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Stability Analysis on Kuramoto Model of Coupled Oscillators ⋆

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Abstract: In this paper we study the problem of stability for one of the most popular models of coupled phase oscillators, the Kuramoto model. The Kuramoto model is used to describe the phenomenon of collective synchronization, in which an enormous system of oscillators spontaneously locks to a common frequency although the oscillators have distinct natural frequencies. In the paper we consider the stability of the Kuramoto model of coupled oscillators with identical natural frequency and provide a stability analysis of phase difference equilibrium. The stability of the phase difference equilibrium make it possible to apply the Kuramoto model in pattern recognition.

Keywords: Nonlinear System, Synchronization, Coupled Oscillator, Kuramoto Model.

1. INTRODUCTION

Collective synchronization phenomenon has long been observed in biological, chemical and physical systems, when the individual frequencies of coupled oscillators converge to a common frequency despite differences in the natural frequencies of the individual oscillators. Biological examples include groups of synchronously flashing fireflies (Buck [1988]) and crickets that chirp in unison (Walker [1969]). One typical physical example is Josephson junctions (see Wiesenfeld et al. [1998]). The phenomenon has drawn considerable attention over the past decade. Firstly, Weiner conjectured its involvement in the generation of alpha rhythms in the brain (Weiner [1958]). It was then taken up by Winfree who used it to study circadian rhythms in living organisms (Winfree [1980]). Winfree's model was significantly extended by Kuramoto who developed results for what is now well known as the Kuramoto model (Kuramoto [1975],Kuramoto [1984]). A comprehensive summary of Kuramoto's work, and later attempts to answer the questions that were raised by his formulations, can be found in Strogatz [2000].

The synchronization of coupled oscillators has triggered a new idea for computation. A novel computing paradigm, emerging from a network of oscillators, has been studied (Acker et al. [2003], Arbib [1995], Hoppensteadt et al. [1997]). Typically, simple analog units such as artificial neurons in the network, process information in parallel. Such networks perform pattern recognition and associative recall via self-organization of neurons. One candidate network is the weakly connected networks of the Kuramoto model that can potentially be implemented with identical coupled lasers (Hoppensteadt et al. [2000]).

In this paper, we first simply describe the model of a laser network that can potentially be used to implement the Kuramoto model. Then we anylyze the stability of the equilibrium in phase differences of the nonlinear system, the Kuramoto model with identical natural frequency. We consider two forms of connectivity among oscillators in the Kuramoto model: a global connectivity and a local connectivity. With the global connectivity every oscillator in the Kuramoto model is connected to all the other oscillators, which forms all-to-all connectivity. With the local connectivity, the oscillators are placed on a circle and each oscillator is only connected to two neighboring oscillators, which forms a ring. A detection technique based on the stability of the equilibrium in phase differences of the Kuramoto model has been proposed and can be found in Wang et al. [2005].

2. THE MODEL OF A LASER NETWORK

The network of coupled lasers has been described in detail (Hoppensteadt et al. [2000]). Below is the brief description. The lasers are written with the dimensionless rate equations as follows:

$$\dot{E}_i = (1+i\alpha)N_i E_i + i\omega E_i + \sum_{j=1}^{n} c_{ij} E_j, \tag{1}$$

$$\dot{N}_i = \mu[P - N_i - (1+2N_i)|E_i|^2], \tag{2}$$

where E_i and N_i are the complex electric field and the excess carrier number of the ith laser, the derivatives $\dot{} = d/ds$, $s = t\tau_p^{-1}$ is the time measured in units of the photon lifetime τ_p, $\mu = \tau_p/\tau_s$ is the ratio of photon to carrier time scales, where τ_s is the carrier lifetime, P is the pumping above threshold, α is the linewidth enhancement factor, ω is normalized optical frequency, and c_{ij} are complex connection coefficients.

It is convenient to use polar coordinates $E_i = r_i e^{i\phi_i}$ and $c_{ij} = s_{ij} e^{i\psi_{ij}}$ to rewrite the model (1) and (2) in the form

$$\dot{\phi}_i = \alpha N_i + \omega + \sum_{j=1}^{n} s_{ij} \frac{r_j}{r_i} sin(\phi_j + \psi_{ij} - \phi_i), \tag{3}$$

$$\dot{r}_i = N_i r_i + \sum_{j=1}^{n} s_{ij} r_j cos(\phi_j + \psi_{ij} - \phi_i), \tag{4}$$

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$$\dot{N}_i = \mu[P - N_i - (1+2N_i)|r_i|^2]. \tag{5}$$

In the case of weak connection, the dynamical analysis of the model (3),(4) and (5) shows that

$$(r_i(t),\ N_i(t)) \longrightarrow (\sqrt{P},\ 0),$$

and the phase $\phi_i(t) \to \omega t + \phi_i^0$, where ϕ_i^0 is determined by the initial conditions. If all $r_i(t) \to r_0$, then the phase model (3) for $\alpha = 0$ has the Kuramoto model form (Kuramoto [1984])

$$\dot{\phi}_i \ = \ \omega + \sum_{j=1, j\neq i}^{n} s_{ij} \sin(\phi_j - \phi_i + \psi_{ij}) \tag{6}$$

where ϕ_i, $i = 1, \cdots, n$, are phase variables taking values in the interval $[-\pi, \pi)$. The parameters s_{ij} and ψ_{ij} are assumed to satisfy $s_{ij} = s_{ji} \geq 0$, $\psi_{ij} = -\psi_{ji}$. The index i, refers to the i^{th} unit and these units are coupled.

Below we give an illustration about this convergence from a single(isolated) laser.

A single laser can be described as follows:

$$\dot{\phi} = \alpha N + \omega, \tag{7}$$

$$\dot{r} = Nr, \tag{8}$$

$$\dot{N} = \mu[P - N - (1+2N)|r|^2]. \tag{9}$$

Then equilibria in the magnitude of the electric field and the excess carrier number (r_e, N_e) are given by solving $\dot{r} = 0$ and $\dot{N} = 0$: $e_1 = (\sqrt{P}, 0)$(minus case is trivial) and $e_2 = (0, P)$.

Jacobian matrix in term of (r, N) is:

$$A(r,N) = \begin{bmatrix} N & r \\ -2\mu(1+2N)r & -\mu(1+2r^2) \end{bmatrix}$$

With the first equilibrium e_1, Jacobian matrix is:

$$A(e_1) = \begin{bmatrix} 0 & \sqrt{P} \\ -2\mu\sqrt{P} & -\mu(1+2P) \end{bmatrix}.$$

The eigenvalues with respect to e_1 are obtained by calculating

$$\begin{aligned} \|\lambda I - A(e_1)\| &= \left\| \begin{bmatrix} \lambda & -\sqrt{P} \\ 2\mu\sqrt{P} & \lambda + \mu(1+2P) \end{bmatrix} \right\| \\ &= \lambda^2 + \mu(1+2P)\lambda + 2\mu P \\ &= 0, \end{aligned}$$

which gives two eigenvalues with negative real part:

$$\lambda_{1,2} = \frac{-\mu(1+2P) \pm \sqrt{\mu^2(1+2P)^2 - 8\mu P}}{2}$$

, So the equilibrium $e_1 = (\sqrt{P}, 0)$ is locally stable.

With the second equilibrium e_2, Jacobian matrix is:

$$A(e_2) = \begin{bmatrix} P & 0 \\ 0 & -\mu \end{bmatrix}.$$

The eigenvalues with respect to e_2 are obtained by calculating

$$\begin{aligned} \|\lambda I - A(e_2)\| &= \left\| \begin{bmatrix} \lambda - P & 0 \\ 0 & \lambda + \mu \end{bmatrix} \right\| \\ &= (\lambda - P)(\lambda + \mu) \\ &= 0, \end{aligned}$$

which gives one negative eigenvalue $\lambda_1 = -\mu$ and one positive eigenvalue $\lambda_2 = P$. So the equilibrium $e_2 = (0, P)$ is unstable.

So for a single laser, $(r(t), N(t))$ locally converges to $(\sqrt{P}, 0)$.

As pointed by Hoppensteadt and Izhikevich in Hoppensteadt et al. [1997], the phase differences, but not phases, play a key role in the neurocomputing mechanism, We are interested in the stability of equilibrium in phase differences of the Kuramoto model. We consider two forms of connectivity: global connectivity and local connectivity. In the next two sections, we analyze the stability of phase difference equilibrium of the Kuramoto model in these forms of connectivity.

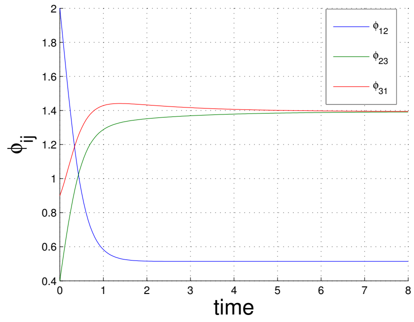

Fig. 1. Phase Differences $\Phi_d(t)$ in 3^{rd} Kuramoto model

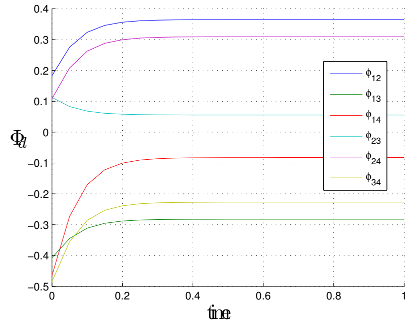

Fig. 2. Phase Differences $\Phi_d(t)$ in 4^{th} order Kuramoto model

3. STABILITY ANALYSIS OF THE KURAMOTO MODEL WITH ALL-TO-ALL CONNECTION

The Kuramoto model with global connectivity can be written in the following form (see Kuramoto [1984]):

$$\dot{\phi}_i \ = \ \omega + \sum_{j=1, j\neq i}^{n} s_{ij} \sin(\phi_j - \phi_i + \psi_{ij}) \tag{10}$$

where ϕ_i, $i = 1, \cdots, n$, are phase variables taking values in the interval $[-\pi, \pi)$. The parameters s_{ij} and ψ_{ij} are assumed to satisfy $s_{ij} = s_{ji} \geq 0$, $\psi_{ij} = -\psi_{ji}$. The index i, refers to the i^{th} unit and these units are coupled.

In pattern recognition with networks of oscillators, phase relations, instead of phases, play an crucial role. It is important to understand the dynamics and stability of the equilibria in phase relation of the nonlinear systems. For simplicity, we denote that $\phi_{ij} = \phi_i - \phi_j$ and WLOG set that $\psi_{ij} = 0$. Denote two vectors as follows:

$$\begin{aligned} \Phi &= [\phi_1\ \ \phi_2\ \ \cdots\ \ \phi_n]^T \\ \Phi_d &= [\phi_{12}\ \ \phi_{13}\ \ \cdots\ \ \phi_{1n}\ \ \phi_{23}\ \ \phi_{24}\ \ \cdots\ \ \phi_{(n-1)n}]^T \end{aligned}$$

then the Kuramoto model can be rewritten as:

$$\dot{\Phi} = -A^T S \sin(\Phi_d) + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \omega$$

and

$$\Phi_d = A\Phi$$

where A is $N \times n$ matrix of rank $n-1$ with $N = \frac{n(n-1)}{2}$. Each row of A corresponding to ϕ_{ij} of Φ_d has 1 and -1 at i^{th} and j^{th} columns and 0 at all others.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

and S is a diagonal matrix and

$$\mathrm{Diag}(S) = [s_{12} \;\; s_{13} \;\; \cdots \;\; s_{1n} \;\; s_{23} \;\; s_{24} \;\; \cdots \;\; s_{(n-1)n}]^T$$

Hence an ODE in phase difference variables can be written as follows:

$$\dot{\Phi}_d = -AA^T S \sin(\Phi_d)$$

If $s_{ij} = s$ (wlog to say $s_{ij} = 1$) and denote $M = -AA^T$, then

$$\dot{\Phi}_d = M \sin(\Phi_d)$$

It is easy to verify that $N \times N$ matrix M has rank $n-1$ and is symmetric and negative semi-definite. So the matrix M has $n-1$ negative eigenvalues and all other eigenvalues are 0. Then there exits an orthonormal matrix $P = [p_1 \; p_2 \cdots p_N]$ such that $P^T MP = \Lambda$ with

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_{(n-1)} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

where $\lambda_i < 0$ for $i \le n-1$. Define by transformation $\alpha = [\alpha_1 \;\; \alpha_2 \;\; \cdots \;\; \alpha_N]^T = P^T \Phi_d$ and therefore we have

$$\dot{\alpha} = P^T M \sin(\Phi_d) = \Lambda P^T \sin(\Phi_d).$$

Hence $\alpha_i = p_i^T \Phi_d$ and $\dot{\alpha}_i = \lambda_i p_i^T \sin(\Phi_d)$ for $i = 1, \cdots, N$. Note that $\dot{\alpha}_i = 0$ for $i \ge n$ since $\lambda_i = 0$ for $i \ge n$. Now we define a function in phase differences as follows:

$$V(\Phi_d) = V(\alpha(\Phi_d)) = \frac{1}{2} \sum_{i=1}^{N} \alpha_i^2(\Phi_d).$$

It is obvious that $V(\Phi_d) = 0$ at $\Phi_d = 0$ and $V(\Phi_d) \ge 0$ at all other Φ_d. Differentiating the function we get

$$\dot{V}(\Phi_d) = \sum_{i=1}^{N} \lambda_i \alpha_i \dot{\alpha}_i = \sum_{i=1}^{n-1} \lambda_i \alpha_i \dot{\alpha}_i = \sum_{i=1}^{n-1} \lambda_i (p_i^T \Phi_d)(p_i^T \sin(\Phi_d)).$$

Because P is orthonormal, then Φ_d and $\sin(\Phi_d)$ can be written as:

$$\begin{aligned} \Phi_d &= \textstyle\sum_{i=1}^{N} c_i p_i \\ \sin(\Phi_d) &= \textstyle\sum_{i=1}^{N} d_i p_i \end{aligned}$$

with $c_i = p_i^T \Phi_d$, $d_i = p_i^T \sin(\Phi_d)$. For $\| \Phi_d \|$ small enough, $c_i d_i \ge 0$. So

$$\dot{V}(\Phi_d) = \sum_{i=1}^{n-1} \lambda_i (p_i^T \Phi_d)(p_i^T \sin(\Phi_d)) = \sum_{i=1}^{n-1} \lambda_i c_i d_i \le 0.$$

Then the set in which $\dot{V}(\Phi_d) = 0$ is

$$E = \{\Phi_d \mid \dot{V}(\Phi_d) = 0\} = \cap_{i=1}^{n-1}(E_{i1} \cup E_{i2}) = \cup_{j_i \in \{1,2\}}(\cap_{i=1}^{n-1} E_{ij_i})$$

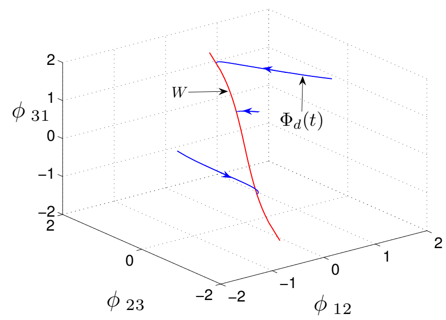

Fig. 3. Convergence of phase difference trajectories to the invariant set W from different initial conditions in 3^{rd} Kuramoto model

with

$$\begin{aligned} E_{i1} &= \{\Phi_d \mid p_i^T \Phi_d = 0\} \\ E_{i2} &= \{\Phi_d \mid p_i^T \sin(\Phi_d) = 0\}. \end{aligned}$$

The tangent plane to the surfaces E_{i1} and E_{i2} at point Φ_{d0} are

$$\begin{aligned} T_{i1} &= \{\Phi_d \mid p_i^T (\Phi_d - \Phi_{d0}) = 0\} \\ T_{i2} &= \{\Phi_d \mid q_i^T (\Phi_d - \Phi_{d0}) = 0\} \end{aligned}$$

where

$$q_i = \begin{bmatrix} p_i^1 & & & \\ & p_i^2 & & \\ & & \ddots & \\ & & & p_i^N \end{bmatrix} \cos(\Phi_{d0}).$$

Note:

$$\dot{\Phi}_d = P\dot{\alpha} = \sum_{i=1}^{n-1} \lambda_i p_i p_i^T \sin(\Phi_d).$$

In the subset of E defined by $\cap_{i=1}^{n-1} E_{ij_i}$ with all $j_i = 2$, it is easy to verify that $\dot{\Phi}_d = 0$ and hence this subset is invariant.

In the subset of E defined by $\cap_{i=1}^{n-1} E_{ij_i}$ with some $j_i = 1$, wlog, to say, $j_i = 1$, $i \le k$ and $j_i = 2$, $i > k$ for $1 < k \le n-1$, the tangent plane to this subset is $\cap_{i=1}^{n-1} T_{ij_i}$ for the same j_i's. Then

$$\dot{\Phi}_d = P\dot{\alpha} = \sum_{i=1}^{k} \lambda_i p_i p_i^T \sin(\Phi_d)$$

is not in the tangent plane since

$$p_i^T \dot{\Phi}_d = \lambda_i p_i^T \sin(\Phi_d) \neq 0$$

which means it is even not in T_{i1} for $i \le k$. So the largest invariant set contained in E is

$$W = \cap_{i=1}^{n-1} E_{i2} = \{\Phi_d \mid p_i^T \sin(\Phi_d) = 0, i = 1, \cdots, n-1\}.$$

By LaSalle's theorem, the solution $\Phi_d(t)$ approaches W as $t \to \infty$.

Fig.1 and 2 show that the phase differences in the 3^{rd} and 4^{th} order Kuramoto models converge to constants respectively. Fig. 3 shows that the phase differences in the 3^{rd} order Kuramoto model converge to the invariant set W in 3 dimensional space. Fig. 4 and 5 show that the the phase differences in the 4^{th} order Kuramoto model converge to the invariant set W in different ways respectively.

Note: The solution $\Phi_d(t)$ moves but stays on the phase plane defined by

$$\sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \phi_i(0)$$

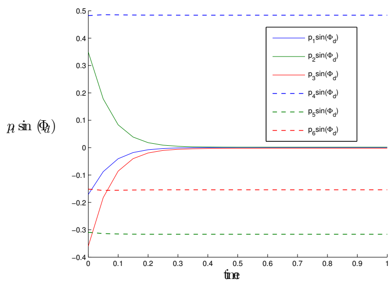

Fig. 4. $P\sin(\Phi_d)$ in 4^{th} order Kuramoto model

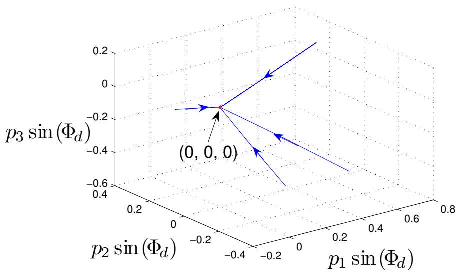

Fig. 5. Convergence of phase difference trajectories to the invariant set W from different initial conditions in 4^{th} Kuramoto model

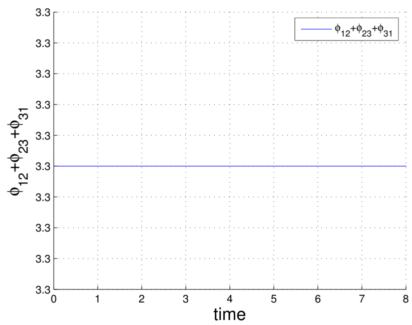

Fig. 6. Trajectory $\phi(t)$ stays on the plane defined by the initial condition in 3^{rd} Kuramoto model

since

$$\sum_{i=1}^{n} \dot{\phi}_i(t) = 0, \text{ all } t.$$

Fig. 6 shows that the sum of phases in the 3^{rd} order Kuramoto model stays on the plane determined by the initial phase condition of the oscillators in the Kuramoto model.

4. STABILITY ANALYSIS OF THE KURAMOTO MODEL IN FORM OF RING

We next consider a Kuramoto model in the form of local connectivity that forms a ring of oscillators. In this case, the Kuramoto model can be written as:

$$\begin{aligned}
\dot{\phi}_1 &= \omega + s_{12}\sin(\phi_2 - \phi_1 + \psi_{12}) + s_{1n}\sin(\phi_n - \phi_1 + \psi_{1n}) \\
\dot{\phi}_i &= \omega + s_{i(i-1)}\sin(\phi_{(i-1)} - \phi_i + \psi_{i(i-1)}) \\
&\quad + s_{i(i+1)}\sin(\phi_{(i+1)} - \phi_i + \psi_{i(i+1)}) \\
&\quad \text{for } i = 2, \cdots, (n-1) \\
\dot{\phi}_n &= \omega + s_{n(n-1)}\sin(\phi_{(n-1)} - \phi_n + \psi_{n(n-1)}) \\
&\quad + s_{n1}\sin(\phi_1 - \phi_n + \psi_{n1})
\end{aligned}$$

where $s_{ij} = s_{ji}$ and $\psi_{ij} = \psi_{ji}$. Define $\phi_{ij} = \phi_i - \phi_j$ and denote the vectors as follows:

$$\begin{aligned}
\Phi &= [\phi_1\ \phi_2\ \cdots\ \phi_n]^T \\
\Phi_d &= [\phi_{12}\ \phi_{23}\ \cdots\ \phi_{(n-1)n}\ \phi_{n1}]^T \\
\Psi &= [\psi_{12}\ \psi_{23}\ \cdots\ \psi_{(n-1)n}\ \psi_{n1}]^T
\end{aligned}$$

Then we can have:

$$\dot{\Phi} = -A^T S \sin(\Phi_d - \Psi)$$

and

$$\Phi_d = A\Phi$$

where A is $n \times n$ matrix of rank $n-1$. Each row of A corresponding to ϕ_{ij} of Φ_d has 1 and -1 at i^{th} and j^{th} columns and 0 at all others.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

S is a diagonal matrix and

$$\text{Diag}(S) = [s_{12}\ \ s_{13}\ \ \cdots\ \ s_{1n}\ \ s_{23}\ \ s_{24}\ \ \cdots\ \ s_{(n-1)n}]^T$$

Then we have an ODE on phase difference variables as follows:

$$\dot{\Phi}_d = -AA^T S \sin(\Phi_d - \Psi) + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \omega$$

If $s_{ij} = s$ (wlog to say $s_{ij} = 1$) and $\Psi = 0$, and denote $M = -AA^T$. Then

$$\dot{\Phi}_d = M\sin(\Phi_d)$$

and

$$\begin{aligned}
M &= -2I_n + \begin{bmatrix} & I_{(n-1)} \\ 1 & \end{bmatrix} + \begin{bmatrix} & 1 \\ I_{(n-1)} & \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}
\end{aligned}$$

It is easy to verify that the $n \times n$ matrix M has rank $n-1$ and is symmetric and negative semi-definite. M has $n-1$ negative eigenvalues and one zero eigenvalue.

Then there exits an orthonormal matrix P such that $P^T MP = \Lambda$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{(n-1)} & \\ & & & & 0 \end{bmatrix}$$

Define by transformation $\alpha = [\alpha_1 \quad \alpha_2 \quad \cdots \ \alpha_n]^T = P^T \Phi_d$. Therefore we have

$$\dot{\alpha} = P^T M \sin(\Phi_d) = \Lambda P^T \sin(\Phi_d).$$

Then $\alpha_i = p_i^T \Phi_d$ and $\dot{\alpha}_i = \lambda_i p_i^T \sin(\Phi_d)$ for $i = 1, \cdots, n$. Note that $\dot{\alpha}_i = 0$ for $i = n$ since $\lambda_i = 0$ for $i = n$.

Define

$$V(\Phi_d) = V(\alpha(\Phi_d)) = \frac{1}{2}\sum_{i=1}^{n} \alpha_i^2(\Phi_d).$$

It is obvious that $V(\Phi_d) = 0$ at $\Phi_d = 0$ and $V(\Phi_d) \geq 0$ at all other Φ_d.

Then

$$\dot{V}(\Phi_d) = \sum_{i=1}^{n} \lambda_i \alpha_i \dot{\alpha}_i = \sum_{i=1}^{n-1} \lambda_i \alpha_i \dot{\alpha}_i = \sum_{i=1}^{n-1} \lambda_i (p_i^T \Phi_d)(p_i^T \sin(\Phi_d))$$

Because P is orthonormal, then

$$\begin{aligned} \Phi_d &= \textstyle\sum_{i=1}^n c_i p_i \\ \sin(\Phi_d) &= \textstyle\sum_{i=1}^n d_i p_i \end{aligned}$$

with $c_i = p_i^T \Phi_d$, $d_i = p_i^T \sin(\Phi_d)$. For $\parallel \Phi_d \parallel$ small enough, $c_i d_i \geq 0$. So

$$\dot{V}(\Phi_d) = \sum_{i=1}^{n-1} \lambda_i (p_i^T \Phi_d)(p_i^T \sin(\Phi_d)) = \sum_{i}^{n-1} \lambda_i c_i d_i \leq 0$$

Then the set

$$E = \{\Phi_d \mid \dot{V}(\Phi_d) = 0\} = \cap_{i=1}^{n-1}(E_{i1} \cup E_{i2}) = \cup_{j_i \in \{1,2\}}(\cap_{i=1}^{n-1} E_{ij_i})$$

with

$$\begin{aligned} E_{i1} &= \{\Phi_d \mid p_i^T \Phi_d = 0\} \\ E_{i2} &= \{\Phi_d \mid p_i^T \sin(\Phi_d) = 0\} \end{aligned}$$

The tangent plane to the surfaces E_{i1} and E_{i2} at point Φ_{d0} are

$$\begin{aligned} T_{i1} &= \{\Phi_d \mid p_i^T(\Phi_d - \Phi_{d0}) = 0\} \\ T_{i2} &= \{\Phi_d \mid q_i^T(\Phi_d - \Phi_{d0}) = 0\} \end{aligned}$$

where

$$q_i = \begin{bmatrix} p_i^1 & & & \\ & p_i^2 & & \\ & & \ddots & \\ & & & p_i^n \end{bmatrix} \cos(\Phi_{d0})$$

Note:

$$\dot{\Phi}_d = P\dot{\alpha} = \sum_{i=1}^{n-1} \lambda_i p_i p_i^T \sin(\Phi_d)$$

In the subset of E defined by $\cap_{i=1}^{n-1} E_{ij_i}$ with all $j_i = 2$, it is easy to verify that $\dot{\Phi}_d = 0$ and hence this subset is invariant.

In the subset of E defined by $\cap_{i=1}^{n-1} E_{ij_i}$ with some $j_i = 1$, wlog, to say, $j_i = 1$, $i \leq k$ and $j_i = 2$, $i > k$ for $1 < k \leq n-1$, the tangent plane to this subset is $\cap_{i=1}^{n-1} T_{ij_i}$ for the same j_i's.

Then

$$\dot{\Phi}_d = P\dot{\alpha} = \sum_{i=1}^{k} \lambda_i p_i p_i^T \sin(\Phi_d)$$

is not in the tangent plane since

$$p_i^T \dot{\Phi}_d = \lambda_i p_i^T \sin(\Phi_d) \neq 0$$

which means it is even not in T_{i1} for $i \leq k$

So the largest invariant set contained in E is

$$W = \cap_{i=1}^{n-1} E_{i2} = \{\Phi_d \mid p_i^T \sin(\Phi_d) = 0, i = 1, \cdots, n-1\}.$$

By LaSalle's theorem, the solution $\Phi_d(t)$ approaches E as $t \to \infty$.

Note: The solution $\Phi_d(t)$ moves but stays on the phase plane defined by

$$\sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \phi_i(0)$$

since

$$\sum_{i=1}^{n} \dot{\phi}_i(t) = 0, \text{ all } t.$$

Furthermore,the solution $\Phi_d(t)$ stays on the plane difference defined by

$$\sum_{i=1}^{n} \Phi_d^i = \sum_{i=1}^{n} \Phi_d^i(0) = 0$$

since

$$\sum_{i=1}^{n} \dot{\Phi}_d^i(t) = 0, \text{ all } t.$$

5. CONCLUSION

In the paper we mainly discussed on the stability of equilibrium in phase difference of the Kuramoto model in two forms of connectivity. We sketched the analysis using LaSalle's theorem. We also introduced the model of laser network and it is relation with the Kuramoto model in this paper.

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