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Why "state" feedback?

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Abstract: We study the linear quadratic control problem from a representation-free point of view, and we show that this formulation brings forth two self-contained and original proofs of the optimality of state feedback control laws; these proofs which do not depend on an *a priori* state-space representation. Moreover, we show an orthogonality property characterizing the set of optimal trajectory of a LQ-control problem.

Keywords: LQ-control problem, data-driven control, Riccati difference equation, state feedback, state maps.

1. INTRODUCTION

The classical approach for solving a control problem is "model-driven"—first a mathematical model of the plant is obtained, and then a model for the controller is computed, based on the model of the plant and on a performance criterion.

In this paper we operate in the *data-driven* paradigm for control design, where the control input signal is determined directly from measurements of the observed variables of the plant and from the performance criterion, without the need to identify explicitly a model of the plant or of the controller. Different approaches for developing controllers or control signals directly from data have been developed by many authors, see Chan (1996); Favoreel et al. (1998); Fujisaki et al. (2004); Ikeda et al. (2001); Woodley (2001). When compared to them, the data-driven paradigm adopted in this paper exhibits two main original aspects. On the one hand, our point of view considers as a starting point *any* system trajectory that completely represents the dynamics of the system, rather than one of a specific nature (e.g. impulse- or step-response). On the other hand, we operate instead in the behavioral framework, where a system is identified by the set of all its trajectories, rather than in *representation-oriented* frameworks such as the transfer-function or state-space approach.

In this paper we give an *intrinsic* proof of the optimality of the state feedback control input in LQ-control: we show that this fact can be *deduced from first principles*, and need not be considered a mere consequence of a set-up essentially based on the use of state-space representations. We also show an orthogonality property of the optimal trajectories, which mirrors the one already known in the context of optimal filtering, and which leads to a simple derivation of the Riccati difference equation. Surprisingly, with the exception of some of the results by Kawamura (see Kawamura (1998)), to the best of our knowledge no explicit condition of this sort has been given in the literature.

The paper is organized as follows. In order to give an intrinsic formulation of the problem, specified at the level of the trajectories of the system, we first introduce some important preliminary results, which are gathered in section 2. We proceed to solve the linear quadratic control problem in section 3. In section 4 we use the data-driven formulation in order to give a self-contained and simple proof of the optimality of state feedback in linear quadratic control problems, without the need to assume *a priori* the notion of state and of state-space representation. In section 5 we illustrate an orthogonality property of the optimal trajectories of the system, and we show that this property implies that the optimal trajectory is a linear function of the state. In this section we also relate the orthogonality property to the Riccati difference equation. Section 6 contains some final remarks.

In this paper we extensively use the conceptual framework and the language of behavioral system and control theory; we refer the reader unfamiliar with the concepts and terminology of the behavioral approach to the book Polderman et al. (1998).

Notation. In this paper we denote the set of nonnegative integers with $\mathbb{Z}_+$, the set of real numbers with $\mathbb{R}$, and that of complex numbers with $\mathbb{C}$. The space of $\mathtt{n}$-dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, and the space of $\mathtt{m} \times \mathtt{n}$ real matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. If $A \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, then $A^\top \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$ denotes its transpose, and A^+ its pseudo-inverse. If A is a matrix, possibly with an infinite number of rows or columns, then $\mathrm{im}(A)$ denotes its image, and $\ker(A)$ its kernel. If $\{A_i\}_{i=1,\dots,N}$ is a set of matrices, then we define

$$\text{block diag}\,(A_i)_{i=1,\dots,N} := \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_N \end{bmatrix}$$

If the sequence $A_i = A$ for $i = 1, \dots, N$, we will be writing block $\mathrm{diag}(A)_{i=1,\dots,N}$. If $\{A_i\}_{i=1,\dots,N}$ is a set of matrices with the same number of columns, then we define

$$\operatorname{col}(A_i)_{i=1,\ldots,N} := \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix}$$

The *Hankel matrix of depth* L associated with a matrix sequence of finite length $w(1),\ldots,w(T)$ is

$$\mathcal{H}_L(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-L+1) \\ w(2) & w(3) & \cdots & w(T-L+2) \\ \vdots & \vdots & \cdots & \vdots \\ w(L) & w(L+1) & \cdots & w(T) \end{bmatrix} \tag{1}$$

The *lower-triangular Töplitz matrix of depth* $L+1$ associated with a matrix sequence $H(0),\ldots,H(T)$ is defined as

$$\mathcal{T}_L(H) := \begin{bmatrix} H(0) & 0 & \cdots & 0 \\ H(1) & H(0) & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ H(L) & H(L-1) & \cdots & H(0) \end{bmatrix} \tag{2}$$

The set consisting of all sequences from $\mathbb{Z}_+$ to $\mathbb{R}^{\mathtt{w}}$ is denoted with $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$. On such space we define the *left*, i.e. *backward*, *shift* defined by $(\sigma w)(t) := w(t+1)$ for all $t \in \mathbb{Z}_+$; and the *right*, i.e. *forward shift* σ^{-1}, defined by

$$\begin{cases} \left(\sigma^{-1}w\right)(t) := w(t-1) & \text{for } t \leq 1 \\ \left(\sigma^{-1}w\right)(0) := 0 & \text{otherwise} \end{cases}$$

We define the *concatenation of two trajectories* $w_1, w_2 \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ *at time* $t > 0$ to be the trajectory $w_1 \underset{t}{\wedge} w_2 \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ defined by

$$\left(w_1 \underset{t}{\wedge} w_2\right)(k) := \begin{cases} w_1(k) & \text{for } k \leq t \\ w_2(k) & \text{for } k > t \end{cases}$$

The set of linear, shift-invariant subspaces ("behaviors") of the space of trajectories from $\mathbb{Z}_+$ to $\mathbb{R}^{\mathtt{w}}$ closed in the topology of pointwise convergence will be denoted with $\mathcal{L}^{\mathtt{w}}$. Equivalently (see Theorem 5 of Willems (1986a)), $\mathfrak{B}$ is the set of trajectories produced by a finite-dimensional, linear, time-invariant system. The subset of $\mathcal{L}^{\mathtt{w}}$ consisting of all controllable behaviors will be denoted with $\mathcal{L}^{\mathtt{w}}_{\text{contr}}$. Associated with a behavior $\mathfrak{B}$ are a number of important "integer invariants" such as the *order of* $\mathfrak{B}$, i.e. the minimal dimension of the state variable in a state-space representation of $\mathfrak{B}$, denoted with $\mathtt{n}(\mathfrak{B})$; the *input cardinality of* $\mathfrak{B}$, i.e. the number of input variables of $\mathfrak{B}$, denoted with $\mathtt{m}(\mathfrak{B})$; the *output cardinality of* $\mathfrak{B}$, i.e. the number of output variables of $\mathfrak{B}$, denoted with $\mathtt{p}(\mathfrak{B})$; and the *lag of* $\mathfrak{B}$, denoted with $\mathtt{L}(\mathfrak{B})$, which we now define. Let $R(\sigma)w = 0$ be a kernel representation of $\mathfrak{B}$. The maximum of the degrees of the polynomial elements of R is called the lag associated with this particular kernel representation. $\mathtt{L}(\mathfrak{B})$ is the smallest possible lag over all kernel representations of $\mathfrak{B}$. It can be proved that there exists a kernel representations of $\mathfrak{B}$ with lags less than or equal to $\mathtt{L}(\mathfrak{B})$.

In the following, when it will be clear from the context which behavior is being referred to, we will drop the explicit dependence on $\mathfrak{B}$ in the invariants' symbols, and write $\mathtt{n}$, $\mathtt{m}$, $\mathtt{p}$.

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. The space of $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that of $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminates ζ and η is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta,\eta]$. Given a polynomial matrix $R(\xi) := R_0 + \cdots + R_L\xi^L \in \mathbb{R}^{n\times m}[\xi]$ with $R_L \neq 0$, we define its *reciprocal matrix* $R^r(\xi)$ as $R^r(\xi) := R_0\xi^L + \cdots + R_L \in \mathbb{R}^{n\times m}[\xi]$.

2. BACKGROUND MATERIAL

This section contains some essential notions needed for the formulation of the data-driven linear quadratic problem. In part this is background material, namely the notion of persistent excitation of an input, and a condition for the identifiability of a linear system from a finite set of measurements (the so-called "Fundamental Lemma"), which have appeared in Willems et al. (2005). The rest of the section is devoted to the formalization of the concept of "initial conditions" in a trajectory setting, using the notion of state map introduced in Rapisarda et al. (1997).

We define a signal $u : [1,T]\cap\mathbb{Z}_+ \to \mathbb{R}^{\mathtt{u}}$ to be *persistently exciting of order* L if the Hankel matrix $\mathcal{H}_L(u)$ of depth L associated with $u(1),\ldots,u(T)$ is of full row rank, i.e. of rank $L\mathtt{u}$.

In Willems et al. (2005) the authors have investigated the following identifiability problem. Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$, and let $w \in \mathfrak{B}$. Assume that a finite set of consecutive values $w(1),\ldots,w(T)$ of w is given, and choose $L < T$; when do the restrictions

$$\begin{aligned} &[w(1), w(2), \ldots, w(L)] \\ &[w(2), w(3), \ldots, w(L+1)] \\ &\quad\vdots \\ &[w(T-L+1), w(T-L+2), \ldots, w(T)] \end{aligned} \tag{3}$$

span the space $\mathfrak{B}|_{[1,L]}$ consisting of all possible restrictions of length L that can be produced by trajectories of $\mathfrak{B}$? A sufficient condition for this is given in the following "fundamental lemma", which is the main result of Willems et al. (2005).

Lemma 1. **(Fundamental Lemma)** Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}_{\text{contr}}$, and let (u,y) be an input-output partition of the external variable w. Denote with $\mathtt{n}(\mathfrak{B})$ the order of $\mathfrak{B}$, and with $\mathcal{H}_L(w)$ the Hankel matrix (1).

Assume that $[w(1), w(2), \ldots, w(T)] \in \mathfrak{B}|_{[1,T]}$. Then

$$\begin{array}{l} [u(1), u(2), \ldots, u(T)] \\ \text{persistently exciting} \\ \text{of order } \ L + \mathtt{n}(\mathfrak{B}) \end{array} \quad \Longrightarrow \quad \operatorname{im}(\mathcal{H}_L(w)) = \mathfrak{B}|_{[1,L]}$$

We proceed to illustrate the concept of state map introduced in Rapisarda et al. (1997) (see also Praagman (1988)), and how it relates to the specification of the "initial conditions" of a trajectory by means of a "prefix" trajectory.

Consider a kernel representation $\mathfrak{B} = \ker(R(\sigma))$; then a polynomial matrix $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ is said to induce a *state map* for $\mathfrak{B}$ if the system with latent variable x defined by the equations

$$R(\sigma)w = 0$$
$$X(\sigma)w = x$$

satisfies the *axiom of state*, i.e. if for all $(w, x) \in \mathfrak{B}_f$ it holds that

$$[x(0) = 0] \Longrightarrow \left[(0 \underset{0}{\wedge} w, 0 \underset{0}{\wedge} x) \in \mathfrak{B}_f\right]$$

Algorithms to compute state maps and minimal state maps are given in Rapisarda et al. (1997).

3. FINITE-HORIZON LQ-CONTROL PROBLEM

The finite-horizon data-driven quadratic optimal control problem for linear time-invariant systems is formulated as follows. Given:

(1) a trajectory $\overline{w} = \mathrm{col}\,(\overline{u}, \overline{y}) \in \mathfrak{B}|_{[1,T_f]}$, where $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}_{\mathrm{contr}}$, with $\overline{u}$ persistently exciting of order greater than or equal to $\mathtt{n}(\mathfrak{B}) + \mathtt{L}(\mathfrak{B})$;
(2) a positive-definite matrix $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$; and
(3) an initial trajectory $w_{\mathrm{ini}} \in \mathfrak{B}|_{[1,T]}$, with $T_f \geq T \geq \mathtt{L}(\mathfrak{B})$.

Find $w^\star \in \mathfrak{B}_{|_{[T,T_f]}}$ such that $w' := w_{\mathrm{ini}} \underset{T}{\wedge} w^\star$ minimizes

$$\sum_{t=1}^{T_f} w'^\top(t)\Phi w'(t)$$
$$\text{subject to } w' \in \mathfrak{B}|_{[1,T_f]}$$

A brief discussion of the set-up described above is in order. The trajectory in point (1) above is the *data* which we use in order to derive information about the plant. It is a set of (input, output) measurements, for example derived from an experiment. The matrix in point (2) is that inducing the cost functional on the external variables of the system. The "prefix" w_{ini} in (3) corresponds to the "initial conditions" of the system, and is analogous to the "initial state" of the classical, i.e. state-space, approach to optimal control problems. In the setting described by (1)-(3), it is required to find among all possible "emanations" of w_{ini}, i.e. among all trajectories in $\mathfrak{B}|_{[1,T_f]}$ whose first T values coincide with w_{ini}, that which minimizes the cost functional over the horizon $[1, T_f]$ (equivalently, considering that the values up to T are fixed, over the horizon $[T, T_f]$).

In order to solve the finite-horizon data-driven control problem, we first compute from the data two matrices with $T_f\mathtt{w}$ rows, denoted respectively $\mathcal{H}_F$ and $\mathcal{H}_Z$, whose columns form a basis for respectively the $[1, T_f]$-free responses and the $[1, T_f]$-zero-initial prefix behavior of the system, which are defined as follows. The set of free responses is:

$$\mathrm{im}\,(\mathcal{H}_F) = \mathfrak{B}_F := \{\ w_{|_{[1,T_f]}} = \mathrm{col}(u, y)_{|_{[1,T_f]}} \in \mathfrak{B}|_{[1,T_f]} \mid u(k) = 0 \text{ for all } 1 \leq k \leq T_f\} \quad (4)$$

Now partition $\mathcal{H}_F$ as $\mathcal{H}_F = \mathrm{col}(\mathcal{H}'_F, \mathcal{H}''_F)$ with $\mathcal{H}'_F \in \mathbb{R}^{\mathtt{w}T\times\bullet}$ and $\mathcal{H}''_F \in \mathbb{R}^{\mathtt{w}(T_f-T)\times\bullet}$, and observe that the trajectory $w_{|_{[1,T_f]}} \in \mathfrak{B}_F$ having w_{ini} as prefix can be obtained from $\mathcal{H}_F$ by finding the unique solution to the linear system of equations $\mathcal{H}'_F\overline{\alpha} = w_{\mathrm{ini}}$ and then defining $w_{\overline{\alpha}} := \mathcal{H}_F\overline{\alpha}$. Observe also that this trajectory $w_{|_{[1,T_f]}}$ is unique.

The zero-initial prefix subbehavior of $\mathfrak{B}$ is defined as

$$\mathfrak{B}_Z := \{\ w_{|_{[1,T_f]}} \in \mathfrak{B}|_{[1,T_f]} \mid \exists\, T' \leq T_f \text{ s.t. } w(k) = 0 \text{ for } 1 \leq k \leq T', T' \geq \mathtt{L}(\mathfrak{B})\}$$

In Markovsky et al. (2007) and in Markovsky et al. (2006) algorithms are given in order to compute a matrix $\mathcal{H}_Z$ with full column rank such that

$$\mathrm{im}\,(\mathcal{H}_Z) = \mathfrak{B}_Z \quad (5)$$

Assuming that $\mathcal{H}_F$ and $\mathcal{H}_Z$ have been computed such that (4) and (5) hold, respectively, then every trajectory $w' \in \mathfrak{B}|_{[1,T_f]}$ with prefix w_{ini} can be written as

$$w' = \mathcal{H}_F\overline{\alpha} + \mathcal{H}_Z\beta \quad (6)$$

where $\overline{\alpha}$ satisfies $\mathcal{H}'_F\overline{\alpha} = w_{\mathrm{ini}}$. Now denote with $\tilde{\Phi}$ the matrix $\tilde{\Phi} := \text{block diag}(\Phi)_{i=1,\ldots,T_f}$, and observe that since Φ is positive definite, also $\tilde{\Phi}$ is positive definite. Then it is easy to see that the minimal cost trajectory is associated with the vector β^* such that

$$\mathcal{H}_Z^\top\tilde{\Phi}\mathcal{H}_F\overline{\alpha} + \mathcal{H}_Z^\top\tilde{\Phi}\mathcal{H}_Z\beta^* = 0 \quad (7)$$

and consequently,

$$w^* = \left[I - \mathcal{H}_Z\left(\mathcal{H}_Z^\top\tilde{\Phi}\mathcal{H}_Z\right)^{-1}\mathcal{H}_Z^\top\tilde{\Phi}\right]\mathcal{H}_F\overline{\alpha} \quad (8)$$

Not surprisingly considering the quadratic nature of the cost functional, w^* is obtained from the free trajectory $\mathcal{H}_F\overline{\alpha}$ with prefix w_{ini} by subtracting from it its projection (in the metric induced by the cost matrix $\tilde{\Phi}$) on the zero-initial prefix subbehavior $\mathfrak{B}_Z$. Another geometrical interpretation of formulas (6)-(7) is given in section 5 of this paper.

In the rest of this paper we examine the implications of formulas (7) and (8), beginning in the next section with its relationship with the concept of "state feedback".

4. WHY "STATE" FEEDBACK?

The notion of state is all-pervasive in system and control theory, all the more so in linear quadratic control, where the fact that the optimal feedback law is a function of the state of the system is rightly considered to be one of the most important results of the framework initiated by Kalman. In this section we show that an alternative approach to linear quadratic control, one that *deduces and does not postulate* the fact that the state is involved in the computation of the optimal trajectory can and, in our opinion, should be taken, if only because of its simplicity and of its pedagogical effectiveness.

In order to justify these claims, we first prove the following result.

Proposition 2. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, let $i \geq \mathtt{L}(\mathfrak{B})$, and denote with $V(i)$ the minimum value of the problem

$$\min \frac{1}{2}\sum_{j=i}^{T_f} w(j)^\top \Phi w(j)$$
$$\text{subject to } w_{|_{[i,T_f]}} \in \mathfrak{B}_{|_{[i,T_f]}}$$
$$w(j) = \overline{w}_j \text{ given, } j = i, \ldots, i + \mathtt{L}(\mathfrak{B}) - 1 \tag{9}$$

Let w^* be such that $V(i) = \frac{1}{2}\sum_{j=i}^{T_f} w^*(j)^\top \Phi w^*(j)$; then for every state map $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ there exists a positive definite matrix $K_i = K_i^\top \in \mathbb{R}^{\bullet\times\bullet}$ such that

$$V(i) = ((X(\sigma)w^*)(i))^\top K_i (X(\sigma)w^*)(i) \tag{10}$$

We now briefly comment the statement of Proposition 2 by comparing it with the results available in the classical framework for control. The fact that in the state-space setting, the optimal performance index for the linear quadratic regulator problem is a quadratic form in the state is a well-known result both in continuous- and in discrete-time, see for example sections 2.2, 2.3 and 2.4 of Anderson et al. (1989). However, the reader should note that the result of Proposition 2 has been obtained from first principles, starting from a description of the system as a set of higher-order difference equations possibly including algebraic constraints among the variables, and not on the basis of a first order representation as the one considered in the state-space setting. Proposition 2 shows that the fact that the optimal cost is a quadratic function of the state is a *direct consequence of the nature of the problem itself, and not of the particular representation used for solving the problem.*

The main result of this section follows in a straightforward manner applying Bellman's optimality principle to the optimization problem defined in (11), and from the result of Proposition 2.

Proposition 3. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and consider the problem

$$\min \frac{1}{2}\sum_{j=1}^{T_f} w(j)^\top \Phi w(j)$$
$$\text{subject to } w_{|_{[1,T_f]}} \in \mathfrak{B}|_{[1,T_f]}$$
$$w(j) = \overline{w}_j \text{ given, } j = 1, \ldots, T \tag{11}$$

For every state map $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ and every $i \in [1, T_f]$ there exists a matrix $L_i \in \mathbb{R}^{\mathtt{w}\times\bullet}$ such that the optimal trajectory w^* satisfies

$$w^*(i) = L_i\left(X(\sigma)w^*\right)(i) \tag{12}$$

Remark. It follows from the proof of Proposition 3 that the value of w^* at any time instant is a function of the past values of w^*. ■

Remark. An additional and alternative proof of the optimality of state feedback laws is given in section 5 of this paper. ■

Remark. It can be shown that the feedback gain L_i of Proposition 3 can be computed recursively, analogously to what happens in the state-space approach to LQ-optimal control with the Riccati difference equation. The simplest way to consider this problem is to study the *L-step ahead recursion*, in which at each step of the iteration one computes the gain- and the optimal cost matrix for the optimal control problem on the horizon $[i, T_f]$ based on the optimal cost matrix for the optimal control problem on the horizon $[i + L, T_f]$. We will not enter into these details here. ■

5. THE ORTHOGONALITY PROPERTY AND THE RICCATI DIFFERENCE EQUATION

On the space $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \bigcap [1,T_f]}$ we define the following inner product induced by the matrix $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$:

$$\langle w_1, w_2\rangle_\Phi := \sum_{k=1}^{T_f} w_1^\top(k)\Phi w_2(k)$$

and we call w_1 orthogonal to w_2 if $\langle w_1, w_2\rangle_\Phi = 0$, written $w_1 \perp_\Phi w_2$. Given a behavior $\mathfrak{B}|_{[1,T_f]}$, we define its Φ-orthogonal, denoted with $\mathfrak{B}|_{[1,T_f]}{}^{\perp_\Phi}$, as

$$\left(\mathfrak{B}|_{[1,T_f]}\right)^{\perp_\Phi} := \{w_{|_{[1,T_f]}} \mid \langle w, v\rangle_\Phi = 0 \text{ for all } v \in \mathfrak{B}|_{[1,T_f]}\}$$

The main result of this section is the following.

Proposition 4. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and consider the finite-horizon LQ-problem defined in section 3. Then $w^* := \{w^*(1), \ldots, w^*(T_f)\}$ with prefix $w_{\rm ini}$ solves the LQ data-driven problem if and only if w^* belongs to $\mathfrak{B}_Z^{\perp_{\tilde\Phi}}$, the Φ-orthogonal behavior of the zero initial prefix behavior.

The proof of this statement is based on the observation that a trajectory $w^* = \mathcal{H}_F\overline{\alpha} + \mathcal{H}_z\beta^*$ is optimal if and only if

$$\mathcal{H}_Z^\top \tilde\Phi \left[\mathcal{H}_F\overline{\alpha} + \mathcal{H}_Z\beta^*\right] = 0 \tag{13}$$

holds true. Equation (13) will be used in the following also to establish an orthogonality property of the optimal trajectories of the system.

The result of Proposition 4 has a straightforward geometric interpretation, illustrated in Figure 1. With reference to the proof of Proposition 4, denote with w_f the "free-response" trajectory $\mathcal{H}_F\overline{\alpha}$ emanating from $w_{\rm ini}$, and with w_c the "control-trajectory" $\mathcal{H}_Z\beta^*$ emanating from the zero initial prefix. Then equation (13) shows that the optimal trajectory w^* is the orthogonal projection (in the metric induced by $\langle\cdot,\cdot\rangle_\Phi$) of w_f onto $\mathfrak{B}_Z^{\perp_\Phi}$, and that it is obtained from w_f by adding to it the trajectory w_c.

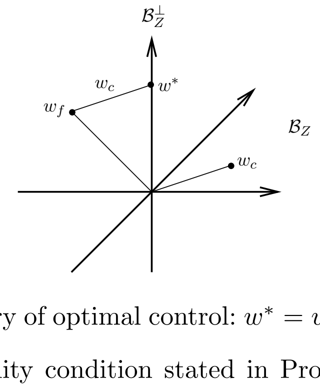

Fig. 1. Geometry of optimal control: $w^* = w_f + w_c \in \mathcal{B}_Z^{\perp_\Phi}$.

The orthogonality condition stated in Proposition 4 can hardly be called surprising, considering the least-squares framework in which the optimization problem is cast; however, to the best of the knowledge of the authors, with the exception of the work of Kawamura (see Kawamura (1998)) discussed later in this section, no explicit condition of this sort has been given in the literature on the linear quadratic optimal control problem. It is interesting to contrast this situation with standard treatments of least-squares estimation (see for example Luenberger (1968)), where orthogonality of random variables plays an important role both from the pedagogical and from the algorithmic point of view.

In the following we show that the orthogonality condition stated in Proposition 4 has some rather interesting consequences. The first one we prove is that it also implies that the optimal trajectory is a linear function of the state (Proposition 3), giving yet another interpretation of the "state feedback" law (12), and an independent proof of the optimality of state-feedback.

Proposition 5. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and consider the finite-horizon LQ-problem defined in section 3. Let $X(\sigma)$ be a state map for $\mathfrak{B}$. If $w^*_{|[1,T_f]} \in \mathfrak{B}|_{[1,T_f]}$ belongs to $\mathfrak{B}_Z^{\perp_\Phi}$, then for every $1 \leq i \leq T_f$ there exists $L_i \in \mathbb{R}^{\mathtt{w}\times\bullet}$ such that $w^*(i) = L_i \left(X(\sigma)w^*\right)(i)$.

Before exploring further the consequences of the result of Proposition 4, we briefly comment on its relationship with the "basic orthogonality condition" of Kawamura (1998). In Theorem 3 of that paper it is shown that in the context of state space systems

$$x(k+1) = Ax(k) + Bu(k)$$
$$z(k) = \begin{bmatrix} Cx(k) \\ Du(k) \end{bmatrix}$$

under the assumptions of stabilizability, detectability, and positive-definiteness of $D^\top D$, a feedback law $u(k) = Gx(k)$ is the infinite-horizon optimal LQ-control law if and only if the impulse response of the *closed-loop* system

$$x(k+1) = (A+BG)x(k) + Bu'(k)$$
$$z(k) = \begin{bmatrix} Cx(k) \\ DG \end{bmatrix} + \begin{bmatrix} 0 \\ D \end{bmatrix} u'(k)$$

is orthogonal in the ℓ_2 sense with any free response of the closed-loop system (see equations (6)-(7) and Theorem 3 of Kawamura (1998)). Observe that instead in Proposition 4 it is stated that any *open-loop* zero-initial prefix trajectory and any free response of the closed-loop (optimal) system are orthogonal.

In Corollary 4 of Kawamura (1998) it is also shown that if the solution to the algebraic Riccati equation is positive semidefinite, then the ARE and the gain equation are equivalent with the orthogonality condition. In the following proposition we state a similar result derived from the orthogonality condition of Proposition 4. For this purpose, we consider the behavior $\mathfrak{B}$ consisting of the trajectories $w = \mathrm{col}(u, x)$ satisfying

$$\sigma x = Ax + Bu \tag{14}$$

and the finite-horizon LQ problem with cost functional induced by

$$\Phi = \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \tag{15}$$

with $R > 0$, $Q \geq 0$.

Proposition 6. The trajectory

$$w^* = (u^*, x^*) \in \mathfrak{B} := \{(u,x) \mid (u,x) \text{ satisfy } (14)\}$$

is orthogonal to $\mathfrak{B}_Z$ in the inner product $\langle \cdot, \cdot \rangle_\Phi$ induced by (15) if and only if

$$u^*(i) = -(R + B^\top K_{i+1} B)^{-1} B^\top K_{i+1} A x^*(i) \tag{16}$$

where

$$\begin{aligned} K_{T_f} &:= Q; \\ K_i &= A^\top K_{i+1} A + Q \\ &\quad - A^\top K_{i+1} B (R + B^\top K_{i+1} B)^{-1} B^\top K_{i+1} A \end{aligned} \tag{17}$$

$i = 1, \ldots, T_f - 1$ and K_i is a positive definite matrix.

6. CONCLUSIONS

In this paper we have presented two independent results on the optimality of state feedback laws in finite-horizon linear quadratic control problems, namely Proposition 3 and Proposition 5. These results have been deduced from first principles: the fact that the optimal cost is a quadratic function of the state has been shown to be a consequence of the nature of the problem itself, and not of the particular representation adopted for the system.

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