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An overview of the CRONE approach in system analysis, modeling and identification, observation and control

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Abstract: The aim of the paper is to present the fundamental definitions connected to fractional differentiation and to present an overview of the CRONE approach in the fields of system analysis, modeling and identification, observation and control. Industrial applications of fractional differentiation are also described in this paper. Some recent developments are also presented.

1. INTRODUCTION

Fractional calculus is a mathematical tool which deals with integrals and derivatives of arbitrary orders.

Fractional calculus may be considered as an *old* and yet a *novel* topic. It is an *old* topic since, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed up to nowadays. See for example the now classical book (Oldham and J. Spanier, 1974), the more formalised books (Samko *et al* 1993) (Miller and Ross, 1993) and (Oustaloup, 1995) and (Podlubny, 1999.a) for a thorough mathematical study of the equations.

However, it may be considered a *novel* topic as well, since only from a little more than twenty years it has been the subject of specialized conferences and treatises. In recent years, a growing interest in fractional calculus has been stimulated by the numerous applications of this mathematical tool in various fields of physics and engineering, including practical applications, and numerical analysis.

Transmission lines, electrical noises (Lewis *et al.* 2004), dielectric polarization and heat transfer phenomena (Battaglia *et al.*, 2000) are only some of the fields having fractional physical laws.

Another interesting application of fractional calculus is in the automatic control area. Some important considerations such as modeling, system identification, stability (Matignon, 1996), controllability, observability (Matignon and D'Andrea-Novel, 1996) and robustness now involve fractional systems. The fractional generalisation of the PID controller (Podlubny, 1999.b, Monje *et al.*, 2004), gave rise these last ten years to a large number of studies. CRONE Control (Oustaloup and Mathieu, 1999), the French acronym for "Commande Robuste d'Ordre Non-Entier", was the first robust control based on fractional differentiation for linear systems. Several extensions of this robust control method now exist and several industrial applications prove its efficiency (Oustaloup *et al.* 2006).

In this paper, an overview of the CRONE group applications in the fields of system analysis, modeling and identification, observation and control is proposed. This paper gives all the materials required to be initiated to fractional differentiation and its applications, since it contains the most usual definitions used in the field of fractional systems and contains numerous references. The interested reader is also invited to read (Tenreiro Machado, 1997), (Baleanu and Muslih, 2005), (Vinagre Jara and Feliu 2007), (Chen *et al.* (2004)).

2. PRELIMINARY DEFINITIONS

The Riemann-Liouville definition of fractional differentiation is here considered but others definitions exist and the interested reader is invited to refer to (Samko *et al.*, 1993). Riemann and Liouville main concern was to extend differentiation by using not only integer but also non-integer (real or complex) orders. They have defined the n^{th} fractional order derivative by the relation (Samko *et al.*, 1993):

$$D^n\left(x(t)\right)=\frac{1}{\Gamma(m-n)}\left(\frac{d}{dt}\right)^m\left(\int_0^t\frac{x(\tau)d\tau}{(t-\tau)^{1-(m-n)}}\right), \qquad (1)$$

with $t > 0$, $n > 0$ and $m=\lfloor n\rfloor+1$. $\lfloor n\rfloor$ means the integer part of n. The Laplace transform of this derivative (Oldham and Spanier, 1974), considering zero initial conditions, is:

$$\mathfrak{L}\left\{\left(D^n x\right)(t)\right\}=s^n X(s)\text{, where } X(s)=\mathfrak{L}\left(x(t)\right). \qquad (2)$$

Using relation (1), a fractional order system can be described by the following fractional order differential equation:

$$\sum_{k=0}^{N} a_k D^{\alpha_k}\left(y(t)\right)=\sum_{k=0}^{M} b_k D^{\beta_k}\left(u(t)\right) \qquad (3)$$

$$a_k\in\mathbb{R}, b_k\in\mathbb{R}, \alpha_k\in\mathbb{R}, \beta_k\in\mathbb{R}, M<N,\ M\in\mathbb{N}, N\in\mathbb{N}$$

where $y(t)$ and $u(t)$ denote respectively the output and the input of the system.
A sub-class of fractional models, namely commensurate order fractional models, is often used in the literature. It involves fractional differentiation orders in (3) multiples of the same commensurate order $\nu \in \mathbb{R}$. Relation (3) then becomes:

$$\sum_{k=0}^{N} \widetilde{a}_k D^{k\nu}\left(y(t)\right) = \sum_{k=0}^{M} \widetilde{b}_k D^{k\nu}\left(u(t)\right). \qquad (4)$$

where, $N\nu = \alpha_N$, $M\nu = \beta_M$, and

$$\begin{cases} \widetilde{a}_k = a_k \text{ if } k\nu = \alpha_k ; \widetilde{a}_k = 0 \text{ otherwise} \\ \widetilde{b}_k = b_k \text{ if } k\nu = \beta_k ; \widetilde{b}_k = 0 \text{ otherwise} \end{cases}$$

Fractional commensurable models can be represented in a transfer function form as:

$$H(s) = \sum_{k=0}^{M} \widetilde{b}_k s^{k\nu} \Big/ \sum_{k=0}^{N} \widetilde{a}_k s^{k\nu} \qquad (5)$$

Fractional models are also based on fractional state-space like representation; see (Matignon, 1996, Oustaloup, 1995, Cois *et al.*, 2001). In the particular case of a commensurate order, this representation can be obtained using $D^{\nu}\left(x_k(t)\right) = x_{k+1}(t)$. The following representation can thus be obtained:

$$\begin{cases} \left(D^{\nu} x\right)(t) = A\, x(t) + B\, u(t) \\ y(t) = C\, x(t) + D\, u(t) \end{cases}. \qquad (6)$$

The modal decomposition of fractional systems is obtained as for classical systems. Applying a similarity transformation to (6), a Jordan canonical form is written as (Cois *et al.*, 2001):

$$\begin{cases} \left(D^{\nu} x_J\right)(t) = J\, x_J(t) + B_J u(t) \\ y(t) = C_J x_J(t) + E_J\, u(t) \end{cases}. \qquad (7)$$

The system eigenvalues are on the diagonal of matrix J. Considering zero initial conditions, the Laplace transform of the fractional "state like" equation of (7) is:

$$s^{\nu} X_J(s) = J\, X_J(s) + B_J\, U(s). \qquad (8)$$

with $U(s) = \mathfrak{L}(\mathrm{u}(t))$ and $X_J(s) = \mathfrak{L}(x_J(t))$. Then, from the output equation of relation (7), system output is:

$$y(t) = \mathfrak{L}^{-1}\left\{ C_J \left(s^n I - J\right)^{-1} B_J \right\} \otimes u(t) + E_J\, u(t), \qquad (9)$$

where $\otimes$ denotes the convolution product (see (Cois *et al.*, 2001)) for detailed expressions of matrices J, B_J, C_J).
System's output $y(t)$ is thus composed of a linear combination of partial fractions called eigenmodes, $\mathfrak{L}^{-1}\left\{ \left(1 \big/ \left(s^{\nu} - \lambda_l\right)\right)^q \right\} \otimes \mathrm{u}(t)$ where q is an integer number depending on the algebraic multiplicity of each eigenvalue λ_l.
In the single input single output case, the transfer function corresponding to system (7) is thus given by:

$$H(s) = \frac{Y(s)}{U(s)} = \sum_{l=1}^{L} \sum_{q=1}^{q_l} H_{l,q}(s), \qquad (10)$$

with $H_{l,q}(s) = K_{l,q} \Big/ \left(s^{\nu} - \lambda_l\right)^q$.

Using the Mellin-Fourier inverse transformation and the residue theorem, the impulse response of an eigenmode can be expressed as:

$$h_{l,q}(t) = \sum_{k=1}^{\text{pole number}} \frac{p_k}{\lambda_l^{\,q}} Q_{q-1}\left(\frac{1}{\nu}, t\, p_k\right) \mathrm{e}^{t p_k} + \frac{1}{\pi} \int_0^{\infty} \frac{\mathrm{e}^{-tx} \sum_{k=0}^{q-1} (-1)^k \binom{q}{k} (\lambda_l)^k x^{\nu(q-k)} \sin\left[\nu\pi(q-k)\right]}{\left[x^{2\nu} - 2\lambda_l x^{\nu} \cos(\nu\pi) + \lambda_l^{\,2}\right]^{q_i}} dx, \qquad (11)$$

where the poles p_k of $H_{l,q}(s)$ (if any) are defined by:

$$\begin{cases} p_k = \left|p_k\right| \mathrm{e}^{\mathrm{j}\theta_k} \\ \theta_k \in \left]-\pi ; \pi\right[\end{cases}, \quad \text{with} \quad \begin{cases} \left|p_k\right| = \left|\lambda_l\right|^{1/\nu} \\ \theta_k = \dfrac{\arg(\lambda_l)}{\nu} + \dfrac{2k\pi}{\nu} \\ -\dfrac{\nu}{2} - \dfrac{\arg(\lambda_l)}{2\pi} < k < \dfrac{\nu}{2} - \dfrac{\arg(\lambda_l)}{2\pi} \end{cases} \qquad (12)$$

and where $Q_{\kappa}(x, y)$ is a 2 variables polynomial defined by:

$$\begin{cases} Q_0(x, y) = x \\ \kappa\, Q_{\kappa}(x, y) = (xy + x - y) Q_{\kappa-1}(x, y) + xy \dfrac{\partial}{\partial y} Q_{\kappa-1}(x, y) \end{cases}. \qquad (13)$$

Equation (11) shows the decomposition of fractional systems into two parts (Oustaloup, 1983):
- the *exponential mode*, resulting from the computation of residue(s) on each pole of $H_{l,q}(s)$ which generates an exponential behavior;
- the *aperiodic multimode*, the main characteristic of fractional systems, resulting from an integral along the negative real axis.
According to relation (11), any stable SISO linear fractional system can be decomposed as the sum of three subsystems:
- an integer linear system denoted E ,
- pure fractional integrators,
- a stable fractional system denoted P .

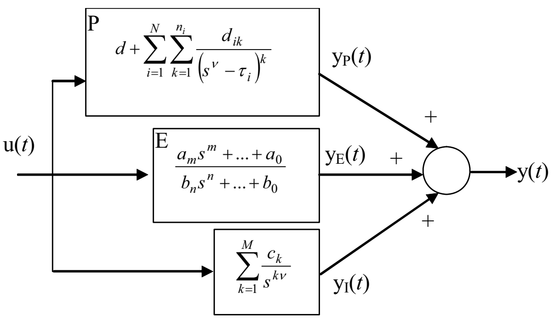

Fig. 1. Structural decomposition of a fractional system

Using the structural decomposition of fractional systems (6) represented by Fig. 1, a stability condition is given by (Matignon, 1996):

$$\left|\arg(\lambda_l)\right| > \frac{\nu\pi}{2}, \quad \text{for } l = 1, .., \dim(x), \qquad (14)$$

where the λ_l are the eigenvalues of matrix A.

2. FRACTIONAL SYSTEMS ANALYSIS

2.1. Fractional systems stability analysis

According to the comments in the previous section, the stability of a fractional system is now a well defined notion. However only recent insights have been proposed for LMI (Linear Matrix Inequality) based stability characterization, often required to apply modern control analysis and control methods. As mentioned in (Sabatier *et al*, 2005), exponential stability cannot be used to characterize asymptotic stability of fractional systems. A new definition must be introduced.

Definition: $t^{-\alpha}$ stability

Trajectory $x(t)=0$ of system $\mathrm{d}^{\nu}x(t)/\mathrm{d}t = f(t,x(t))$ is $t^{-\alpha}$ asymptotically stable if there is a positive real α such that :
$\forall \left\|x(t)\right\|$ with $t \le t_0$, $\exists$ $N(x(t),\ t \le t_0)$ such that $\forall\ t \ge t_0$, $\left\|x(t)\right\| \le N\, t^{-\alpha}$. □

The fact that the components of $x(t)$ slowly decay towards 0 following $t^{-\nu}$ leads to fractional systems sometimes being called long memory systems.
The main issue when dealing with LMI is the convexity of the optimization set. In figure 2, which shows the stability domain of a fractional system according to the value of fractional order ν, note that the stability domain is not convex when $0<\nu<1$.

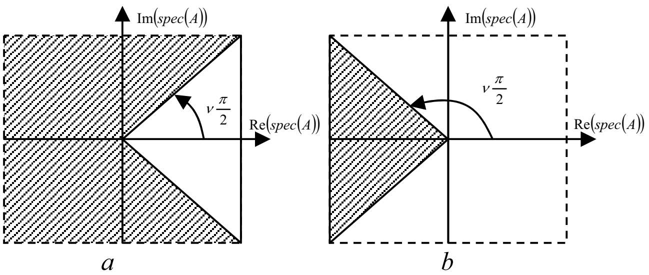

Fig. 2. Stability domain (▨) for: a) $0<\nu<1$, b) $1\le\nu<2$

As the stability domain of a fractional system with order $1\le\nu<2$ is a convex set, various LMI methods for defining such a region have already been developed. Hence a LMI based theorem for the stability of a fractional system with order $1\le\nu<2$ can be formulated as follows (Moze *et al*, 2005).

Theorem 1: *Fractional system described by (6) with order* $1\le\nu<2$ *is asymptotically stable if and only if there exists a matrix* $P=P^T>0$, $P\in\mathbb{R}^{M\times M}$, *such that the following LMI holds:*

$$\begin{pmatrix} \left(A^TP+PA\right)\sin\left(\nu\dfrac{\pi}{2}\right) & \left(A^TP-PA\right)\cos\left(\nu\dfrac{\pi}{2}\right) \\ \left(PA-A^TP\right)\cos\left(\nu\dfrac{\pi}{2}\right) & \left(A^TP+PA\right)\sin\left(\nu\dfrac{\pi}{2}\right) \end{pmatrix}<0. \quad (15)$$

□

For a fractional system with order $0\le\nu<1$, it can be demonstrated (Moze *et al*, 2005) that the arguments of the eigenvalues of matrix A belong to $\left]-\pi,-\nu\pi/2\right]\cup\left]\nu\pi/2,\pi\right]$ if and only if

$$-\left(-A\right)^{\frac{1}{2-\nu}} \quad \text{is stable.} \quad (16)$$

Stability of system (6) can thus be deduced by applying the Lyapunov stability condition to a fictive integer system with state matrix $-\left(-A\right)^{\frac{1}{2-\nu}}$.

Theorem 2: *Fractional system (6) is* $t^{-\alpha}$ *stable if and only if a positive definite matrix P exists such that*

$$\left(-\left(-A\right)^{\frac{1}{2-\nu}}\right)^{*}P+P\left(-\left(-A\right)^{\frac{1}{2-\nu}}\right)<0. \quad (17)$$

□

This condition is a necessary and sufficient condition as opposed to the condition previously proposed in (Momani and El-Khazali, 2001) which is only a sufficient condition. However, LMI condition (17) is expressed with a non-linear function of the fractional system state matrix. This thus reduces the utility of this condition. To overcome this problem, another stability condition is proposed. As the stability domain is non-convex for $0<\nu<1$, the unstability domain is now identified using LMI (Moze *et al*, 2005).

Theorem 3: *Fractional system (6) is* t^{α} *stable if and only if there does not exist any non-negative rank one matrix* $Q\in\mathbb{C}^{n\times n}$ *such that*

$$\begin{pmatrix} \left(AQ+QA^T\right)\sin\left(\nu\dfrac{\pi}{2}\right) & \left(AQ-QA^T\right)\cos\left(\nu\dfrac{\pi}{2}\right) \\ \left(AQ-QA^T\right)\cos\left(\nu\dfrac{\pi}{2}\right) & \left(AQ+QA^T\right)\sin\left(\nu\dfrac{\pi}{2}\right) \end{pmatrix}\ge 0. \quad (18)$$

□

2.2. H_2 norm computation of the impulse response

To measure the quality of a fractional system approximation by a rational model, an analytical method for determining the H_2 norm of any fractional explicit commensurate transfer function is proposed in (Malti *et al*, 2003). It generalizes the well-know results by Aström (1970), allowing to compute the H_2 norm of any rational transfer function.

Theorem 4: *Let H(s) be a commensurable transfer function, with a commensurable order equal to ν, of a BIBO-stable system, defined in (5). Let q and ρ be respectively integer and non-integer parts of 1/ν. Define*

$$G\left(\omega^{\nu}\right)=H\left(j\omega\right) \quad \text{and} \quad \frac{A(x)}{B(x)}=G(x)\overline{G(x)}. \quad (19)$$

Then, the H_2 *norm of H(s), namely* $\left\|H\right\|_2$, *depends on its relative degree (Nν - Mν) and is given according to ρ as:*

- *if* $(N\nu - M\nu) \leq 1/2$, *then* $\|H\|_2 = \infty$,
- *if* $(N\nu - M\nu) > 1/2$ *and* $\rho \neq 0$ *then*

$$\|H\|_2^2 = \frac{\sum_{k=1}^{r}\sum_{l=1}^{\gamma_k}(-1)^{l-1} a_{k,l} s_k^{\rho-1}\binom{\rho-1}{l-1}}{\nu \sin(\rho\pi)} \tag{20}$$

where $a_{k,l}$, $(-s_k)$ *and* γ_k *are respectively coefficients, poles and their multiplicity of the partial fraction expansion of*

$$x^q \frac{A(x)}{B(x)} = \sum_{k=1}^{r}\sum_{l=1}^{\gamma_k} \frac{a_{k,l}}{(x+s_k)^l} \tag{21}$$

- *if* $(N\nu - M\nu) > 1/2$ *and* $\rho = 0$ *then*

$$\|H\|_2^2 = \sum_{k=1}^{r} \frac{c_k(\ln(s_k) - \ln(s_1))}{\nu\pi(s_k - s_1)} + \sum_{k=1}^{r}\sum_{l=2}^{\gamma_k} \frac{b_{k,l} s_k^{1-l}}{\nu\pi(l-1)} \tag{22}$$

where $(-s_k)$ *and* γ_k *are poles and their multiplicities of* $x^{q-1}\frac{A(x)}{B(x)}$. s_1 *is an arbitrary chosen pole.* c_k *and* $b_{k,l}$ *represent coefficients of the following expansion of:*

$$x^{q-1}\frac{A(x)}{B(x)} = \sum_{k=2}^{r} \frac{c_k}{(x+s_1)(x+s_k)} + \sum_{k=1}^{r}\sum_{l=1}^{\gamma_k} \frac{b_{k,l}}{(x+s_k)^l}. \tag{23}$$

All poles of order 1 are gathered in the left sum; all poles of order greater than one are gathered in the right double sum.

□

Proof of this theorem and some other remarks can be found in (Malti *et al*, 2003).

2.3. H_∞ norm computation

Transfer matrix $H(s)$ of system (6), denoted S_f, is given by

$$H(s) = C\left(s^\nu I - A\right)^{-1} B + D. \tag{24}$$

Let γ denotes a real positive number satisfying

$$\gamma > \sigma_{\max}(D). \tag{25}$$

System S_f H_∞-norm is bounded by γ if and only if

$$\sup_{\omega} \sigma_{\max}(H(j\omega)) < \gamma. \tag{26}$$

Equivalently, it can be demonstrated that (Sabatier *et al*, 2005) H_∞-norm of system S_f is bounded by γ if and only if

$$\forall \omega \in \mathbb{R}, \quad \left(\gamma^2 I - H(-j\omega)^T H(j\omega)\right)^{-1} \text{ exists.} \tag{27}$$

As state matrix of the state space description associated with transfer matrix $\left(\gamma^2 I - H(-s)^T H(s)\right)^{-1}$ is:

$$A_\gamma = \begin{pmatrix} A + B\left(\gamma^2 I - D^T D\right)^{-1} D^T C & e^{-\nu j\pi} B\left(\gamma^2 I - D^T D\right)^{-1} B^T \\ C^T\left(I + D\left(\gamma^2 - D^T D\right)^{-1} D^T\right) C & e^{-\nu j\pi}\left(A^T + C^T D\left(\gamma^2 I - D^T D\right)^{-1} B^T\right) \end{pmatrix}, \tag{28}$$

the following theorem can hence be stated (Sabatier *et al*, 2005).

Theorem 5: *H_∞-norm of system S_f, whose transfer function is given by relation (24), is bounded by a real positive number γ if and only if the eigenvalues of matrix A_γ given by relation (28) do not lie on* $\mathbb{C}_{0_\nu} = \left\{(j\omega)^\nu, \omega \in \mathbb{R}\right\}$.

□

Let $y_P(t)$ and $y_E(t)$ denote the integer subsystem P output and the stable subsystem E output. According to Fig. 1

$$y(t) = y_P(t) + y_E(t). \tag{29}$$

The H_∞-norm of a fractional system S_f is thus bounded by γ if S_f is stable and

$$\forall T \geq 0, \quad 2\int_0^T y_E(t)^T y_E(t)dt \leq \left(\gamma^2 - 2r^2\right)\int_0^T u(t)^T u(t)dt. \tag{30}$$

where r is a bound of the L_1-norm of the impulse response of fractional subsystem P.

Such an analysis permits to derive a real bounded lemma based lemma (Sabatier *et al*, 2005).

Lemma*: Fractional system S_f is stable and its H_∞-norm is bounded by γ if there exists a symmetric positive definite matrix P such that*

$$\begin{pmatrix} {A_E}^T P + P A_E & P B_E & {C_E}^T \\ {B_E}^T P & -\left(\gamma^2 - 2r\right)^{1/2} I & {D_E}^T \\ C_E & D_E & -\left(\gamma^2 - 2r\right)^{1/2} I \end{pmatrix} < 0, \tag{31}$$

where (A_E, B_E, C_E, D_E) is the state space description of the integer subsystem E.

□

3. FRACTIONAL SYSTEMS MODELING

Studies on real systems such as thermal or electrochemical (Battaglia *et al.*, 2000, Cois *et al.*, 2000), reveal fractional behavior. The use of classical models, based on integer order differentiation, is thus inappropriate in representing these fractional systems. This section present, through four examples, some fields for which an approach based on fractional differentiation permits to obtain models with a low number of parameters as compared to integer models.

3.1 – Porous dyke

Dyke builders have long been aware of the damping properties of the very irregular dykes, in particular those with cavities or depressions imprisoning pockets of air which can be compressed by the oncoming water.
Given that nature is an inexhaustible source of solutions, we have studied the attenuation of the motion of water on this type of dyke.
To tackle such a study which concerns the physics of the complex media, we aimed to establish the differential equation which governs a water mass, M, of speed $V(t)$, in horizontal

relaxation on a dyke, said porous, exerting a reaction force $F(t)$ (Fig. 3).

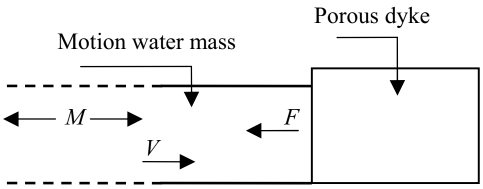

Fig. 3. Study system

The dyke is then interpreted as a porous medium such as it makes it possible to conceive a model which achieves a good compromise between reality and simplicity. The porosity assumption is indeed motivated by a set of properties which favor this compromise.
The dimensionless and heterogeneous properties of porosity, implies an indefinite number of pores of different sizes.
The unordered character of porosity implies an indeterminate distribution of the different size pores. This distribution justifies the uniformity of the dynamic pressure and of the flow density on the cross-section of flow which defines the water-dyke interface (just upstream of the dyke). Such a study configuration permits characterizing the interface by a hydraulic admittance.
Porous rock is not necessarily permeable, which suggests considering pores not linked between them which give them an alveolar character (dead-end).
Such a character allows describing a pore by its constituent elements which originate two distinct physical phenomena:
- an orifice or canal which laminates water, which can be characterized by a hydraulic resistance, and where a dissipative energy is located;
- a cavity or alveolus which imprisons air compressible by the moving water, which can be characterized by a pneumatic capacitance, and where an elasticity potential energy is located; the corresponding elasticity reduces the dynamic pressure peaks at the water-dyke interface.
Due to the additivity of the flows in the pores (Fig. 4) namely $Q = \sum_i Q_i$, the admittance of the water-dyke interface is that of a parallel arrangement of resistance-capacitance hydropneumatic cells. Each cell illustrates both, for each pore, the energy dissipation through viscosity (and turbulence) and the elastic potential energy through air compression (case of the non saturated porous media).

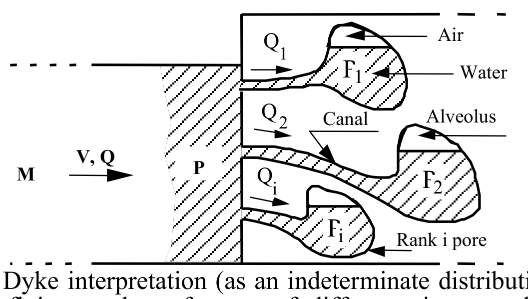

Fig. 4. Dyke interpretation (as an indeterminate distribution of an indefinite number of pores of different sizes not linked between them)

The fractal character attributed by Benoît Mandelbrot to porosity because of its property of self-similarity (Mandelbrot, 1975), suggests the (simplifying) assumption of recursivity on the distribution of the resistances and capacitances, thus leading to a recursive parallel arrangement of series RC cells, with constant ratios α and η between the resistances and the capacitances of two consecutive cells, α and η being greater than one and called recursive factors (Fig. 5).

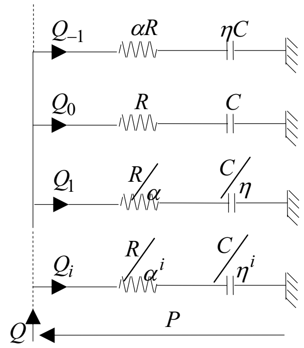

Fig. 5. Recursive parallel arrangement of series RC cells

These recursive factors condition the form of the Bode asymptotic diagrams of admittance $Y(j\omega)$:
- the gain asymptotic diagram results from a regular sequence of steps ;
- the phase asymptotic diagram results from a regular sequence of crenels.
The smoothing of these asymptotic diagrams can be defined by a non integer admittance of the form $Y(j\omega) = (j\omega/\omega_0)^m$ with $0<m<1$, ω_0 denoting the frequency for which the gain smoothing straight line intersects the axis 0 dB, called unit gain frequency or transition frequency

The idea of the CRONE suspension in hydropneumatic technology results from the transposition, in vibration insulation, of the porous dyke interpretation.
To pass from this interpretation to the hydropneumatic concept of the CRONE suspension of a quarter vehicle (Fig. 6), it suffices:
- to rotate of a quarter of turn the study system of Fig. 4;
- to replace air by nitrogen;
- to replace water by oil;
- to set oil in motion with a piston linked to the wheel.

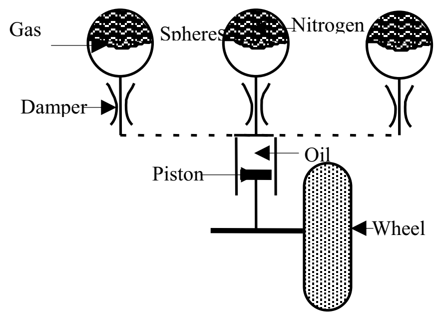

Fig. 6 - Idea of the multisphere CRONE suspension

3.1 – Viscoelasticity

Fractional differentiation is often used to model viscoelastic material. In this section, a car silent-block is studied.

Fig. 7 presents a single degree-of-freedom system which consists of a rigid body (mass M) connected to a silent-block.

x(t) is the horizontal displacement of mass *M* defined from static equilibrium position and force solicitation *f(t)*.

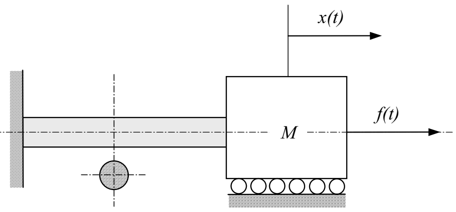

Fig. 7 : One-degree-of-freedom model

In the most general case, the relationship between uniaxial stress $\sigma(t)$ and strain $\varepsilon(t)$ in a viscoelastic material may be expressed by a linear combination of integer derivatives, called the standard viscoelastic model (Bagley and Torvik, 1985), namely:

$$\sigma(t)+\sum_{i=1}^{i=I} e_i\, D^{q_i}\sigma(t) = E_0\,\varepsilon(t)+\sum_{j=1}^{j=J} E_j\, D^{r_j}\varepsilon(t)\ , \quad (32)$$

in which the factors e_i, E_0 and E_j together with integer orders derivative q_i and r_j are model parameters. The parameters are determined from frequency identification by least-squares fit to experimental data.

For viscoelastic materials having mechanical properties that are strongly frequency dependent over many decades of frequency (Bagley and Torvik, 1986), the number of parameters is large. As a result, computations on the model are time consuming and produces high-order differential equations.

The fractional derivative model for a uniaxial solicitation is a generalization of (32):

$$\sigma(t)+\sum_{i=1}^{i=I} e_i\, D^{m_i}\sigma(t) = E_0\,\varepsilon(t)+\sum_{j=1}^{j=J} E_j\, D^{n_j}\varepsilon(t)\ , \quad (33)$$

with $0 < m_i < 1$ and $0 < n_i < 1$ (Oldham and Spanier, 1971).

It has been observed (Bagley and Torvik, 1985) that only five parameters often suffice in the stress-strain relationship (33) to get a good fit to experimental data, namely:

$$\sigma(t)+e_1\, D^{m_1}\sigma(t) = E_0\,\varepsilon(t)+E_1\, D^{n_1}\varepsilon(t)\ . \quad (34)$$

To obtain a thermodynamically well-behaved model some constraints have to be placed on the parameters of the model. In (Bagley and Torvik, 1986), these constraints are:

$$E_0 \ge 0,\ \ E_1 \ge 0,\ \ e_1 \ge 0,\ \ \frac{E_1}{E_0} \ge e_1, \qquad (35)$$
$$n_1 = m_1 = m \ \ and\ \ 0 \le m \le 1\ .$$

In other words, the orders of the two fractional derivatives in the model have to be equal, which means that the original five-parameter model reduces to a four-parameter model, namely:

$$\sigma(t)+e_1\, D^{m}\sigma(t) = E_0\,\varepsilon(t)+E_1\, D^{m}\varepsilon(t)\ . \quad (36)$$

The fractional derivative model has several very attractive features. The first is that it produces a compact analytic representation, well behaved in both the frequency and time domains. The second is that the mathematical form has its foundation in accepted molecular theories governing the mechanical behaviour of viscoelastic media (Bagley and Torvik, 1986). Finally, the fractional derivative model may be viewed as an extension of the standard viscoelastic model in the sense that the derivatives are no longer limited to being of integer order.

Using $\quad \sigma(t) = \dfrac{u(t)}{S} \quad$ and $\quad \varepsilon(t) = \dfrac{x(t)}{L}$, (37)

where S and L are, respectively, the cross section and the length of the material elastomer, relation (36) can be rewritten as:

$$u(t)+e_1\, D^m u(t) = \frac{S}{L}E_0\, x(t)+\frac{S}{L}E_1\, D^m x(t)\ . \quad (38)$$

The corresponding rheological model to relation (38), given in Fig. 8, is composed of two elastic elements and one fractional element.

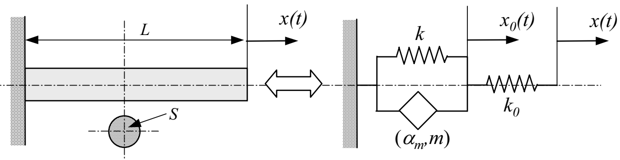

Fig. 8. Rheological model of the silent-block

3.2 – Thermal systems

The thermal system considered is a semi-infinite plane homogenous medium with a conductivity λ and diffusivity α. At rest, the system is considered to be at ambient temperature. All simulations are carried out starting from this equilibrium position. A flux density $\varphi(t)$ is applied on the outgoing normal surface $\vec{n}$ of the medium. Losses on the surface where the thermal flux is applied are neglected.

If x denotes the abscissa of the measurement slot in the medium (Fig. 9), 1D heat transfer in a semi-infinite plane medium is governed by the following partial differential equations (Crank, 1957):

$$\begin{cases} \dfrac{\partial T(x,t)}{\partial t} = \alpha \dfrac{\partial^2 T(x,t)}{\partial x^2},\ \ 0 < x < \infty,\ t > 0 \\ -\lambda \dfrac{\partial T(x,t)}{\partial n} = \varphi(t),\ \ x = 0,\ t > 0 \\ T(x,t) = 0,\ \ 0 \le x < \infty,\ t = 0 \end{cases} \quad . \quad (39)$$

Laplace transform of the first equation leads to:

$$\frac{\partial^2 \overline{T}(x,s)}{\partial x^2} - \frac{s}{\alpha}\overline{T}(x,s) = 0\ , \quad (40)$$

where $\quad \overline{T}(x,s) = \mathscr{L}\{T(x,t)\}$. (41)

This relation is a differential equation of the variable x. Solution of this equation is given by:

$$\overline{T}(x,s) = K_1(s)e^{-x\sqrt{s/\alpha}} + K_2(s)e^{x\sqrt{s/\alpha}}\ . \quad (42)$$

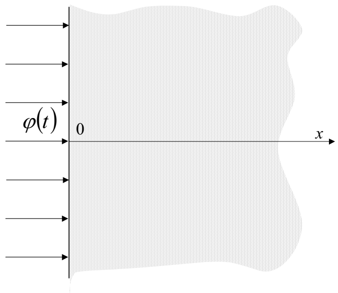

Fig. 9. Semi-infinite plane medium

Tacking into account boundary conditions, the following transfer function can be expressed:

$$H(x,s)=\frac{\overline{T}(x,s)}{\overline{\varphi}(s)}=\frac{1}{\sqrt{s}\sqrt{\lambda\,\rho\,C_p}}\,\mathrm{e}^{-x\sqrt{s/\alpha}}\,. \tag{43}$$

Two studies can then be done. The first one corresponds to the medium thermal impedance obtained for $x = 0$:

$$H(0,s)=\frac{\overline{T}(0,s)}{\overline{\varphi}(s)}\,. \tag{44}$$

The second one corresponds to the study of the transfer obtained for $x > 0$ defined by:

$$H(x,s)=\frac{\overline{T}(x,s)}{\overline{\varphi}(s)}\,. \tag{45}$$

If $x = 0$, relation (43) becomes:

$$H(0,s)=\frac{\overline{T}(0,s)}{\overline{\varphi}(s)}=\frac{1}{\sqrt{s}\sqrt{\lambda\,\rho\,C_p}}\,, \tag{46}$$

or, in time domain, initial conditions being considered null:

$$T(0,t)=\frac{1}{\sqrt{\lambda\,\rho\,C_p}}\,{}_0I_t^{0.5}\varphi(t)\,. \tag{47}$$

This relation highlights that the thermal impedance of a semi-infinite media is based on a 0.5 fractional order integrator. This result allows an analytic expression of the temperature $T(0,t)$ by a 0.5 fractional integral of $\varphi(t)$. The compactness property of the fractional order operator is thus shown. In comparison with a spatio-temporal discretisation of the media (finite element method) leading to large dimensional integer model, the fractional approach leads to a simple model with few parameters.

If the temperature is measured at an abscissa $x > 0$ inside the medium, transmittance $H(x,s)$ is given by relation (43).

Given that :

$$H(x,s)=\frac{\mathrm{e}^{-x\sqrt{s/\alpha}}}{\sqrt{s}\sqrt{\lambda\,\rho\,C_p}}=\frac{1}{\sqrt{s}\sqrt{\lambda\,\rho\,C_p}}\,\frac{\mathrm{e}^{-\frac{x}{2}\sqrt{s/\alpha}}}{\mathrm{e}^{\frac{x}{2}\sqrt{s/\alpha}}}\,, \tag{48}$$

using Taylor series expansion of the exponential function, the following approximation is obtained :

$$\widetilde{H}_K(x,s)=\frac{1}{\sqrt{\lambda\,\rho\,C_p}}\,\frac{\sum_{k=0}^{K} a'_k\,s^{k/2}}{\sum_{k=0}^{K}\left|a'_k\right|s^{(k+1)/2}} \tag{49}$$

with
$$a'_k=(-1)^k\,\frac{(x/2)^k}{\alpha^{k/2}k!}\,, \tag{50}$$

and demonstrates that $\widetilde{H}_K(x,s)$ can be expressed, in time domain, by fractional order differential equations whose orders are multiples of 0.5. Their approximation by integer admittance is thus impossible on the entire frequency domain.

3.4 – Electrochemistry

The Randles' model is frequently used in the literature for modeling lead acid batteries. This model results from a simplified solution of the electrochemical diffusion equation in batteries (Fick's law) (Sathyanarayana, *et al*, 1979). If $u(t)$ denotes voltage variations from the open circuit voltage and if $i(t)$ denotes the battery current, then Randles' model is defined by the electric circuit shown in Fig. 10.

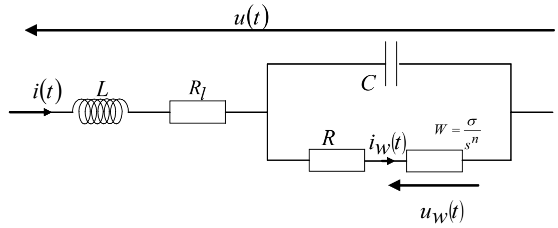

Fig. 10. Randles' model of a lead acid battery

Fractional behavior of Randles' model is due to fractional impedance $W(s)$. This impedance is known as Warburg cell and is a fractional order integrator of order $n = 0.5$.

Using Fig. 10, Randles' model transfer function (for L=0) is given by:

$$H(s)=\frac{U(s)}{I(s)}\underset{L=0}{=}\frac{R_lRCs^{n+1}+R_l\sigma Cs+(R_l+R)s^n+\sigma}{s^n\left(RCs+\sigma Cs^{1-n}+1\right)}\,. \tag{51}$$

Relation (51) thus demonstrates that Randles' model frequently used for battery modeling is a fractional order differential model.

It also demonstrates that low frequency asymptotic behavior (behavior obtained as Laplace variable s tends towards 0) of the Randles' model is that of a fractional integrator of order 0.5. However, this analysis is in contradiction with the time domain behavior of the battery in discharge.

To overcome this problem, a new model whose transmittance is denoted $H_{CRONE}(s)$ has been proposed in (Sabatier *et al*, 2006). This model is not constructed using physical arguments as the Randles model was (there is no physical justification). It was constructed only using frequency responses (obtained by harmonic analysis) obtained for various batteries, for various states of charge and operating temperatures. This new model is made of two limited frequency band integrators and has:

- a low frequency asymptotic behavior of order 0, in accordance with observed batteries time responses in discharge;

- a frequency behavior compatible with the frequency behavior of a lead acid battery, whatever its state of charge and its operating temperature.

Transmittance of this new model is given by :

$$H_{CRONE}(s) = K\left(1+\frac{s}{\omega_b}\Big/1+\frac{s}{\omega_r}\right)^{-n_1}\left(1+\frac{s}{\omega_r}\Big/1+\frac{s}{\omega_h}\right)^{-n_2} \quad (52)$$

This model is based on implicit fractional order integration, fractional order n_1 and n_2 being not directly applied to Laplace variable s.This model has been validated by several trails, using frequency and time data (Sabatier *et al*, 2006).

Figure 11 presents model and battery response (filtered) comparison for a state of charge equal to 80% at temperature T = 20°C (similar comparisons have been done for others state of charges and temperatures). This comparison permits to conclude that the model $H_{CRONE}(s)$ is a powerful tools for battery behavior modeling.

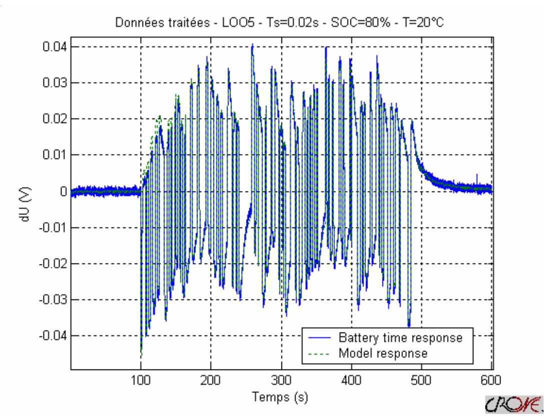

Fig. 11. Comparisons of battery response and obtained model response for S0C =80% and at 20°C

4. FRACTIONAL SYSTEMS IDENTIFICATION

The aim of system identification is to establish a mathematical model capable of reproducing system's physical behaviour as faithfully as possible from a series of observations. Many methods have been developed using continuous time models (Young, 1981), (Unbehauen and Rao, 1987) (Neumann and Isermann, 1988). Such models are based on classical differential equations whose parameters are usually estimated by minimizing a given criterion. In the context of identifying systems based on diffusive processes, the use of these 'classical' models, based on integer order differentiation, appears to be inappropriate because of the inherent fractional behaviour of such systems. Thus, models described by (3) have been used.

One of the goal of the identification procedure consists now in estimating not only coefficients $[a_0 \ \cdots \ a_N \ \ b_0 \ \cdots \ b_M]^T$, but also differentiation orders $[\alpha_0 \ \cdots \ \alpha_N \ \ \beta_0 \ \cdots \ \beta_M]^T$. In this context, 2 types of method have been developed:

- equation error methods where only the coefficients are estimated, differentiation orders being fixed according to a prior knowledge;
- output error methods where both coefficients and differentiation orders are estimated.

4.1. Equation error methods

The developed Equation Error methods are based on the use of the Linear Least Square optimization technique. Recall that in this case, the goal is to estimate only the coefficients of fractional differential equation (3), the differentiation orders being supposed known by the user (as it is the case for many thermal systems) (Battaglia et al., 2000).

First method: Model discretization. A first method, developed in (Oustaloup et al., 1996b), (Oustaloup, 2005), consists in replacing the continuous fractional derivatives by their discrete Grünwald approximation (Oustaloup, 1995):

$$D^n f(t) = \frac{1}{T_s}\sum_{k=0}^{K}(-1)^k\binom{n}{k} f\big((K-k)T_s\big), \quad (55)$$

where T_s denotes the sampling period, and $t = KT_s$. A recursive equation, called fractional ARX model, is obtained. A discrete-time linear regression form can be built, and the parameters are then estimated by a classical Prediction Error Method (PEM).

Second method: Fractional State Variable Filters. A second method is based on the use of fractional state variable filters (Cois et al., 2001). In this case, a continuous-time linear regression is directly built from (53) filtered by a fractional state variable filter:

$$\begin{aligned} D^{n_{a_0}} y_f(t) &= -a_1 D^{n_{a_1}} y_f(t) - \ldots - a_L D^{n_{a_L}} y_f(t) + b_0 D^{n_{b_0}} u_f(t) + \ldots + b_M D^{n_{b_M}} u_f(t) \\ &= \left[-D^{n_{a_1}} y_f(t) \ \cdots \ -D^{n_{a_L}} y_f(t) \ \ D^{n_{b_0}} u_f(t) \ \cdots \ D^{n_{b_M}} u_f(t)\right]\boldsymbol{\theta} \\ &= \boldsymbol{\varphi}_f(t)\,\boldsymbol{\theta} \end{aligned} \quad (56)$$

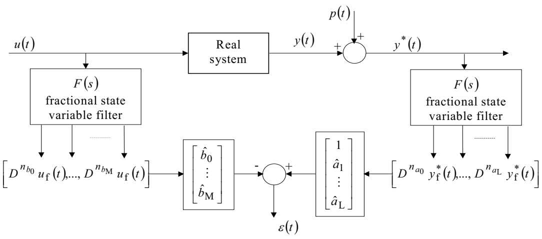

Fig. 12. Use of fractional state variable filters for equation error methods

The parameter estimation is then obtained by using the classical Least Square applied on filtered input/output data:

$$\theta_{opt} = \left(\boldsymbol{\Phi}_f^{*T}\,\boldsymbol{\Phi}_f^*\right)^{-1}\boldsymbol{\Phi}_f^{*T}\,D^{n_{a_0}}\mathbf{Y}_f^* \quad (57)$$

where

$$\begin{cases} \boldsymbol{\Phi}_f^* = \left[\boldsymbol{\varphi}_f^{*T}(k_0 h) \ \ \boldsymbol{\varphi}_f^{*T}((k_0+1)h) \ \cdots \ \boldsymbol{\varphi}_f^{*T}((k_0+N-1)h)\right]^T \\ \mathbf{Y}_f^* = \left[y_f^*(k_0 h) \ \ y_f^*((k_0+1)h) \ \cdots \ y_f^*((k_0+N-1)h)\right] \end{cases}. \quad (58)$$

4.2 - Output error methods

When the fractional differentiation orders cannot be easily determined by the user, Output Error Methods provide estimation of both coefficients and differentiation orders using Non Linear Optimization Techniques, such as the Marquardt

Algorithm. The procedure consists in optimizing the model parameters by minimizing a quadratic criterion of the output error. Two types of model structure have been developed:
- frequency bounded fractional models;
- modal fractional models.

Frequency bounded fractional models. Developed by Trigeassou (Trigeassou *et al*, 1999), the frequency bounded fractional models are based on the use of the following frequency bounded Fractional Integrator:

$$I^n(s) = \frac{1}{s} C_0 \left(\frac{1 + \frac{s}{\omega_\mathrm{b}}}{1 + \frac{s}{\omega_\mathrm{h}}} \right)^{1-n} . \tag{59}$$

The advantage of such a structure lies in the fact that it makes the time-domain simulation easier, by approximating fractional integrators through recursive distributions of zeros and poles.

Modal fractional models. Modal fractional models are based on the diagonal form of representation (7) (Cois et al., 2000). Under such a form, the model can be expressed as the sum of fractional eigenmodes characterized by 3 parameters: a differentiation order, a coefficient and an eigenvalue.

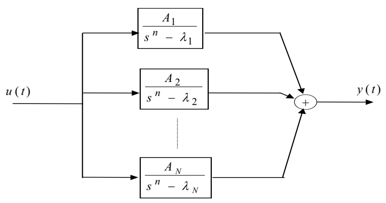

Fig. 13. Natural mode decomposition of a non integer model

The estimation procedure consists then in optimizing the modal parameters with respect to the model output error.

4.3 - Example

The example concerns a thermal application in the field of machining by turning (Battaglia, 2001). The goal was to estimate, during machining the heat flux through the tool, through using an inverse model obtained off-line using system identification (as direct measurement of heat flux is not possible).

To obtain the machining tool model, a thermocouple (type T) is embedded close to the tip of the insert tool. Heat flux is then generated using a heat resistor formed by a platinum film (10 µm) placed on a thin ceramic substrate (250 µm) that allows to neglect the thermal inertia of such a resistor compared to the sampling interval, h=0.4s, of the experiment.

Two sequences have been generated, one for parameter estimation, the other for model validation. Parameter estimation is performed using the instrumental variable method based on fractional SVF described in section 4.1 that leads to the 5 parameter model:

$$\begin{gathered} T(t) + 10.90 D^{0.5} T(t) - 1.04 D^1 T(t) + 39.81 D^{1.5} T(t) = \\ 6.22 \phi(t) - 1.65 D^{0.5} \phi(t) \end{gathered} . \tag{61}$$

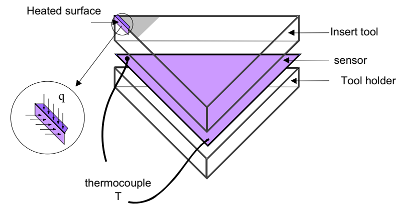

Fig. 14. Description of operation

Fig. 15 shows residue and output error signals. As it clearly appears, a very small number of parameters are required in order to fit the measured temperature. As shown in model validation results of Fig. 16, the model output correctly fits the measured temperature. This proves the reliability of the model during the time concerned by the identification procedure.

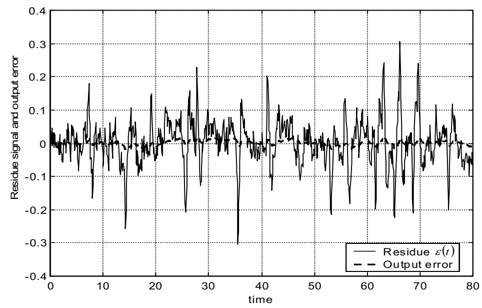

Fig. 15. Residue signal and output error

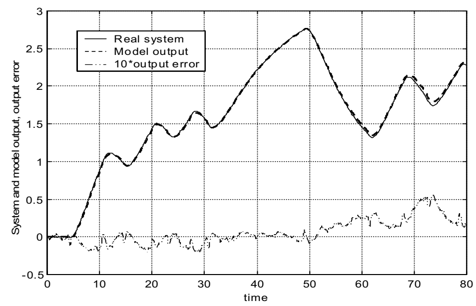

Fig. 16. Model validation results

4.4 – Others applications

System identification methods previously described has been used to model several others fractional systems. Among these applications, the most representative ones are:
- lead acid batteries state of charge estimation (Sabatier *et al*, 2006),
- gastrocnemius frog muscle modeling (Sommacal *et al.*, 2006)
- car driver dynamic behavior modeling.

5 – Control and observation

5.1. CRONE Control

The CRONE control-system design is based on the common unity-feedback configuration (Fig. 17). The robust controller or the open-loop transfer function is defined using fractional order integro-differentiation. The required robustness is that of both stability margins and performance, and particularly peak value M_r (called resonant peak) of the common complementary sensitivity function $T(s)$.

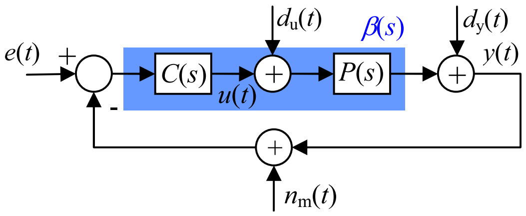

Fig.17. Common CRONE control-system diagram

Three CRONE control design methods have been developed, successively extending the application field.
The third CRONE control generation, based on complex fractional integration, must be used when the plant frequency uncertainty domains are of various types (not only gain-like). It is based on the definition of a generalized template described as a straight line in the Nichols chart of any direction (complex fractional order integration), or by a multi-template (or curvilinear template) defined by a set of generalized templates.
An optimization allows to determine the independent parameters of the open loop transfer function. This optimization is based on the minimization of the stability degree variations, while respecting other specifications taken into account by constraints on sensitivity function magnitude. The complex fractional order permits parameterization of the open-loop transfer function with a small number of high-level parameters. The optimization of the control is thus reduced to only the search for the optimal values of these parameters. As the form of uncertainties taken into account is structured, this optimization is necessarily nonlinear. It is thus very important to limit the number of parameters to be optimized. After this optimization, the corresponding CRONE controller is synthesized as a rational fraction only for the optimal open-loop transfer function.
The third generation CRONE CSD methodology, the most powerful one, is able to design controllers for plants with positive real part zeros or poles, time delay, and/or with lightly damped modes (Oustaloup *et al*.,1995). Associated with the w-bilinear variable change, it also permits to design digital controllers. The CRONE control has also been extended to linear time variant systems and nonlinear systems whose nonlinear behaviors are taken into account by sets of linear equivalent behaviors (Pommier *et al*., 2002). For MIMO (multivariable) plants, two methods have been developed (Lanusse et *al*., 2000). The choice of the method is made through an analysis of the coupling rate of the plant. When this rate is reasonable, one can choose the simple multi SISO approach.

Within a frequency range $[\omega_A, \omega_B]$ around open-loop gain-crossover frequency ω_{cg}, the Nichols locus of a third generation CRONE open-loop is defined by an any-angle straight line segment, called a generalized template (Fig. 18).
The generalized template can be defined by an integrator of complex fractional order n whose real part determines its phase location at frequency ω_{cg}, that is $-\Re e_{/i}(n)\pi/2$, and whose imaginary part determines its angle to the vertical (Fig. 18).

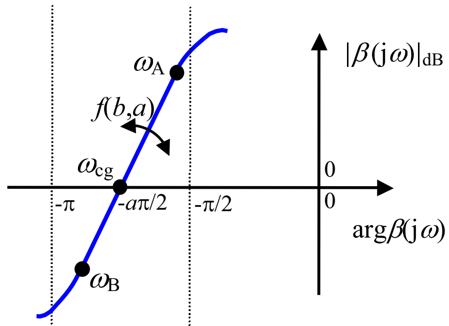

Fig. 18. Generalized template in the Nichols plane

The transfer function including complex fractional order integration (Oustaloup *et al*, 2000) is:

$$\beta(s)=\left(\cosh\left(b\frac{\pi}{2}\right)\right)^{\mathrm{sign}(b)}\left(\frac{\omega_{cg}}{s}\right)^{a}\left(\mathrm{Re}_{/i}\left(\left(\frac{\omega_{cg}}{s}\right)^{ib}\right)\right)^{-\mathrm{sign}(b)} \tag{62}$$

with $n = a + ib \in \mathbb{C}_i$ and $\omega \in \mathbb{C}_j$, and where $\mathbb{C}_i$ and $\mathbb{C}_j$ are respectively time-domain and frequency-domain complex planes. The definition of the open-loop transfer function including the nominal plant must take into account:
- accuracy specifications at low frequencies;
- the generalized template around frequency ω_{cg};
- plant behaviour at high frequencies while respecting the control effort specifications at these frequencies.
Thus, the open-loop transfer function is defined by a transfer function based on band-limited complex fractional order integration:

$$\beta(s)=\beta_1(s)\bar{\beta}(s)\beta_h(s), \tag{63}$$

with:

$$\bar{\beta}(s)=C^{\mathrm{sign}(b)}\left(\alpha\frac{1+s/\omega_h}{1+s/\omega_l}\right)^{a}\left(\Re e_{/i}\left\{\left(\alpha\frac{1+s/\omega_h}{1+s/\omega_l}\right)^{ib}\right\}\right)^{-q\mathrm{sign}(b)} \tag{64}$$

$$\alpha_0=\left(1+\left(\frac{\omega_r}{\omega_0}\right)^2\bigg/1+\left(\frac{\omega_r}{\omega_1}\right)^2\right)^{1/2} \tag{65}$$

- where $\beta_l(s)$ is an integer order n_l proportional integrator:

$$\beta_l(s)=C_l(\omega_l/s+1)^{n_l} \tag{66}$$

- where $\beta_h(s)$ is a low-pass filter of integer order n_h:

$$\beta_h(s)=C_h\big/(s/\omega_h+1)^{n_h} \tag{67}$$

with

$$C_l=\left(\frac{\omega_{cg}{}^2}{\omega_l{}^2+\omega_{cg}{}^2}\right)^{n_l/2} \text{ and } C_h=\left(\frac{\omega_{cg}{}^2}{\omega_h{}^2}+1\right)^{n_h/2}. \tag{68}$$

The optimal open loop transfer function is obtained by the minimization of the robustness cost function

$$J=\sup_{\omega,P}\left|T(j\omega)\right|-M_{r0}\ , \tag{30}$$

where M_{r0} is the resonant peak set for the nominal parametric state of the plant, while respecting the following set of inequality constraints for all plants (or parametric states of the plant) and for $\omega \in \mathbb{R}^+$:

$$\inf_P |T(\mathrm{j}\omega)| \geq T_\mathrm{l}(\omega) \text{ and } \sup_P |T(\mathrm{j}\omega)| \leq T_\mathrm{u}(\omega) ,$$
$$\sup_P |S(\mathrm{j}\omega)| \leq S_\mathrm{u}(\omega) , \sup_P |CS(\mathrm{j}\omega)| \leq CS_\mathrm{u}(\omega) \tag{69}$$

and
$$\sup_P |PS(\mathrm{j}\omega)| \leq PS_\mathrm{u}(\omega) , \tag{70}$$

with
$$\begin{cases} T(s) = \dfrac{C(s)P(s)}{1+C(s)P(s)} & S(s) = \dfrac{1}{1+C(s)P(s)} \\ CS(s) = \dfrac{C(s)}{1+C(s)P(s)} & PS(s) = \dfrac{G(s)}{1+C(s)P(s)} \end{cases} \tag{71}$$

As the uncertainties are taken into account by the least conservative method, a non-linear optimization method must be used to find the optimal values of three independent parameters. The parameterization of the open-loop transfer function by complex fractional orders, then simplifies the optimization considerably. During optimization a complex order has the same effect as a large set of parameters found in common rational controllers.

When the optimal nominal open-loop transfer is determined, the fractional controller $K_F(s)$ is defined by its frequency response:

$$K_\mathrm{F}(\mathrm{j}\omega) = \beta(\mathrm{j}\omega)/P_0(\mathrm{j}\omega), \tag{72}$$

where $P_0(\mathrm{j}\omega)$ is the nominal frequency response of the plant.

The parameters of a rational transfer function $K_R(s)$ with a predefined low-order structure are tuned to fit the ideal frequency response $K_F(\mathrm{j}\omega)$. The rational integer model on which the parametric estimation is based, is given by:

$$K_\mathrm{R}(s) = B(s)/A(s) , \tag{73}$$

where $B(s)$ and $A(s)$ are polynomials of specified integer degrees n_B and n_A. Any frequency-domain system-identification technique can be used. Whatever the complexity of the control problem, satisfactorily low values of n_B and n_A, usually around 6, can be used without performance reduction.

5.2. CRONE observer as an application of CRONE Control

Dynamic output feedback based observer concept was introduced in (Marquez, 2003) and (Marquez and Riaz, 2005) in which the observation problem is solved using the feedback diagram of Fig. 19. The plant P, the model M and the dynamic controller K are supposed single input / single output systems represented by the state space descriptions:

$$P : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{74}$$

$$M : \begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bv(t) = A\hat{x}(t) + B(u(t) + w(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{75}$$

$$K : \begin{cases} \dot{x}_K(t) = A_k x_K(t) - B_K \varepsilon(t) \\ \qquad = A_K x(t) + B_K C(x(t) - \hat{x}(t)) \\ w(t) = C_K x_K(t) \end{cases} \tag{76}$$

State $x(t)$ is supposed not measurable and $\hat{x}(t)$ denotes the estimated state. All the elements of matrices and vectors in (74) to (76) are supposed to be real.

Fig. 19 clearly shows that the goal of the used feedback structure is to cancel the observation error $\chi(t) = x(t) - \hat{x}(t)$ by cancelling the error signal $\varepsilon = \hat{y}(t) - y(t)$.

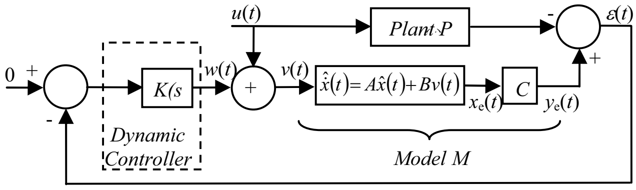

Fig. 19. Dynamic output feedback based observer

Robustness considerations versus plant perturbation are also addressed in (Marquez, 2003) in an H_∞ framework for the synthesis of a dynamic output feedback based observer. In the CRONE observer synthesis methodology, plant perturbation is taken into account like with CRONE Control, thus leading to a new formulation of of the CRONE control-system design methodology. That thus permits to profit from the advantages of CRONE control versus H_∞ (it turn out that in practice a CRONE controller permits to obtain better performance than an H_∞ one on the same plants, see for instance (Landau, et al, 1995) for a comparison on a benchmark based on robust digital control of a flexible transmission system).

6 – Conclusion

Various activities of the CRONE group in the field of fractional systems began about thirty years ago with an application of fractional order controllers for robust control of continuous colorant lasers. Since then, many others applications have been developed in many others fields (Oustaloup, 1995) and industrial applications now exist (Oustaloup *et al.* 2006). This paper only gives an overview of the most recent CRONE group applications in the field of system analysis, modeling and identification, observation and control. Beyond this presentation, this paper is also an introduction to fractional differentiation and fractional systems. It could be a starting point for researchers willing to work in the field of fractional differentiation and its applications.

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