Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Componentwise stabilization of interval systems

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Abstract: The componentwise stability of a linear system is a special type of asymptotic stability induced by the existence of exponentially decreasing rectangular sets that are invariant with respect to the free response. An interval system $\dot{x}(t) = Ax(t) + Bu(t)$, $A \in [A^-, A^+]$, $B \in [B^-, B^+]$, is componentwise stabilizable if there exists a constant feedback $u(t) = Fx(t)$ that ensures the componentwise stability of the whole family of linear systems defined by $\dot{x}(t) = (A + BF)x(t)$. The paper formulates computable necessary and sufficient conditions for the componentwise stabilizability of interval systems. It is shown that the componentwise stabilizing feedback matrices define the solution set of two equivalent linear inequalities These results are further exploited to construct a linear programming problem for which (i) the absence of a feasible solution means the componentwise stabilization is not possible, (ii) a feasible solution provides a componentwise stabilizing feedback matrix. The applicability of the theoretical development is illustrated by a numerical example.

1. INTRODUCTION

A continuous-time state-space model

$$\dot{x}(t) = Ax(t) + Bu(t),\ t \geq 0, \tag{1}$$

where the entries of $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $B = (b_{ik}) \in \mathbb{R}^{n\times m}$ present interval-type uncertainties of the form

$$\begin{aligned} &a_{ij}^- \leq a_{ij} \leq a_{ij}^+,\ b_{ik}^- \leq b_{ik} \leq b_{ik}^+, \\ &i = 1,\ldots,n,\ j = 1,\ldots,n,\ k = 1,\ldots,m, \end{aligned} \tag{2}$$

is called interval matrix system, or interval dynamical system, or shorter, *interval system* (abbreviated IS). The matrix componentwise inequalities

$$\begin{aligned} &A^- \leq A \leq A^+,\ B^- \leq B \leq B^+, \\ &A^- = (a_{ij}^-),\ A^+ = (a_{ij}^+) \in \mathbb{R}^{n\times n}, \\ &B^- = (b_{ik}^-),\ B^+ = (b_{ik}^+) \in \mathbb{R}^{n\times m}, \end{aligned} \tag{3}$$

provide a compact writing equivalent to the conditions expressed by (2).

The asymptotic stability of IS (1)&(2) with $u(t) \equiv 0$, was one of the most intensively explored properties of ISs, as reflected by the results reported in literature – see the reference list in (Mao and Chu, 2003) and the more recent publications (Chen and Lin, 2004), (Yamaç and Bozkurt, 2004), (Kolev and Petrakieva, 2005), (Zhang *et al.*, 2006). The greater part of these papers provides sufficient conditions, showing that the necessity is valid only for some particular classes of interval matrices. Necessary and sufficient conditions for the stability of arbitrary interval matrices are formulated in (Wang *et al.*, 1994), (Yedavalli, 1999), (Yedavalli, 2001), (Mao and Chu, 2003), (Zhang *et al.*, 2006). In a broader context, by regarding interval matrices as matrix polytopes, we should also mention researches on the stability of polytopic systems, such as (Geromel *et al.*, 2006), (Grman *et al.*, 2005), (Liu and Molchanov, 2002), (Kau *et al.*, 2005), (Molchanov and Liu, 2002).

The *componentwise stability* of IS (1)&(2) with $u(t) \equiv 0$ is a special type of asymptotic stability induced by the positive invariance of exponentially decreasing rectangular sets with respect to the free response of the IS. This concept was introduced and characterized in (Pastravanu and Voicu, 2004); it is briefly reviewed bellow as constituting the background of our current work. Let us consider the family of contractive sets

$$\begin{aligned} &X_{de^{ct}}^{\varepsilon} = \{x \in \mathbb{R}^n \mid \| [d_1^{-1}x_1 \cdots d_n^{-1}x_n]^T \|_\infty \leq \varepsilon e^{ct}\}, \\ &d_i > 0,\ i = 1,\ldots,n,\ c < 0,\ \varepsilon > 0,\ t \geq t_0, \end{aligned} \tag{4}$$

where T denotes transposition. Denote by $d \in \mathbb{R}^n$ the positive vector $d = [d_1 \cdots d_n]^T$ built with the constants $d_i > 0,\ i = 1,\cdots,n,$ used in (4). The vector valued function de^{ct} can be interpreted as generating the family of sets (4).

Definition 1. IS (1)&(2) with $u(t) \equiv 0$ is *componentwise stable* relative to the vector valued function de^{ct} (abbreviated as CW$_{de^{ct}}$ *-stable*) if the sets $X_{de^{ct}}^{\varepsilon}$, $c > 0$, are invariant with respect to the state-space trajectories of (1) with $u(t)\equiv 0$, for any $A\in[A^-,A^+]$. □

The $\mathrm{CW}_{de^{ct}}$-stability of IS (1)&(2) with $u(t)\equiv 0$ can be characterized by the help of a single matrix $\bar{A}=(\bar{a}_{ij})\in\mathbb{R}^{n\times n}$, built from the entries of A^-, A^+:

$$\begin{aligned}&\bar{a}_{ii}=a_{ii}^+,\ i=1,\dots,n,\\ &\bar{a}_{ij}=\sup_{a_{ij}^-\le a_{ij}\le a_{ij}^+}|a_{ij}|,\ i\ne j,\ i,j=1,\dots,n.\end{aligned} \tag{5}$$

Theorem 1 (see Corollary 2 in (Pastravanu and Voicu, 2004)) IS (1)&(2) with $u(t)\equiv 0$ is $\mathrm{CW}_{de^{ct}}$-stable if and only if the constant $c<0$ and the vector $d>0$ satisfy the inequality:

$$\bar{A}\,d\le cd\,. \qquad \square \tag{6}$$

In the current paper we consider the $\mathrm{CW}_{de^{ct}}$-stabilization of IS (1)&(2) by using a linear constant feedback:

$$u(t)=Fx(t),\ F\in\mathbb{R}^{m\times n}\,. \tag{7}$$

The closed-loop system

$$\dot{x}(t)=(A+BF)x(t)\,, \tag{8}$$

incorporates the interval uncertainties (2) of both matrices $A\in[A^-,A^+]$ and $B\in[B^-,B^+]$ used in equation (1). Thus, for a certain feedback matrix $F\in\mathbb{R}^{m\times n}$, each element of the matrix $A+BF$ is also defined by an interval. In other words, the closed-loop system (8) together with the uncertainties (2) represent an IS, which will be referred to as the *closed-loop* IS (8)&(2).

Inequality (6) provides an easy to apply procedure for the analysis of the $\mathrm{CW}_{de^{ct}}$-stability of the open-loop IS (1)&(2). However, if we refer to the closed-loop IS (8)&(2), then resolving inequality (6) (i.e. $\overline{A+BF}\,d\le cd$, $A\in[A^-,A^+]$, $B\in[B^-,B^+]$) with respect to the matrix $F\in\mathbb{R}^{m\times n}$ is a cumbersome task.

The literature on the IS stabilization is rather scarce, compared with IS stability. It exploits the properties of nonnegative systems (Shafai and Hollot, 1991), quadratic stability and LMIs (Mao and Chu, 2003), (Zhang *et al.*, 2006), generalized antisymmetric stepwise configurations (Hu and Wang, 2000), (Wei, 1994), controllability and spectrum allocation for (A,B) interval pairs (Shashikhin, 2002) and arithmetic intervals (Smagina and Brewer, 2002). The papers (Blanchini, 1995), (Blanchini and Miani, 1999) also deserve a definite interest, since they consider nonquadratic constraints for the state variables and the feedback control is applied to polytopic systems. Nevertheless a proper comparison cannot be developed between the cited papers (Blanchini, 1995), (Blanchini and Miani, 1999) and our work, even if our context would be adapted to the matrix polytope approach. This is because the cited papers design control laws with variable structure, whereas our objective is to synthesize constant feedbacks for the $\mathrm{CW}_{de^{ct}}$-stabilization.

The main contribution of the current work is a technique for testing the $\mathrm{CW}_{de^{ct}}$-stabilizability of IS (1)&(2), which also provides a $\mathrm{CW}_{de^{ct}}$-stabilizing matrix F whenever the $\mathrm{CW}_{de^{ct}}$-stabilization is possible. The technique relies on a computable necessary and sufficient condition for the existence of matrices F that ensure the $\mathrm{CW}_{de^{ct}}$-stability of the closed-loop IS (8)&(2). The key point consists in developing a result equivalent to Theorem 1 applied to $\overline{A+BF}$, but suitable for numerical tractability, in order to circumvent the direct solving of the inequality $\overline{A+BF}\,d\le cd$, $A\in[A^-,A^+]$, $B\in[B^-,B^+]$. To the best of our knowledge, no technique for the $\mathrm{CW}_{de^{ct}}$-stabilization of ISs has been proposed by now.

The exposition in the current paper is organized as follows. Section 2 introduces a set of notations required by the manipulation of some matrices and vectors with special structures. Section 3 considers the class of feedback matrices that ensure the $\mathrm{CW}_{de^{ct}}$-stability of the closed-loop IS (8)&(2) and gives a characterization of this class in terms of matrix inequalities. Section 4 exploits the results of the previous section in order to formulate a linear programming (LP) problem for which there exist two possibilities: (i) if feasible, then the solution contains the values of a $\mathrm{CW}_{de^{ct}}$-stabilizing feedback matrix, (ii) if unfeasible, then the IS is not $\mathrm{CW}_{de^{ct}}$-stabilizable. A numerical example is given in Section 5 for illustrating the applicability of the proposed technique.

2. NOTATIONS

Throughout the paper we use a set of notations for handling matrices and vectors with special structures. These notations have been chosen to support a quick understanding of the contextual message, despite the complexity of the computational approach we intend to develop.

Let $p,q,\pi,\rho,\eta\in\mathbb{N}$.

$0_{p\times q}$ is the null matrix of size $p\times q$. I_p is the identity matrix of order p.

- For a real matrix $M=(m_{ij})\in\mathbb{R}^{p\times q}$, we introduce the following notations:

$M_{(:,j)}=[m_{1j}\cdots m_{pj}]^T\in\mathbb{R}^p$ is a vector containing the j-th column of M, $j=1,\cdots,q$.

$M\,|^{\mathrm{vec}}_{pq\times 1}=[M_{(:,1)}^T\dots M_{(:,q)}^T]^T\in\mathbb{R}^{pq}$ is a vector reshaping the elements of the matrix M taken columnwise in the ascending order of the column subscript.

If $r=\{r_1,\cdots,r_\eta\}$, $r_1,\ldots,r_\eta \in \mathbb{N}$, $1\le r_1<\ldots<r_\eta\le p$, is a set of row subscripts for *M*, then $<M>_r \in \mathbb{R}^{(p-\eta)\times q}$ is the matrix obtained from *M* by deleting the rows subscripted $r_1,\ldots,r_\eta$.

If $M\in\mathbb{R}^{p\times p}$ is a square matrix, then $M^{\text{off}}\in\mathbb{R}^{p\times p}$ preserves the off-diagonal elements of *M* and has zeros on the main diagonal.

- For a real vector $v=(v_i)\in\mathbb{R}^\rho$, we introduce the following notations:

If $\rho\in\mathbb{N}$ and $\rho=pq$, then

$$v|_{p\times q}^{\text{mat}}=\begin{bmatrix} v_1 & \cdots & v_{(q-1)p+1} \\ v_2 & \cdots & v_{(q-1)p+2} \\ \vdots & \vdots & \vdots \\ v_p & \cdots & v_{qp}\end{bmatrix}\in\mathbb{R}^{p\times q}$$

is a matrix reshaping the elements of the vector v taken in the ascending order of their subscript.

If $r=\{r_1,\ldots,r_\eta\}$, $r_1,\ldots,r_\eta \in \mathbb{N}$, $1\le r_1<\ldots<r_\eta\le p$, is a set of element subscripts for *v*, then $<v>_r\in\mathbb{R}^{\rho-\eta}$ is the vector obtained from *v* by deleting the elements subscripted $r_1,\ldots,r_\eta$.

- For the construction of a matrix that contains columns selected from two different matrices M^-, $M^+\in\mathbb{R}^{p\times q}$, we introduce the following notations:

If $M^-_{(:,j)}$, $M^+_{(:,j)}$ are the *j*-th column of M^- and, respectively, M^+, and $s_j\in\{-1,+1\}$, $j=1,\ldots,q$, then

$$M^{s_j}_{(:,j)}=\begin{cases} M^-_{(:,j)},\ s_j=-1 \\ M^+_{(:,j)},\ s_j=+1\end{cases}$$

is the column selected in accordance with the value of s_j.

If $s=[s_1\ldots s_q]$, $s_j\in\{-1,+1\}$, $j=1,\ldots,q$, then the matrix $M^s=[M^{s_1}_{(:,1)}M^{s_2}_{(:,2)}\cdots M^{s_q}_{(:,q)}]$ has the columns selected in accordance with the entries of the vector $s\in\{-1,+1\}^q$.

- For two matrices $M\in\mathbb{R}^{p\times q}$, $\mathbf{\Omega}\in\mathbb{R}^{\pi\times\rho}$, the Kronecker product is denoted by $M\otimes\mathbf{\Omega}$ and defined (in a block form) as

$$M\otimes\mathbf{\Omega}=\begin{bmatrix} m_{11}\mathbf{\Omega} & m_{12}\mathbf{\Omega} & \cdots & m_{1q}\mathbf{\Omega} \\ m_{21}\mathbf{\Omega} & m_{22}\mathbf{\Omega} & \cdots & m_{2q}\mathbf{\Omega} \\ \cdots & \cdots & \cdots & \cdots \\ m_{p1}\mathbf{\Omega} & m_{p2}\mathbf{\Omega} & \cdots & m_{pq}\mathbf{\Omega}\end{bmatrix}\in\mathbb{R}^{p\pi\times q\rho}.$$

- For any interval square matrix defined as in (2) or (3), the bar operator $\overline{(\ )}$ defines a unique constant matrix whose entries are given by (5).

3. COMPONENTWISE-STABILIZING FEEDBACK MATRICES AS A SOLUTION SET OF LINEAR INEQUALITIES

In this section we consider the closed-loop IS (8)&(2) and look for a common characterization of all feedback matrices $F=(f_{kj})\in\mathbb{R}^{m\times n}$ that ensure the $\text{CW}_{de^{ct}}$ - stability of IS (8)&(2). We show that all these matrices (regarded as a matrix class) define the solution set of some linear inequalities.

Definition 2. (a) A feedback matrix $F=(f_{kj})\in\mathbb{R}^{m\times n}$ is called *componentwise-stabilizing* relative to the vector valued function de^{ct} (abbreviated as $\text{CW}_{de^{ct}}$ - *stabilizing*) for IS (1)&(2), if the closed loop IS (8)&(2) is $\text{CW}_{de^{ct}}$ -stable (in the sense of Definition 1).

(b) Denote by $\mathcal{F}_{\text{CW}_{de^{ct}}}$ the set of all $\text{CW}_{de^{ct}}$ -stabilizing feedback matrices for IS (1)&(2). IS (1)&(2) is called *componentwise-stabilizable* relative to the vector function de^{ct} (abbreviated as $\text{CW}_{de^{ct}}$ *-stabilizable*) if $\mathcal{F}_{\text{CW}_{de^{ct}}}\neq\varnothing$. □

Remark 1. According to Definition 2 and Theorem 1, the set $\mathcal{F}_{\text{CW}_{de^{ct}}}$ can be defined as $\mathcal{F}_{\text{CW}_{de^{ct}}}=\{F\in\mathbb{R}^{n\times n}|\ \overline{A+BF}\,d\le cd$, $A\in[A^-,A^+]$, $B\in[B^-,B^+]\}$. However, the inequality $\overline{A+BF}\,d\le cd$ is nonlinear and, hence, inappropriate to numerical computation - as already commented in the introductory section. Therefore below we propose a new characterization for $\mathcal{F}_{\text{CW}_{de^{ct}}}$, exclusively based on linear inequalities. □

Theorem 2. Consider the matrix inequalities

$$\forall s\in\{-1,+1\}^m,\ A^+ + B^sF\le G, \tag{9}$$
$$\forall s\in\{-1,+1\}^m,\ -G^{\text{off}}\le\ (A^- + B^sF)^{\text{off}}\ , \tag{10}$$
$$Gd\le cd, \tag{11}$$

where $F=(f_{kj})\in\mathbb{R}^{m\times n}$, $G=(g_{ij})\in\mathbb{R}^{n\times n}$. Denote by $\mathcal{F}_{(9)-(11)}=\{F\in\mathbb{R}^{m\times n}\ |\ \exists G\in\mathbb{R}^{n\times n}:(9)-(11)\text{ true}\}$ the set of all matrices $F=(f_{kj})\in\mathbb{R}^{m\times n}$ for which there exists $G=(g_{ij})\in\mathbb{R}^{n\times n}$ such that inequalities (9)-(11) are satisfied. Then $\mathcal{F}_{(9)-(11)}=\mathcal{F}_{\text{CW}_{de^{ct}}}$.

Proof: Consider an arbitrary $F\in\mathbb{R}^{m\times n}$ defining the state feedback (7), and denote by $\mathbf{\Theta}=(\theta_{ij})$, $i,j=1,\ldots,n$, the interval matrix

$$\mathbf{\Theta}=A+BF,\ A\in[A^-,A^+],\ B\in[B^-,B^+], \tag{12}$$

of the closed-loop IS (8)&(2). Let $\theta_{ij}^- \le \theta_{ij}^+$ be the lower and upper bounds of the interval associated with θ_{ij}, $i, j = 1,\dots,n$, and consider the corresponding matrices $\Theta^- = (\theta_{ij}^-) \in \mathbb{R}^{n\times n}$, $\Theta^+ = (\theta_{ij}^+) \in \mathbb{R}^{n\times n}$. For any θ_{ij}, we can write

$$\begin{aligned} \theta_{ij}^- = a_{ij}^- + \sum_{k=1}^{m} b_{ik}^{-s_{kj}} f_{kj} \le \theta_{ij} = a_{ij} + \sum_{k=1}^{m} b_{ik} f_{ik} \le \\ \le a_{ij}^+ + \sum_{k=1}^{m} b_{ik}^{+s_{kj}} f_{kj} = \theta_{ij}^+, \end{aligned} \tag{13}$$

where, according to the notations in Section 2,

$$b_{ik}^{+s_{kj}} = \begin{cases} b_{ik}^+, & f_{kj} \ge 0 \\ b_{ik}^-, & f_{kj} < 0 \end{cases} \text{ and } b_{ik}^{-s_{kj}} = \begin{cases} b_{ik}^-, & f_{kj} \ge 0 \\ b_{ik}^+, & f_{kj} < 0 \end{cases}.$$

The expressions of elements θ_{ij}^-, θ_{ij}^+ given by (13) show that matrices Θ^+ and Θ^- can be described columnwise by

$$\begin{aligned} \Theta_{(:,j)}^- &= A_{(:,j)}^- + \sum_{k=1}^{m} B_{(:,k)}^{-s_{kj}} f_{kj}, \\ \Theta_{(:,j)}^+ &= A_{(:,j)}^+ + \sum_{k=1}^{m} B_{(:,k)}^{+s_{kj}} f_{kj},\ j = 1,\dots,n. \end{aligned} \tag{14}$$

Let $F \in \mathcal{F}_{(9)-(11)}$. Since inequalities (9), (10) involve all matrices B^s built columnwise for $s \in \{-1,+1\}^m$, the fulfillment of (9) and (10) ensures $\Theta_{(:,j)}^+ \le G_{(:,j)}$ and $<-G_{(:,j)}>_{\{j\}} \le <\Theta_{(:,j)}^->_{\{j\}}$, for all $j = 1,\dots,n$. This is equivalent to $\theta_{ij}^+ \le g_{ij}$, $i, j = 1,\dots,n$, and $-g_{ij} \le \theta_{ij}^-$, $i \ne j$, $i, j = 1,\dots,n$, respectively. By using the $\overline{(\ )}$ notation, we get $\bar{\theta}_{ij} \le g_{ij}$, $i, j = 1,\dots,n$. In a compact writing, we have $\bar{\Theta} \le G$, which, together with inequality (11), imply $\bar{\Theta} d \le cd$. Theorem 1 guarantees the $\mathrm{CW}_{d^e}^{ct}$-stability of the closed-loop IS (8)&(2), i.e. $F \in \mathcal{F}_{\mathrm{CW}_{d^e}^{ct}}$. Thus, we have proven that $\mathcal{F}_{(9)-(11)} \subseteq \mathcal{F}_{\mathrm{CW}_{d^e}^{ct}}$.

For the counterpart, let us consider $F \in \mathcal{F}_{\mathrm{CW}_{d^e}^{ct}}$, i.e. the closed-loop IS (8)&(2) is $\mathrm{CW}_{d^e}^{ct}$-stable. Theorem 1 ensures the fulfillment of the inequality $\bar{\Theta} d \le cd$.

On the other hand, the expressions of $\theta_{ij}^-, \theta_{ij}^+$ given by (13) show that $\theta_{ij}^- \le a_{ij}^- + \sum_{k=1}^{m} b_{ik}^{\sigma_{ik}} f_{kj}$, $a_{ij}^+ + \sum_{k=1}^{m} b_{ik}^{\sigma_{ik}} f_{kj} \le \theta_{ij}^+$ for any choice of $\sigma_{ik} \in \{-1,+1\}$, which includes the case of a common choice per column, i.e. for all $i = 1,\dots,n$, $\sigma_{ik} = s_k \in \{-1,+1\}$. This means

$$\forall s \in \{-1,+1\}^m:\ A^+ + B^s F \le \Theta^+,\ \Theta^- \le A^- + B^s F. \tag{15}$$

Since $\Theta^+ \le \bar{\Theta}$ and $(-\bar{\Theta})^{\mathrm{off}} \le (\Theta^-)^{\mathrm{off}}$, the inequalities (9) – (11) are satisfied for the considered F and $G = \bar{\Theta}$, i.e. $F \in \mathcal{F}_{\mathrm{CW}_{d^e}^{ct}}$. Thus, we have proven $\mathcal{F}_{\mathrm{CW}_{d^e}^{ct}} \subseteq \mathcal{F}_{(9)-(11)}$. □

Remark 2. Theorem 2 shows that the problem of $\mathrm{CW}_{d^e}^{ct}$ stabilization for IS (1)&(2) does not require the exploration of all 2^{mn} vertices of the polytope defined by the interval matrix $[B^-, B^+]$. The proof of Theorem 2 reveals that from 2^{mn} possible tests, only 2^m tests are meaningful. In other words, by checking those 2^m vertices specified by the theorem, one gets complete information about the extreme values of the interval entries of the closed-loop matrix $A + BF$. □

4. COMPUTATIONAL APPROACH

Although inequalities (9)-(11) are linear, their matrix form is still inconvenient for the automatic manipulation in a scientific software environment. Therefore we reorganize the matrix inequalities (9)-(11) in the standard form of a linear inequality with appropriate dimensions $M\omega \le v$, $M \in \mathbb{R}^{p\times q}$, $\omega \in \mathbb{R}^q$, $v \in \mathbb{R}^p$, where the vector ω collects the elements of the matrices F and G.

Theorem 3. Consider the inequality

$$\left[\begin{array}{cc} I_n \otimes B^{[+1\cdots+1]} & -I_{n^2} \\ \vdots & \vdots \\ I_n \otimes B^{[-1\cdots-1]} & -I_{n^2} \\ \hline -<I_n \otimes B^{[+1\cdots+1]}>_r & -<I_{n^2}>_r \\ \vdots & \vdots \\ -<I_n \otimes B^{[-1\cdots-1]}>_r & -<I_{n^2}>_r \\ \hline 0_{n\times mn} & d^T \otimes I_n \end{array}\right] \begin{bmatrix} \varphi \\ \gamma \end{bmatrix} \le \left[\begin{array}{c} -A^+|_{n^2\times 1}^{\mathrm{vec}} \\ \vdots \\ -A^+|_{n^2\times 1}^{\mathrm{vec}} \\ \hline <A^-|_{n^2\times 1}^{\mathrm{vec}}>_r \\ \vdots \\ <A^-|_{n^2\times 1}^{\mathrm{vec}}>_r \\ \hline cd \end{array}\right] \tag{16}$$

where $r = \{r_1,\dots,r_j,\dots,r_n\}$, $r_j = (j-1)n + j$, $j = 1,\dots,n$, and $\varphi \in \mathbb{R}^{nm}$, $\gamma \in \mathbb{R}^{n^2}$. Denote by $\Phi_{(16)}$ the set of all vectors $\varphi \in \mathbb{R}^{nm}$ for which there exists $\gamma \in \mathbb{R}^{n^2}$ such that inequality (16) is satisfied, i.e $\Phi_{(16)} = \{\varphi \in \mathbb{R}^{mn} \mid \exists \gamma \in \mathbb{R}^{n^2} : (16) \text{ true}\}$. Define $\mathcal{F}_{(16)} = \{F \in \mathbb{R}^{m\times n} \mid F = \varphi|_{m\times n}^{\mathrm{mat}}, \varphi \in \Phi_{(16)}\}$ the set of matrices obtained by reshaping the vectors from $\Phi_{(16)}$.

Then $\mathcal{F}_{(16)} = \mathcal{F}_{\mathrm{CW}_{d^e}^{ct}}$.

Proof: It consists in showing that inequality (16) is equivalent to inequalities (9)-(11). □

Once we know that $\mathcal{F}_{\mathrm{CW}_{d^e}^{ct}}$ represents the solution set of the linear inequality (16), we are interested in developing a computational procedure for finding a concrete $\mathrm{CW}_{d^e}^{ct}$-stabilizing feedback matrix $F \in \mathcal{F}_{\mathrm{CW}_{d^e}^{ct}}$, when $\mathcal{F}_{\mathrm{CW}_{d^e}^{ct}} \ne \varnothing$.

We propose a linear programming (LP) approach since the linear inequality (16) can be exploited for defining the constraints, and the minimization can refer to the decreasing rate of the invariant sets (4) which ensure the componentwise stability of the closed loop IS.

Theorem 4. Consider the LP problem that minimizes the objective function:

$$J(\varphi,\gamma,\lambda)=\lambda \tag{17}$$

with the constraints

$$\left[\begin{array}{ccc} I_n \otimes B^{[+1\cdots+1]} & -I_{n^2} & 0_{n^2\times 1} \\ \vdots & \vdots & \vdots \\ I_n \otimes B^{[-1\cdots-1]} & -I_{n^2} & 0_{n^2\times 1} \\ \hdashline -<I_n \otimes B^{[+1\cdots+1]}>_r & -<I_{n^2}>_r & 0_{(n^2-n)\times 1} \\ \vdots & \vdots & \vdots \\ -<I_n \otimes B^{[-1\cdots-1]}>_r & -<I_{n^2}>_r & 0_{(n^2-n)\times 1} \\ \hdashline 0_{n\times mn} & d^T \otimes I_n & -d \\ 0_{1\times mn} & 0_{1\times n^2} & 1 \end{array}\right] \begin{bmatrix} \varphi \\ \gamma \\ \lambda \end{bmatrix} \le \left[\begin{array}{c} -A^+|^{\mathrm{vec}}_{n^2\times 1} \\ \vdots \\ -A^+|^{\mathrm{vec}}_{n^2\times 1} \\ \hdashline <A^-|^{\mathrm{vec}}_{n^2\times 1}>_r \\ \vdots \\ <A^-|^{\mathrm{vec}}_{n^2\times 1}>_r \\ \hdashline 0_{n\times 1} \\ c \end{array}\right] \tag{18}$$

where $r=\{r_1,\ldots,r_j,\ldots,r_n\}$, $r_j=(j-1)n+j$, $j=1,\ldots,n$, and $\varphi\in\mathbb{R}^{nm}$, $\gamma\in\mathbb{R}^{n^2}$, $\lambda\in\mathbb{R}$. Denote by $\boldsymbol{\Phi}_{LP}$ the set of all vectors $\varphi\in\mathbb{R}^{nm}$ which are solutions to LP. Define the set of matrices $\mathcal{F}_{LP}=\{F\in\mathbb{R}^{m\times n} \mid F=\varphi|^{\mathrm{mat}}_{mn\times 1}, \varphi\in\boldsymbol{\Phi}_{LP}\}$. Then

(a) $\mathcal{F}_{LP}\subseteq\mathcal{F}_{\mathrm{CW}_{de^{ct}}}$.

(b) $\mathcal{F}_{LP}\equiv\varnothing \Rightarrow \mathcal{F}_{\mathrm{CW}_{de^{ct}}}\equiv\varnothing$.

Proof: a) Let $F\in\mathcal{F}_{LP}$ and consider the corresponding solution $\varphi\in\mathbb{R}^{nm}$, $\gamma\in\mathbb{R}^{n^2}$, $\lambda\in\mathbb{R}$ of the LP problem. This means λ fulfills the constraint $\lambda\le c$, and $\lambda d\le cd$. Consequently, $\varphi\in\mathbb{R}^{nm}$, $\gamma\in\mathbb{R}^{n^2}$, satisfy the inequality (16) i.e. $F\in\mathcal{F}_{(16)}$, and Theorems 3 guarantees $F\in\mathcal{F}_{\mathrm{CW}_{de^{ct}}}$.

b) Assume that $\mathcal{F}_{LP}\equiv\varnothing$, but $\mathcal{F}_{\mathrm{CW}_{de^{ct}}}\ne\varnothing$. According to Theorem 3, inequality (16) has solution(s) $\varphi\in\mathbb{R}^{nm}$, $\gamma\in\mathbb{R}^{n^2}$, and these vectors together with $\lambda=c$ also satisfy the constraints (18) of the LP problem. Hence, the LP problem is feasible and $\mathcal{F}_{LP}\ne\varnothing$, fact which, by contradicting the hypothesis, completes the proof. □

Remark 3. Theorem 4 provides a computable necessary and sufficient condition for the $\mathrm{CW}_{de^{ct}}$ - stabilizability of IS (1)&(2). The robust numerical tractability of the LP problems ensures the practical applicability of the result. The LP solver returns an unfeasible solution if and only if IS (1)&(2) is not $\mathrm{CW}_{de^{ct}}$ - stabilizable; otherwise any feasible solution can be used as a $\mathrm{CW}_{de^{ct}}$ - stabilizing feedback matrix. □

Remark 4. The numerical approach to $\mathrm{CW}_{de^{ct}}$ - stabilizability of IS (1)&(2) can also rely on Theorem 3, by considering a constant objective function $J(\varphi,\gamma)$ with constraints given by inequality (16). The usage of an LP solver is still possible in the same manner as commented in Remark 3 with regard to Theorem 4. However the great advantage of Theorem 4 consists in finding (whenever it exists) a feedback matrix that ensures the fastest decreasing rate for the invariant sets (4). Thus, if the LP solver used in the context of Theorem 4 returns a solution $\varphi\in\mathbb{R}^{nm}$, $\gamma\in\mathbb{R}^{n^2}$, $\lambda\in\mathbb{R}$, with $\lambda<c$, then the feedback matrix $F=\varphi|^{\mathrm{mat}}_{m\times n}$ guarantees the invariance of the sets $X^{\varepsilon}_{de^{\lambda t}}=\{x\in\mathbb{R}^n \mid \|[d_1^{-1}x_1\cdots d_n^{-1}x_n]^T\|_\infty\le\varepsilon e^{\lambda t}\}$, with the same $d_i>0$, $i=1,\ldots,n$, as in (4), but approaching the equilibrium faster. Thus, the minimization of λ may improve the stability margin (in the standard sense) of the closed-loop IS, by left shifting the eigenvalues of $A+BF$, $A\in[A^-,A^+]$, $B\in[B^-,B^+]$. □

5. ILLUSTRATIVE EXAMPLE

To illustrate our approach, we consider the IS (1)&(2) for the interval matrices defined by

$$A^-=\begin{bmatrix} -2 & -5 \\ -2.75 & -1 \end{bmatrix},\ A^+=\begin{bmatrix} 2.50 & -4.50 \\ -2 & 2 \end{bmatrix}, \qquad b^-=[-2 \quad 1]^T, \qquad b^+=[-1.75 \quad 1.20]^T. \tag{19}$$

Notice that IS (1)&(2) with $u(t)\equiv 0$ is not asymptotically stable since there exist matrices $A\in[A^-,A^+]$ that are not Hurwitz stable (for example, A^+).

For $d=[1 \quad 2]^T$ and $c=-0.1$ as design specifications, we want to find a feedback matrix F that allows the $\mathrm{CW}_{de^{ct}}$ - stabilization of the considered IS. We use Theorem 4 and the LP problem is solved by using the **linprog** function from the Optimization Toolbox for MATLAB.

The numerical solution is $\varphi\in\mathbb{R}^2$, $\varphi^T=[2.1986,-2.5333]^T$, $\gamma\in\mathbb{R}^4$, $\gamma^T=[-1.3475, 0.6383, 0.5667, -0.5333]^T$ and $\lambda=-0.2142$. Taking Remark 4 into account, we conclude that the feedback matrix $F=\varphi|^{\mathrm{mat}}_{1\times 2}=[2.1986 \ -2.5333]$ guarantees that the closed-loop IS (8)&(2) is not only $\mathrm{CW}_{de^{ct}}$ -stable, but also $\mathrm{CW}_{de^{\lambda t}}$ -stable, with $\lambda=-0.2142<c=-0.1$. In terms of positive invariance, the sets $X^{\varepsilon}_{de^{\lambda t}}$ decrease faster than $X^{\varepsilon}_{de^{ct}}$.

For the above F, the matrix $\bar{\boldsymbol{\Theta}}$ of the closed-loop IS (8)&(2) is given by the elements of vector γ reshaped as $\bar{\boldsymbol{\Theta}}=G=\gamma|^{\mathrm{mat}}_{2\times 2}=\begin{bmatrix} -1.3475 & 0.5667 \\ 0.6383 & -0.5333 \end{bmatrix}$.

6. CONCLUSIONS

Compared with the standard concept of stabilization, the $CW_{d e^{ct}}$ -stabilization ensures supplementary properties to the closed-loop trajectories of an IS. These properties are related to the existence of exponentially decreasing sets of rectangular form, which are invariant with respect to the closed-loop trajectories. In colloquial terms, despite the interval type uncertainties, the $CW_{d e^{ct}}$ -stabilization “obliges” all the trajectories initialized inside a box to remain inside a homothetic box which reduces its size exponentially.

Thus, by $CW_{d e^{ct}}$ -stabilization, the exponential decrease of the invariant boxes offers a global characterization for all the trajectories, which does not necessarily result from the asymptotic stability of the closed-loop IS. This discussion explains why the $CW_{d e^{ct}}$ -stabilization may be regarded as conservative in comparison with the standard stabilizability, but, at the same time, why it deserves the designers' attention.

The possibility of monitoring the entire evolution of an IS by means of invariant boxes that decrease exponentially creates an evident benefit for applications. On the other hand, our computer oriented work to $CW_{d e^{ct}}$ -stabilization also motivates the approach to concrete problems.

The proposed technique is numerically robust and operates in a single step, either providing a feedback matrix that achieves the $CW_{d e^{ct}}$ -stabilization of the IS, or deciding that the IS is not $CW_{d e^{ct}}$ -stabilizable.

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