightarrow \tilde{u}(x(\cdot)),$$

in which $x(\cdot)$ denotes the solution to the differential equation $\dot{x} = f(x, \tilde{u}(x(t)), v(t))$ starting at x_0. Then α is a non-anticipative strategy.

Remark 2. Non-anticipative strategies are useful mathematical tools but they are not very practical from application point of view. Fortunately, they can be approximated numerically using feedbacks of the form $\hat{u} : (x, v) \to u$. This provides insight on the fact that the player who uses non-anticipative strategies in a sense "plays second" and has an advantage because he/she knows the action of the other player.

We can define two games, depending on which player uses a non-anticipative strategy (the other player uses open loop control). When using a differential game setting for studying robust control, it is often legitimate to consider that the control uses a non-anticipative strategy. The other game provides a guaranteed conservative approach.

Definition 3. Let $K \subset \mathbb{R}^n$ and $C \subset K$ be closed sets.

- $\mathrm{Disc}_f(K, C)$ denotes the *discriminating kernel of K with target C*. It is the set of initial conditions in K from which the control can find a strategy such that for all possible disturbances, the trajectory is viable in K with target C.
- $\mathrm{Lead}_f(K, C)$ denotes the *leadership kernel of K with target C*. It is the set of initial conditions in K for which for any strategy played by the disturbance, for all $\epsilon > 0$ and for all $T > 0$, the control can generate a trajectory such that
$$\forall t \leq \min\{T, \inf\{s : x(s) \in C_\epsilon\}\}, \quad x(T) \in K_\epsilon,$$
in which $K_\epsilon = K + \epsilon B$ and $C_\epsilon = \mathbb{R}^n \setminus ((\mathbb{R}^n \setminus C) + \epsilon B)$, with B a ball in $\mathbb{R}^n$.

Remark 3. The rather cumbersome definition of the leadership kernel is the only one which ensures good mathematical properties in the general case.

Remark 4. We have by definition

$$\mathrm{Lead}_f(K, C) \subset \mathrm{Disc}_f(K, C).$$

Equality holds under Isaacs condition, namely

$$\forall p \in \mathbb{R}^n, \quad \sup_{v \in V} \inf_{u \in U} \langle p, f(x, u, v)\rangle = \inf_{u \in U} \sup_{v \in V} \langle p, f(x, u, v)\rangle,$$

which holds true for dynamics (1).

Under Hypothesis 1, $\mathrm{Disc}_f(K, C) = \mathrm{Lead}_f(K, C)$ is a closed set which is the largest closed subset of K which enjoys robust viability with target C for the relevant game setting. For instance, if $C = \emptyset$, $\mathrm{Disc}_f(K)$ is the set of initial conditions in K such that the control, playing with a non-anticipative strategy, can keep the state in K. As it is the case for viability and invariance kernels, discriminating and leadership kernel can be used to characterize the epigraph of reaching time functions.

Numerical approaches exist also for the approximation of discriminating and leadership kernels (Cardaliaguet et al. [1999]). They are more computationally intensive than invariance or viability kernel computation as will be shown in Section 4.

3. SOLVING A ROBUST CONTROL PROBLEM

From the previous section, we know that robust control problems related to reachability and invariance can be solved using the notion of discriminating and leadership kernels. Moreover, with dynamics (1), Isaacs condition is always satisfied and both kernels are equal. In this section, we first prove that the discriminating kernel can be bounded using viability and invariance kernels. Then we further use the particular form of dynamics (1) to simplify also the computation of these bounds. Our aim is to obtain a rough approximation of the discriminating kernel using several lower-complexity computation steps. This computation can then be refined using the full discriminating kernel algorithm of Cardaliaguet et al. [1999].

3.1 Approximation of the game solution

Proposition 2. Under Assumption 1, we have

$$\mathrm{Inv}_{F_2}(K, C) \subset \mathrm{Disc}_f(K, C) \subset \mathrm{Viab}_{F_1}(K, C).$$

Proof. In order to prove the left-hand inclusion, let $x_0 \in \mathrm{Inv}_{F_2}(K, C)$ and let the control play the constant strategy

$$\forall v(\cdot) \in \mathcal{V}, \quad \alpha(v(\cdot)) = 0.$$

Then by definition, the disturbance cannot force the state out of K before its reaching the target. For the right-hand side inclusion, let $x_0 \in \mathrm{Disc}_f(K, C)$ and let α be a strategy which ensures control victory. Let the disturbance play

$$\forall t \geq 0, \quad v(t) = 0.$$

Then $x(\cdot; x_0, \alpha(v(\cdot)), v(\cdot)) \subset S_{F_1}(x_0)$ and by definition of α, it is viable in K with target C.

Proposition 3. Let us consider two set-valued maps G_1 and G_2 defined on $\mathbb{R}^n$ with non-empty compact convex values and linear growth. Let us assume moreover that

$$\forall x \in K, \quad 0 \in G_2(x).$$

Then we have

$$\mathrm{Viab}_{G_1+G_2}(K, C) = \mathrm{Viab}_{G_1+G_2}(K, \mathrm{Viab}_{G_1}(K, C))$$
$$\mathrm{Inv}_{G_1+G_2}(K, C) = \mathrm{Inv}_{G_1+G_2}(\mathrm{Inv}_{G_1}(K, C), C)$$

Proof. The inclusion

$$\text{Viab}_{G_1+G_2}(K,C) \subset \text{Viab}_{G_1+G_2}(K, \text{Viab}_{G_1}(K,C))$$

stems from the definition of the viability kernel. The set of constraint is the same, and the target is larger, so more initial conditions can satisfy the viability conditions. In order to prove the converse inclusion, let us set $x_0 \in \text{Viab}_{G_1+G_2}(K, \text{Viab}_{G_1}(K,C))$ and let $x(\cdot) \in S_{G_1+G_2}(x_0)$ be viable in K with target $\text{Viab}_{G_1}(K,C)$. Let us set

$$T = \inf\{t : x(t) \in \text{Viab}_{G_1}(K,C)\}.$$

Then

$$\forall t \in [0,T), \quad x(t) \in K.$$

Therefore, if $T = \infty$, we have $x_0 \in \text{Viab}_{G_1+G_2}(K,C)$. If $T < \infty$, $x(T) \in \text{Viab}_{G_1}(K,C)$. Let us set

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if } t \leq T \\ x_1(t-T) & \text{otherwise,} \end{cases}$$

with $x_1(\cdot) \in S_{G_1}(x(T))$ which is viable in K with target C. Then $\tilde{x}(\cdot) \in S_{G_1+G_2}(x_0)$ because

$$\forall x, \quad 0 \in G_2(x).$$

By construction, $\tilde{x}(\cdot)$ is viable in K with target C. The proof for the invariance kernel is similar.

This leads to the following result using the fact that $\text{Viab}_{f_0}(K,C) = \text{Inv}_{f_0}(K,C)$.

Corollary 4. Under Assumption 1,

$$\begin{aligned} \text{Inv}_{F_2}(K,C) &= \text{Inv}_{F_2}(\text{Viab}_{f_0}(K,C),C) \\ \text{Viab}_{F_1}(K,C) &= \text{Viab}_{F_1}(K, \text{Viab}_{f_0}(K,C)) \end{aligned} \tag{3}$$

Let us mention that $\text{Viab}_{f_0}(K,C)$ is easy to compute given that the dynamics is single-valued. Moreover, in certain cases, an analytic solution may be available.

Using Proposition 2 and Corollary 4, we obtain the main result of this section

Theorem 5. Under Assumption 1,

$$\text{Disc}_f(K,C) = \text{Disc}_f\left(\text{Viab}_{F_1}(K,\widehat{C}) \ , \ \text{Inv}_{F_2}(\widehat{C},C)\right),$$

with $\widehat{C} = \text{Viab}_{f_0}(K,C)$.

Proof. Let $x_0 \subset \text{Disc}_f(K,C)$, and let α denote a strategy which ensures control success. Let $v(\cdot) \in \mathcal{V}$ and set

$$T = \inf\{s : x(s; x_0, \alpha(v(\cdot)), v(\cdot)) \in C\}.$$

Then by definition of the discriminating kernel,

$$\forall t \leq T, \quad x(t; x_0, \alpha(v(\cdot)), v(\cdot)) \in \text{Disc}_f(K,C).$$

From Proposition 2,

$$\forall t \leq T, \quad x(t; x_0, \alpha(v(\cdot)), v(\cdot)) \in \text{Viab}_{F_1}(K,\widehat{C}).$$

Now, we have $C \subset \text{Inv}_{F_2}(\widehat{C},C)$. Therefore,

$$\exists \theta \leq T, \quad x(\theta; x_0, \alpha(v(\cdot)), v(\cdot)) \in \text{Inv}_{F_2}(\widehat{C},C).$$

We have proved that the trajectory $x(\cdot; x_0, \alpha(v(\cdot)), v(\cdot))$ is viable in $\text{Viab}_{F_1}(K,\widehat{C})$ with target $\text{Inv}_{F_2}(\widehat{C},C)$. Therefore,

$$\text{Disc}_f(K,C) \subset \text{Disc}_f\left(\text{Viab}_{F_1}(K,\widehat{C}) \ , \ \text{Inv}_{F_2}(\widehat{C},C)\right).$$

In order to prove the converse inclusion, let $x_0 \in \text{Disc}_f\left(\text{Viab}_{F_1}(K,\widehat{C}), \text{Inv}_{F_2}(\widehat{C},C)\right)$ and let α denote a strategy which ensures control success. Let $v(\cdot) \in \mathcal{V}$ and set

$$T = \inf\{s : x(s; x_0, \alpha(v(\cdot)), v(\cdot)) \in \text{Inv}_{F_2}(\widehat{C},C)\}.$$

Then

$$\forall t \leq T, \quad x(t; x_0, \alpha(v(\cdot)), v(\cdot)) \in \text{Viab}_{F_1}(K,\widehat{C}) \subset K.$$

Now, let us define the strategy

$$\hat{\alpha} : v(\cdot) \longrightarrow u(t) = \begin{cases} \alpha(v(\cdot)) & \text{if } t \leq T \\ 0 & \text{otherwise} \end{cases}$$

Then $\hat{\alpha}$ is non-anticipative and

$$\forall t \leq T, \quad x(T; x_0, \alpha(v(\cdot)), v(\cdot)) = x(T; x_0, \hat{\alpha}(v(\cdot)), v(\cdot))$$

Now, $x(T; x_0, \hat{\alpha}(v(\cdot)), v(\cdot)) \in \text{Inv}_{F_2}(\widehat{C},C)$ Therefore we have proved that $x(\cdot; x_0, \hat{\alpha}(v(\cdot)), v(\cdot))$ is viable in K with target C, which completes the proof.

3.2 Approximation methodology

The approximation methodology can be summarized as follows:

(1) Compute $\widehat{C} = \text{Viab}_{f_0}(K,C)$
(2) Compute $\text{Viab}_{F_1}(K,\widehat{C})$
(3) Compute $\text{Inv}_{F_2}(\widehat{C},C)$
(4) Compute

$$\text{Disc}_f(K,C) = \text{Disc}_f\left(\text{Viab}_{F_1}(K,\widehat{C}) \ , \ \text{Inv}_{F_2}(\widehat{C},C)\right)$$

This methodology can be applied with any technique for computing the different stages. In the next section, we explain how it can save computational effort when using grid algorithms. Analytical solutions may exist for simple cases, especially for computing $\text{Viab}_{f_0}(K,C)$. The main idea is to make use of *heavy* computational tools only when necessary.

4. NUMERICAL CONSIDERATIONS

In this section, we assume that the approximation procedure described above is applied using algorithms described in Cardaliaguet et al. [1999]. Our aim is to prove that our approach reduces the overall computational effort while retaining the convergence properties.

4.1 Viability-based algorithms

The algorithms described in Cardaliaguet et al. [1999] require a time discretization of the dynamics, and the projection of this discrete-time dynamics on a grid in order to obtain a fully discrete dynamics. We provide here a simplified version of this approximation procedure in the case when f is bounded by a constant M. Let us denote by l the Lipschitz constant of the function f, by $\epsilon > 0$ a time-step for time discretization, and by $h > 0$ a grid step for the discretization of the state space. Let us set

$$G_\epsilon(x,v) = x + \epsilon(f(x,U,v) + Ml\mathcal{B}),$$

in which $\mathcal{B}$ denotes the unit ball in $\mathbb{R}^n$. Then G_ϵ provides a good approximation of f in the sense that for an initial condition x_0 and for all controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, we have

$$\forall k \in \mathbb{N}, \quad x((k+1)\epsilon) \in G_\epsilon(x(k\epsilon), v(k_\epsilon)),$$

with the notation $x(\cdot) = x(\cdot; x_0, u(\cdot), v(\cdot))$.

Now we define X_h and V_h locally finite subsets of $\mathbb{R}^n$ and V such that

$$\begin{cases} \forall x \in \mathbb{R}^n, & \exists x_h \in X_h \text{ such that } ||x_h - x|| \leq h \\ \forall v \in \mathbb{V}, & \exists v_h \in V_h \text{ such that } ||v_h - v|| \leq h \end{cases}$$

and we set

$$\Gamma_{\epsilon,h}(x_h, v_h) = [G_\epsilon(x_h, v_h) + 2(1 + l\epsilon)h\mathcal{B}] \cap X_h.$$

Then $\mathrm{Disc}_f(K, C)$ can be approximated using the following algorithm

$$\begin{aligned} K^0_{\epsilon,h} &= (K + h\mathcal{B}) \cap X_h \\ K^{k+1}_{\epsilon,h} &= \{x \in K^k_{\epsilon,h} : \forall v_h \in V_h, \Gamma_{\epsilon,h}(x_h, v_h) \cap K^k_{\epsilon,h} \neq \emptyset\} \\ &\quad \cup C^{\epsilon,h} \end{aligned}$$

with $C^{\epsilon,h} = (C + (M\epsilon + h)\mathcal{B}) \cap X_h$. If K_h is finite, the algorithm converges in finite time. We denote by $K^\infty_{\epsilon,h}$ this limit. Then

$$\lim_{\epsilon\to 0^+,\, \frac{h}{\epsilon}\to 0^+} \mathcal{K}^\infty_{\epsilon,h} = \mathrm{Disc}_f(K, C) \subset \mathcal{K}^\infty_{\epsilon,h} + h\mathcal{B}, \quad \forall(\epsilon, h).$$

The same algorithm can be used to compute a viability kernel with $V_h = \emptyset$, or an invariance kernel if $f(x, U, v) = \{\hat{f}(x, v)\}$. Computing a viability kernel is less intensive than computing an invariance kernel, which is less intensive than computing a discriminating kernel.

4.2 Speeding up computation

Let us denote by $\Gamma^0_{\epsilon,h} : X_h \to X_h$, $\Gamma^1_{\epsilon,h} : X_h \to X_h$, $\Gamma^2_{\epsilon,h} : X_h \times V_h \to X_h$ and $\Gamma_{\epsilon,h} : X_h \times V_h \to X_h$ the discrete functions which approximate $f_0(\cdot)$, $F_1(\cdot)$, $f_0(\cdot)+f_2(\cdot)v$ and $F_1(\cdot) + f_2(\cdot)v$. We assume that $0 \in V_h$ and that

$$\forall x_h \in X_h, \quad \begin{cases} \Gamma^2_{\epsilon,h}(x_h, 0) = \Gamma^0_{\epsilon,h}(x_h) \\ \Gamma_{\epsilon,h}(x_h, 0) = \Gamma^1_{\epsilon,h}(x_h) \end{cases}$$

Moreover, we assume that

$$\forall x_h \in X_h, \quad \begin{cases} \Gamma^0_{\epsilon,h}(x_h) \subset \Gamma^1_{\epsilon,h}(x_h) \\ \Gamma^2_{\epsilon,h}(x_h, v_h) \subset \Gamma_{\epsilon,h}(x_h, vh),\ \forall v_h \in V_h \end{cases}$$

The implementation of the algorithm described above requires to parse the discrete space. In order to save time, we can combine the two first steps of the approximation procedure

$$\begin{aligned} H^0_{\epsilon,h} &= (K + h\mathcal{B}) \cap X_h \\ L^0_{\epsilon,h} &= (K + h\mathcal{B}) \cap X_h \\ H^{k+1}_{\epsilon,h} &= \{x \in H^k_{\epsilon,h} : \Gamma^0_{\epsilon,h}(x_h) \cap H^k_{\epsilon,h} \neq \emptyset\} \cup C_{\epsilon,h} \\ L^{k+1}_{\epsilon,h} &= \{x \in L^k_{\epsilon,h} : \Gamma^1_{\epsilon,h}(x_h) \cap L^k_{\epsilon,h} \neq \emptyset\} \cup H^{k+1}_{\epsilon,h} \end{aligned}$$

Similarly, we can combine the two last computation steps

$$\begin{aligned} I^0_{\epsilon,h} &= H^\infty_{\epsilon,h} := \lim_{k\to\infty} H^k_{\epsilon,h} \\ J^0_{\epsilon,h} &= L^\infty_{\epsilon,h} := \lim_{k\to\infty} L^k_{\epsilon,h} \\ I^{k+1}_{\epsilon,h} &= \{x \in I^k_{\epsilon,h} : \forall v_h \in V_h, \Gamma^2_{\epsilon,h}(x_h, v_h) \cap I^k_{\epsilon,h} \neq \emptyset\} \\ &\quad \cup C_{\epsilon,h} \\ J^{k+1}_{\epsilon,h} &= \{x \in J^k_{\epsilon,h} : \forall v_h \in V_h, \Gamma_{\epsilon,h}(x_h, v_h) \cap J^k_{\epsilon,h} \neq \emptyset\} \\ &\quad \cup I^k_{\epsilon,h} \end{aligned}$$

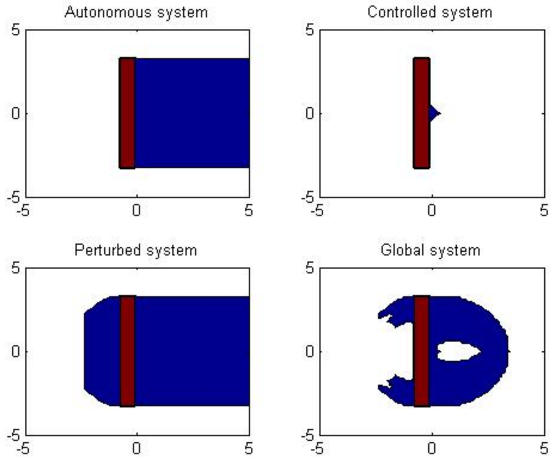
Proposition 6.

$$\forall(\epsilon, h), \qquad J^\infty_{\epsilon,h} = K^\infty_{\epsilon,h}.$$

5. NUMERICAL RESULTS

We present here early numerical results. We are currently working on statistics in order to measure the computational effort saved by our approach.

In order to illustrate our approach, we use the pursuit-evasion game in $\mathbb{R}^2$ described in Cardaliaguet et al. [1999].



Complementary of the evasion set computed using the 4 step approximation procedure. We have used the following parameters: $V_P = 1$, $V_E = 1.1$, $R_P = 0.8$, $r = 0.5$, $D = 3.5$.

Fig. 1. The 4 steps of the approximation procedure

The pursuer has a constant speed V_P, the orientation of which can change with a minimum turning radius R_P. The evader can choose its speed in a ball of radius V_E except when closer to the pursuer than r, then it becomes proportional to the separation. Writing the game in coordinates relative to the pursuer leads to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = V_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{V_P}{R_P} \begin{pmatrix} -y \\ x \end{pmatrix} v + \min(||(x, y)||, r)u,$$

with U the unit ball in $\mathbb{R}^2$ and $V = [-1\ 1]$. We define $\Omega = \{(x, y) : -0.2 < x < 0 \text{ and } |y| < D\}$ the capture set, in which D is the detection radius. From the evader point of view (the control) the objective is to stay in $K := \mathbb{R}^n \setminus \Omega$. We consider that the evader wins if it can reach the set $C := \{(x, y) : \min(x, y) \geq 5\}$.

The results of the four steps of the approximation procedure are displayed in Figure 1. We have plotted the victory domain of the pursuer in the four related games that we use for our analysis.

- On the top left picture, the evader cannot move and the pursuer cannot turn; the victory domain of the pursuer is straightforward.
- On the top right picture, the evader can fully control its trajectory while the pursuer has to keep constant velocity with constant orientation. It can be seen, as expected, that the evader can escape except when it is very close to the pursuer and has therefore reduced speed.
- On the bottom left picture, the evader cannot move while the pursuer can change its direction of pursuit. Therefore the pursuer wins except when the maneuver necessary to reach the evader takes it too far away (the evader is then in C).

The result of the full game is displayed on the bottom right picture. What is not straight forward is the *hole* in the victory domain of the pursuer which means that the discriminating kernel is not connected. The hole represents positions from which the evader cannot reach C but can stay in K forever by maintaining its distance to the pursuer.

From the computational point of view, we have observed a reduction of around 15% of the time needed to compute the discriminating kernel when using our sequential approximation procedure instead of using directly the discriminating kernel algorithm. For comparison purposes, we have used the same grid (100×100) and the simplest possible implementation without any optimization. The gain would become more significant when solving problems in higher dimensions.

Moreover, our approach enables the use of insight on the dynamics in order to speed up computation. The disconnectedness of the discriminating kernel in our example demonstrates how difficult it can be to gain insight on game dynamics. However, the first three steps of the approximation procedure concerns control systems. In our example, $\widehat{C} = \text{Viab}_{f_0}(K, C)$ can be determined analytically, and a little analysis shows that $\text{Viab}_{F_1}(K, \widehat{C})$ and $\text{Inv}_{F_2}(\widehat{C}, C)$ can be computed in reduced domains. This can reduce drastically the number of points on the grid which have to be considered when carrying out the last step of the procedure.

6. CONCLUDING REMARKS

We have presented a method for solving a robust control problem of reachability or viability in a differential game setting when the dynamics is affine in both inputs. This method is based on existing tools from viability theory. Our contribution is the decomposition of the computation in four independent steps in order to take advantage of the particular form of the dynamics. This allows the use of tools developed for the single input problems (controlled dynamics or perturbed dynamics) in order to reduce the size of the problem (in number of points) which has to be treated using complex algorithms.

Computation of the discriminating kernel is seldom easy. The example that we have provided shows that the discriminating kernel can be disconnected, and this challenges the methods which rely on convexity assumptions. To the best of our knowledge, the discriminating kernel algorithm is the only available approach which requires only mild regularity assumptions. Our approach, in an effort toward the breaking of the curse of dimensionality, provides a preprocessing of the problem likely to reduce the computational load involved for a large class of dynamics.

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