Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

A Balancing Approach to Model Reduction of Polynomial Nonlinear Systems ⋆

Hardy B. Siahaan *,**

* *International Research Institute of Stavanger (IRIS), Thormohlensgate 55, N-5008 Bergen, Norway (e-mail: hsi@iris.no)*
** *Department of Engineering Cybernetics, Norwegian University of Science and Technology (NTNU), 7491 Trondheim, Norway*

Abstract: This paper considers a computational approach to obtain a reduced order model for polynomial nonlinear systems. The approach is based on balancing generalized gramians of polynomial systems and truncating the systems based on the balanced generalized gramians. The approach utilizes sum of squares programming for its computational purposes.

1. INTRODUCTION

Model reduction by balanced truncation for linear systems was introduced by Moore [1981]. This method is systematic on its construction of reduced model and very popular among other reduction schemes for linear systems due to its simplicity. On the other hand, there is no similar systematic procedure of balanced truncation like in linear systems which can be implemented for nonlinear systems as the problem in nonlinear systems becomes more difficult to understand and solve and it is still subject to further research. However there exist several approaches in the literature dealing with nonlinear systems. In particular, a mechanism for balancing nonlinear systems is given by Scherpen [1993] and an empirical approach for truncating nonlinear systems is given by Lall et al. [2002].

One of the main advantage of the empirical approach introduced in Lall et al. [2002] is that its computational scheme to construct a reduced order model is not expensive as it only requires linear matrix computations. Its limitation is on the resulting reduced order model. The reduced model is expected to work well within a working region of state space as the method relies heavily on the snapshots of data. In this case the quality of its reduced model depends on the collection of data obtained through generating trajectories of the original system.

On the other hand the approach in Scherpen [1993] does not depend on the snapshots of data. The method first computes a controllability function and an observability function from Hamilton-Jacobi equations. In general those functions are not entirely balanced. By seeking a coordinate transformation the original system is made balance to some extent of definition. A reduced order model is obtained by truncating the balanced system. The drawback of this method is on the computation of the controllability and observability functions which in most cases are very difficult.

One way to avoid this problem is introduced in Prajna & Sandberg [2005] where the authors consider generalized controllability and observability functions which are obtained through Hamilton-Jacobi inequalities instead of Hamilton-Jacobi equalities. Despite the fact that the truncation scheme based on these generalized functions will not guarantee to give a stable reduced order model for a stable original system, the advantage of this approach is that it exploits the use of sum of squares programming (Parrilo [2003]) to compute the generalized functions and thus is amenable to computer solution in case the original system to be reduced has polynomial vector fields. This approach shows promising direction in developing computational scheme for model reduction of polynomial systems. However, this approach still leaves an open problem on how to balance the generalised functions which are non-quadratic.

This paper introduces an approach of balancing the non-quadratic generalised functions which are obtained through the same procedure in Prajna & Sandberg [2005]. In this case, we use another quadratic functions which can be viewed as conservative version of non-quadratic generalised functions. Instead of balancing the generalised functions, we focus on the conservative quadratic functions as the basis of performing truncation for polynomial nonlinear systems.

The following notation will be used throughout the paper. The set of real numbers is denoted by $\mathbb{R}$. The collection of all real matrices of size $n \times m$ is denoted by $\mathbb{R}^{n\times m}$. The set of polynomial in x with real coefficient is denoted by $\mathbb{R}[x]$. The superscript $'$ stands for matrix transposition. A scalar function $w(x)$ is said to be positive definite if $w(0) = 0$ and $w(x) > 0$ for all $x \neq 0$. The notation $\|\cdot\|$ means the Euclidean norm of the vector involved.

2. GENERALIZED FUNCTIONS

We consider polynomial nonlinear system in the form

$$\dot{x} = f(x) + B(x)\,u, \tag{1a}$$

$$y = h(x), \tag{1b}$$

where $x = [x_1, \ldots, x_n]^T \in D_x \subset \mathbb{R}^n$ is the state vector of the system, $u \in \mathbb{R}^{n_u}$ is the input to the system and $y \in \mathbb{R}^{n_y}$ is the output of the system. The functions $f : D_x \to \mathbb{R}^n$,

⋆ This work was supported by Statoil and the Norwegian Research Council.

$g : D_x \to \mathbb{R}^{n \times n_u}$ and $h : D_x \to \mathbb{R}^{n_y}$ are polynomials in x and therefore smooth. We assume that the origin of the unforced system $\dot{x} = f(x)$ is asymptotically stable on D_x.

We define two functions which characterize the minimum energy of the input to reach particular state and the minimum energy of the output generated by a particular initial condition.

Definition 1. A positive definite function $W_o(x)$ with $W_o(0) = 0$ is an observability function to the system (1) if it satisfies

$$\frac{\partial W_o(x)}{\partial x} f(x) + \frac{1}{2} h'(x) h(x) = 0 \qquad (2)$$

for all $x \in D_x$.

Definition 2. A positive definite function $W_c(x)$ with $W_c(0) = 0$ is a reachability function to the system (1) if it satisfies

$$\frac{\partial W_c(x)}{\partial x} f(x) + \frac{1}{2} \frac{\partial W_c(x)}{\partial x} g(x) g(x)' \frac{\partial W_c(x)}{\partial x} = 0. \qquad (3)$$

for all $x \in D_x$.

The reachability function satisfies

$$W_c(x_0) = \frac{1}{2} \int_{-\infty}^{0} \|u_{\min}(t)\|^2 \, dt$$

where $u_{\min} \in L_2(-\infty, 0]$, is the input with minimum energy required for the system (1) with $x(-\infty) = 0$ such that $x(0) = x_0$. The observability function satisfies

$$W_o(x_0) = \frac{1}{2} \int_{0}^{\infty} \|y(t)\|^2 \, dt$$

where y is the output of the system (1) with $x(0) = x_0$ and $u(t) = 0$ for $t \in [0, \infty)$.

For more details on the existence of solution of $W_o(x)$ and $W_c(x)$ the reader can consult Scherpen [1993]. As we move to practicality we will encounter difficulty in computing the observability and reachability functions as there is no tractable computational scheme serving the purpose yet. Instead of computing both functions for the purpose of model reduction through balanced truncation, we consider an approach where we use generalized observability and rechability functions as introduced in Prajna & Sandberg [2005].

Definition 3. A positive definite polynomial function $L_o(x)$ with $L_o(0) = 0$ is a generalized observability function to the system (1) if it satisfies

$$\frac{\partial L_o(x)}{\partial x} f(x) + \frac{1}{2} h'(x) h(x) \leq 0 \qquad (4)$$

for all $x \in D_x$.

Definition 4. A positive definite polynomial function $L_c(x)$ with $L_c(0) = 0$ is a generalized reachability function to the system (1) if it satisfies

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{\partial L_c(x)}{\partial x} g(x) u - \frac{1}{2} u' u \leq 0 \qquad (5)$$

for all $x \in D_x$ and $u \in \mathbb{R}^{n_u}$.

The degree of polynomial generalized functions $L_o(x)$ and $L_c(x)$ should not be less than two to guarantee positive definiteness and being vanished at the origin.

Some properties pertaining to the generalized functions are summarized as follows.

- A generalized observability function $L_o(x)$ to the system (1) satisfies
$$L_o(x) \geq W_o(x)$$
for all $x \in D_x$.
- A generalized reachability function $L_c(x)$ to the system (1) satisfies
$$L_c(x) \leq W_c(x)$$
for all $x \in D_x$.

It is important to note that there are many choices of generalized functions which satisfy (4-5). A means to classify a closer representation of the functions $L_o(x)$ and $L_c(x)$ with the functions $W_o(x)$ and $W_c(x)$ is by introducing

$$\gamma_{opt} = \sup_{0 \neq x \in D_x} \frac{W_o(x)}{W_c(x)}.$$

Then the generalized functions $L_o(x)$ and $L_c(x)$ which satisfy (4-5) are computed such that

$$L_o(x) \leq \gamma L_c(x) \qquad (6)$$

for all $x \in D_x$ where $\gamma > 0$. Hence γ is an upper bound for the gain γ_{opt}. In this case we minimize the constant γ so that the upper bound is as tight as possible.

3. BALANCED TRUNCATION BASED ON APPROXIMATE GENERALIZED FUNCTIONS

The generalized functions characterize the states, in the domain of interest, which are weakly observable and reachable in the following ways. The states which have small value of $L_o(x)$ are considered to be less observable. The states which have larger value of $L_c(x)$ are considered to be less reachable.

Now let us consider the class of input with

$$\int_0^T \|u(t)\|^2 dt \leq K_u,$$
$$u(t) = 0 \ \forall t > T,$$

for all $T \in [0, \infty)$. Suppose that we generate the trajectories from the origin with this class of input then for $t \in [0, T]$ all the trajectories of the system will be inside of the set

$$\mathcal{R}_o = \left\{ x \in \mathbb{R}^n \ | L_c(x) \leq \frac{1}{2} K_u \right\}.$$

It is easy to see that the trajectories will also remain in $\mathcal{R}_o$ for $t \geq T$ because $\mathcal{R}_o$ is a positively invariant set for zero input. Hence for the given class of input with initial condition at the origin all the trajectories of the system will be inside of $\mathcal{R}_o$. Then for any $x \in \mathcal{R}_o$ we have $L_o(x) \leq \frac{1}{2}\gamma K_u$.

For the reachability set

$$\mathcal{R}_c = \left\{ x \in \mathbb{R}^n | \ L_c(x) \leq \frac{1}{2} K_u \right\}$$

which is compact a smaller value of $L_c(x)$ means that the state x is easier to reach for the given admissible input. For the observability set

$$\mathcal{R}_o = \left\{ x \in \mathbb{R}^n | \ L_o(x) \leq \frac{1}{2} \gamma K_u \right\}$$

which is compact a larger value of $L_o(x)$ means that the state x is easier to observe from the output. In this paper, instead of dealing with the sets $\mathcal{R}_c$ and $\mathcal{R}_o$ to analyse the most/least important part of the system which is reachable and observable, we will consider another sets Ω_c and Ω_o as the estimate of the reachability set $\mathcal{R}_c$ and observability set $\mathcal{R}_o$, respectively, in a way that $\mathcal{R}_c \subseteq \Omega_c$ and $\mathcal{R}_o \subseteq \Omega_o$ where the sets Ω_c and Ω_o are to be as close as possible to the sets $\mathcal{R}_c$ and $\mathcal{R}_o$, respectively. The sets Ω_c and Ω_o will be characterised by quadratic functions $\hat{L}_o(x)$ and $\hat{L}_c(x)$ such that

$$\Omega_c = \left\{ x \in \mathbb{R}^n | \; \hat{L}_c(x) = \frac{1}{2} x' Y^{-1} x \leq 1 \right\},$$
$$\Omega_o = \left\{ x \in \mathbb{R}^n | \; \hat{L}_o(x) = \frac{1}{2} x' X x \leq 1 \right\}$$

where $X, Y \succ 0$ are symmetric matrices.

Though the sets Ω_c and Ω_o are more conservative they are easier to analyse because the sets are in the form of hyperellipsoid. It is easy to see from the principal axes of the hyperellipsoid Ω_c that a longer axis represent a more reachable part of the state. On the other hand, from the principal axes of the hyperellipsoid Ω_o, we can see that a shorter axis represent a more observable part of the state. Equivalently, we can analyse the length of the axes from the eigenvalues of matrices Y^{-1} and X. The associated eigenvector of a smaller eigenvalues of matrix Y^{-1} represents the direction of a longer axis of hyperellipsoidal set Ω_c. (Or equivalently, the associated eigenvector of a larger eigenvalues of matrix Y represents the direction of a longer axis of hyperellipsoidal set Ω_c.) Similarly, the associated eigenvector of a larger eigenvalues of matrix X represents the direction of a shorter axis of hyperellipsoidal set Ω_o. Thus, the associated eigenvector of a larger eigenvalues of matrices Y and X represent the direction of the states which are more strongly reachable and observable, respectively.

To satisfy the containment $\mathcal{R}_c \subseteq \Omega_c$ and $\mathcal{R}_o \subseteq \Omega_o$ we use the following sufficient conditions which are easy to implement numerically.

Proposition 5. Let $L(x)$ be a positive definite polynomial function with $L(0) = 0$. Let

$$\mathcal{R} = \{ x \in D_x | \; L(x) \leq \epsilon \},$$
$$\Omega = \left\{ x \in D_x | \; \frac{1}{2} x' \Phi x \leq 1 \right\},$$

for positive constant $\epsilon \in \mathbb{R}$ and positive definite matrix $\Phi = \Phi' \in \mathbb{R}^{n \times n}$. If there exists a positive definite polynomial $s(x)$ such that

$$1 - \frac{1}{2} x' \Phi x + s(x) \left(L(x) - \epsilon \right) \geq 0 \tag{7}$$

for all $x \in D_x$ then $\mathcal{R} \subseteq \Omega$.

Proof. For any $x \in \mathcal{R}$ we have $L(x) - \epsilon \leq 0$. It follows that $0 \leq 1 - \frac{1}{2} x' \Phi x + s(x) \left(L(x) - \epsilon \right) \leq 1 - \frac{1}{2} x' \Phi x$ or $x \in \Omega$. □

To reduce conservatism of the set Ω we require that the set $\mathcal{R}$ should be contained in as small Ω as possible.

We now consider a change of basis $x = \Gamma z$ where Γ is given by

$$\Gamma = Y^{\frac{1}{2}} U S^{-\frac{1}{2}}$$

where matrices U and S are obtained from the singular value decomposition of

$$Y^{\frac{1}{2}} X Y^{\frac{1}{2}} = U S^2 U'.$$

It is easy to see that

$$\begin{aligned} \Gamma' X \Gamma &= \left(S^{-\frac{1}{2}} U' Y^{\frac{1}{2}} \right) X \left(Y^{\frac{1}{2}} U S^{-\frac{1}{2}} \right) \\ &= S^{-\frac{1}{2}} U' U S^2 U' U S^{-\frac{1}{2}} = S \end{aligned}$$

and

$$\begin{aligned} \Gamma^{-1} Y \left(\Gamma^{-1} \right)' &= \left(S^{\frac{1}{2}} U' Y^{-\frac{1}{2}} \right) Y \left(Y^{-\frac{1}{2}} U S^{\frac{1}{2}} \right) \\ &= S. \end{aligned}$$

With respect to the change of basis the system (1) can be transformed into

$$\dot{z} = \hat{f}(z) + \hat{B}(z) u, \tag{8a}$$
$$y = \hat{h}(z), \tag{8b}$$

where

$$\hat{f}(z) = \Gamma^{-1} f(\Gamma z), \; \hat{g}(z) = \Gamma^{-1} g(\Gamma z), \; \hat{h}(z) = h(\Gamma z).$$

The generalized functions for the system (8) are given as follows.

Proposition 6. For a coordinate transformation $x = \Gamma z$, which brings the system (1) into (8) we define $\hat{L}_o(z) = L_o(\Gamma z)$ and $\hat{L}_c(z) = L_c(\Gamma z)$. Then $\hat{L}_o(z)$ and $\hat{L}_c(z)$ are generalized observability and reachability functions, respectively for (8).

Proof. For reachability

$$\begin{aligned} \phi_c(x) =& \frac{\partial \hat{L}_c(z)}{\partial z} \left(\hat{f}(z) + \hat{g}(z) u \right) - \frac{1}{2} u' u \\ =& \left(\frac{\partial L_c(\nu)}{\partial \nu} \frac{\partial \nu}{\partial z} \left(\Gamma^{-1} f(\Gamma z) + \Gamma^{-1} g(\Gamma z) u \right) \right)_{\nu = \Gamma z} \\ & - \frac{1}{2} u' u \\ =& \frac{\partial L_c(\nu)}{\partial \nu} f(\nu) + \frac{\partial L_c(\nu)}{\partial \nu} g(\nu) u - \frac{1}{2} u' u \leq 0. \end{aligned}$$

For observability

$$\begin{aligned} \phi_o(x) =& \frac{\partial \hat{L}_o(z)}{\partial z} \hat{f}(z) + \frac{1}{2} \hat{h}(z)' \hat{h}(z) \\ =& \left(\frac{\partial L_o(\nu)}{\partial \nu} \frac{\partial \nu}{\partial z} \Gamma^{-1} f(\Gamma z) \right)_{\nu = \Gamma z} + \frac{1}{2} h(\Gamma z)' h(\Gamma z) \\ =& \frac{\partial L_o(\nu)}{\partial \nu} f(\nu) + \frac{1}{2} h(\nu)' h(\nu) \leq 0. \end{aligned}$$

□

Furthermore it is easy to see that

$$\begin{aligned} 0 &\leq 1 - \frac{1}{2} x' Y^{-1} x + s_c(x) \left(L_c(x) - \frac{1}{2} K_u \right) \\ &= 1 - \frac{1}{2} z \Gamma' Y^{-1} \Gamma z + \hat{s}_c(z) \left(\hat{L}_c(z) - \frac{1}{2} K_u \right) \\ &= 1 - \frac{1}{2} z S^{-1} z + \hat{s}_c(z) \left(\hat{L}_c(z) - \frac{1}{2} K_u \right) \end{aligned}$$

where $\hat{s}_c(z) = s_c(\Gamma z)$ are SOS in z and

$$\begin{aligned} 0 &\leq 1 - \frac{1}{2} x' X x + s_o(x) \left(L_o(x) - \frac{1}{2} \gamma K_u \right) \\ &= 1 - \frac{1}{2} z' \Gamma' X \Gamma z + \hat{s}_o(z) \left(\hat{L}_o(z) - \frac{1}{2} \gamma K_u \right) \\ &= 1 - \frac{1}{2} z' S z + \hat{s}_o(z) \left(\hat{L}_o(z) - \frac{1}{2} \gamma K_u \right) \end{aligned}$$

where $\hat{s}_o(z) = s_o(\Gamma z)$ are SOS in z. Using the same argument like in Proposition 5 it follows that

$$\left\{z \in \mathbb{R}^n | \hat{L}_c(z) \le \frac{1}{2}K_u\right\} \subseteq \left\{z \in \mathbb{R}^n | \frac{1}{2}zS^{-1}z \le 1\right\},$$
$$\left\{z \in \mathbb{R}^n | \hat{L}_o(z) \le \frac{1}{2}\gamma K_u\right\} \subseteq \left\{z \in \mathbb{R}^n | \frac{1}{2}z'Sz \le 1\right\},$$

which indicate that the transformed system (8) have a balanced representation in that the states which are more strongly reachable and observable are more or less in the same direction. In this case the associated eigenvector of a larger eigenvalue of matrix S represent the direction of the states which is both more strongly reachable and observable.

Next, we partition the part of size n into two parts of size n_r and $n - n_r$ with $n_r < n$ based on the following

$$z = \left[\, z'_{[1]}\;\; z'_{[2]} \,\right]',$$
$$\Gamma = [\,\Gamma_1\;\; \Gamma_2\,],$$
$$\Gamma^{-1} = [\,\Upsilon_1\;\; \Upsilon_2\,].$$

By removing the weakly reachable and observable part $z_{[2]}$ we have the dynamic of our new reduced model $x_r = z_{[1]}$ of dimension n_r given by

$$\dot{x}_r = f_r(x_r) + B_r(x_r)u, \tag{9a}$$
$$y_r = h_r(x_r), \tag{9b}$$

where

$$f_r(x_r) = \Upsilon_1 f(\Gamma_1 x_r),\; B_r(x_r) = \Upsilon_1 B(\Gamma_1 x_r),$$
$$h_r(x_r) = h(\Gamma_1 x_r).$$

To sum up, we have obtained a reduced order model (9) for the system (1) where the least reachable and observable parts in (1) are removed while the most influential parts are preserved in (9).

4. SUM OF SQUARES FORMULATION

We define a polynomial in the form

$$p(x) = \sum_i p_i^2(x)$$

as a sum of squares (SOS) polynomial when $p_i(x)$ are polynomials. It is obvious that any polynomial which can be expressed as an SOS of other polynomials is nonnegative everywhere. One way to express an SOS equivalently is by

$$p(x) = z'(x)Mz(x)$$

where M is a positive semidefinite symmetric matrix and $z(x)$ is monomial of degree less than or equal to half of the degree of $p(x)$. For the same monomial $z(x)$ it might be possible to have similar representation with different M with M being not positive semidefinite. Thus if the intersection of $\{M \in S_n | p(x) = z'(x)Mz(x)\}$ with $\{M \in S_n | M \succeq 0\}$ is not empty then $p(x) = z'(x)Mz(x)$ is an SOS. Within this direction, in Parrilo [2003] the author showed that determining whether a polynomial can be expressed as an SOS is an LMI problem. Hence, the problem of testing whether a polynomial is sum of squares becomes relatively easy as it can be computed using semidefinite programming. In view of the fact that verifying nonnegativity of a polynomial is very difficult, throughout the paper, we will relax most polynomial inequalities by replacing nonnegativity with SOS condition.

Since we are concerned with polynomial systems, computation of the generalized functions $L_o(x)$ and $L_c(x)$ can be done efficiently by relaxing the left hand side of inequalities (4-5) being SOS. For the case $D_x = \mathbb{R}^n$, computational scheme forthe generalized functions can be summarized as follows.

- Minimize $\gamma \ge 0$ and find positive definite polynomials $L_o(x)$ and $L_c(x)$ with $L_o(0) = 0$ and $L_c(0) = 0$ such that

$$-\frac{\partial L_o(x)}{\partial x} f(x) - \frac{1}{2}h(x)' h(x) \text{ is SOS}, \tag{10}$$
$$-\frac{\partial L_c(x)}{\partial x} f(x) - \frac{\partial L_c(x)}{\partial x} g(x)u + \frac{1}{2}u'u \le 0 \text{ is SOS}, \tag{11}$$
$$\gamma L_c(x) - L_o(x) \text{ is SOS}. \tag{12}$$

For containment in Proposition 5 where the set $\mathcal{R}$ should be contained in as small Ω as possible we can maximize the trace of matrix Φ so that the volume of Ω is minimized. Hence, we can summarize our computational approach to get the sets Ω_c and Ω_o as follows.

(1) Maximize trace Y^{-1} such that

$$1 - \frac{1}{2}x'Y^{-1}x + s_c(x)\left(L_c(x) - \frac{1}{2}K_u\right) \text{ is SOS}$$

where $s_c(x)$ is a sum of squares.

(2) Maximize trace X such that

$$1 - \frac{1}{2}x'Xx + s_o(x)\left(L_o(x) - \frac{1}{2}\gamma K_u\right) \text{ is SOS}$$

where $s_o(x)$ is a sum of squares.

For the case where D_x is a semialgebraic set given by

$$D_x = \{x \in \mathbb{R}^n |\; p_i(x) \ge 0; i = 1, \ldots, m\}$$

where $p_i \in \mathbb{R}[x]$ we can use the following relaxation which is deduced from the Positivestellensatz in Stengle [1974].

Proposition 7. If there exists sum of squares $s_i(x)$ for $i = 1, \ldots, m$ such that

$$q(x) - p_1(x)s_1(x) - \ldots - p_m(x)s_m(x) \text{ is sum of squares}$$

then $q(x) \ge 0\; \forall x \in D_x$.

This proposition can be easily applied for feasibility test of (4-5), (6) and (7) using sum of squares programming.

5. EXAMPLE

In this section, two numerical examples are given to illustrate the applicability of the proposed approach.

5.1 Example 1

Consider the system from Siahaan et al. [2007]

$$\dot{x}_1 = -x_2 - x_3 - x_1(x_1^2 + x_2^2 + x_3^2 + 1),$$
$$\dot{x}_2 = x_1 - x_3 - x_2(x_1^2 + x_2^2 + x_3^2 + 1),$$
$$\dot{x}_3 = x_1 + x_2 - x_3(x_1^2 + x_2^2 + x_3^2 + 1) + u,$$
$$y = x_1.$$

We want to compute a reduced model of order two for the global case $D_x = \mathbb{R}^3$. Feasibility tests are carried out

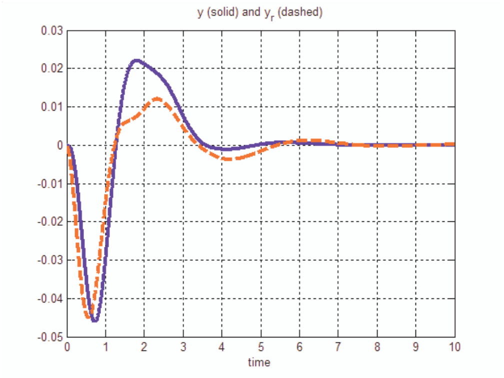

Fig. 1. Response of the system in Example 1 to the input $u(t) = e^{-1.5t}sin(5t)$

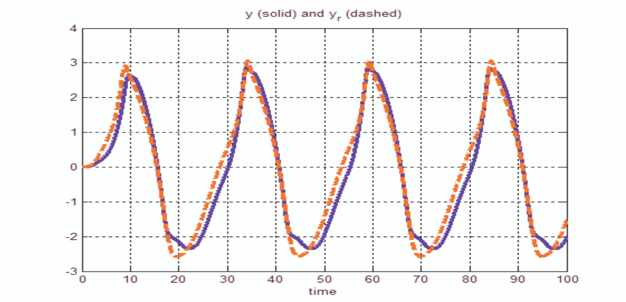

Fig. 2. Response of the system in Example 2 to the sinusoidal input $u(t) = 2.5sin(0.25t)$

using SOS programming tool Yalmip of Lofberg [2004]. The method gives transformation Γ

$$\Gamma = \begin{bmatrix} 0.3128 & -0.2998 & 0.4152 \\ 0.8310 & 0.3536 & -0.0721 \\ -0.9193 & 1.1060 & 0.7230 \end{bmatrix}$$

and truncation of the transformed system gives

$$\begin{aligned} \dot{x}_{r1} &= -1.8505x_{r2} - x_{r1}\left(1.6335x_{r1}^2 - 1.6334x_{r1}x_{r2}\right. \\ &\quad \left.+1.4382x_{r2}^2 + 0.0799\right) - 0.1602u, \\ \dot{x}_{r2} &= 1.4356x_{r1} - x_{r2}\left(1.6335x_{r1}^2 - 1.6334x_{r1}x_{r2}\right. \\ &\quad \left.+1.4382x_{r2}^2 + 1.0699\right) + 0.4703u, \\ y_r &= 0.3128x_{r1} - 0.2998x_{r2}. \end{aligned}$$

The response of the system and the reduced model to inputs $u = e^{-1.5t}sin(5t)$ can be seen in Fig. 1. Qualitatively, our scheme outperforms the one in Siahaan et al. [2007].

5.2 *Example 2*

The following system is taken from Prajna & Sandberg [2005]

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1x_2 - 3x_2x_3 - x_1x_4, \\ \dot{x}_2 &= x_3 + 0.5x_1x_2 + 0.5x_2x_3 + x_1x_4, \\ \dot{x}_3 &= x_4 + 0.5x_1x_2 + 0.5x_2x_3 - 0.25x_1x_4, \\ \dot{x}_4 &= -x_1 - 3x_2 - 5x_3 - 7x_4 - 3x_1x_2 + 0.1x_2x_3 \\ &\quad + 0.3x_1x_4 + u, \\ y &= x_1, \end{aligned}$$

where $D_x = \left\{x \in \mathbb{R}^4 | \; 12 - \|x\|^2 \geq 0\right\}$. For this system we can compute quadratic generalized functions but for higher order of generalized functions we can get a lower value of γ which is an upper bound for the Hankel norm. We compute generalized functions of order six with bound $\gamma = 1.2$. Choosing higher order than six will not give significant improvement of bound γ. By applying the truncation scheme we obtain a reduced model of order three whose response to the sinusoidal input $u(t) = 2.5sin(0.25t)$ can be seen in Fig. 2.

6. CONCLUDING REMARKS

This paper introduces a novel approach to balancing a polynomial nonlinear system and truncates the balanced representation to obtain a reduced order model. The method utilizes the power of sum of squares programming which is efficient numerically. Future investigation should focus on the structure preservation because it is not yet clear what kind of property, in general, the reduced order model preserves from the original system.

REFERENCES

S. Lall, J. Marsden and S. Glavaski. A subspace approach to balanced truncation for model reduction of nonlinear control systems. *International Journal of Robust and Nonlinear Control*, vol. 12, no. 6, pp. 519-535, 2002.

J.Lofberg. YALMIP: a toolbox for modeling and optimization in Matlab. *Proceedings 2004 CACSD Conference*, Taipei, Taiwan, 2004. Available at `http://control.ee.ethz.ch/~joloef/yalmip.php`

B.C. Moore. Principal component analysis in linear systems: controllability, observability and model reduction. *IEEE Transaction on Automatic Control*, vol. 26, pp. 17-32, 1981.

P.A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming Ser. B*, 96:2, pp. 293-320, 2003.

S. Prajna and H. Sandberg. On model reduction of polynomial dynamical systems. *Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference 2005*, Seville, Spain, pp. 1666-1671, 2005.

J. Scherpen. Balancing for nonlinear systems. *Systems and Control Letters*, 21, pp. 143-153, 1993.

H.B. Siahaan, O.M. Aamo and B.A. Foss. Reachability-based approach for $\mathcal{L}_2$ gain approximation of polynomial systems. *Proceedings of European Control Conference 2007*, Kos, Greece, pp. 1119-1125, 2007.

G. Stengle. A Nullstellensatz and a Positivestellensatz in semialgebraic geometry. *Math. Ann.*, 207, pp. 87-97, 1974.