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Nonlinear H∞ Robust Control for Six DOF equations of motion of Rigid Body with Mass Uncertainty

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Abstract: This paper derives the analytic solution of nonlinear H∞ robust controller for a system with mass and moments of inertia uncertainties and investigates the implementation using control surface inverse algorithm. A special Lyapunov function with mass and moments of inertia uncertainties is introduced to solve the associated Hamilton-Jacobi partial differential inequality (HJPDI). The HJPDI is solved analytically, resulting in a nonlinear H∞ robust controller with simple proportional feedback structure. The control surface inverse algorithm (CSIA) is employed to determine the angles of control surface deflection from the nonlinear H∞ control command. The ranges that guarantee stability and robustness of nonlinear *H*∞ flight control system implemented by vehicle actuators are derived. Numerical simulation is carried out and the results show that the responses still show good convergence for large initial perturbation.

1. INTRODUCTION

The existing applications of nonlinear H∞ flight control are almost restricted to the longitudinal or lateral motion alone. Accounting complete six degrees-of-freedom (DOF) motion (including both longitudinal and lateral directions) is still a challenge for all nonlinear flight control design methods. The main difficulty encountered in the six DOF nonlinear H∞ flight control design is to solve an associated Hamilton-Jacobi partial differential inequality (HJPDI). This difficulty can be conquered by a methodology recently developed by (Yang *et al*., 2000) where an analytical solution of HJPDI was derived for general flight vehicles with six DOF motions. However, it is not clear that how to implement this nonlinear H∞ flight control command by using the aerodynamic control surface and engine thrust in that paper. Furthermore, there is no discussion of the effect of parameter uncertainties on the controller performance.

In this paper, we re-drive the analytical solution of HJPDI with mass uncertainty and try to implement it to general vehicle by control surfaces. Along the three perpendicular body axes, we decompose nonlinear H_∞ flight control command into three force commands and three moment commands, and find that it at least needs six independent control surfaces (and the related actuators) to generate the required forces and moments. However, flight vehicle generally has only five or four independent control surfaces, and that those have the limitations of saturation. It is thus unavoidable that the six DOF nonlinear H_∞ flight control command can not be implemented exactly by actuator systems. Therefore the key of flight control implementation issue is to minimize the tracking errors between the H_∞ commands and the actually achievable control forces and moments. The control surface inverse algorithm (CSIA) developed in this paper is just aimed at this purpose, which is based on the Moore-Penrose generalized inverse formulas that can be found in many other applications such as tracking control (Robinett *et al*., 1996), and redundancy optimization (Roberts *et al*., 1991). The proposed CSIA algorithm computes the best deflection angles at each sampling instant so as to produce the control forces and moments with the deviations from the H∞ commanded values being as small as possible. While one may wonder if the system is still stability or not when the control forces and moments are generated by CSIA. In this paper, the applying range of CSIA to guarantee robustness and stability is driven.

Otherwise, it does occur that the nonlinear H∞ commanded amplitude and rate of control surface deflection exceed the operation ranges of aircraft actuating systems. A proper design of command prefilter to avoid this saturation of control system is therefore indispensable. Before the computed nonlinear H∞ command is fed into the aircraft flight control system, it need be reshaped by a command prefilter. Parameters in command prefilters (Reigelsperger *et al*., 1998) can be designed to vary with flight conditions so as to optimally reflect the maneuverability of the aircraft and to achieve the best flight qualities. In this paper, the range of control command prefiler is designed such that the control surface deflections commanded by nonlinear H∞ controller can be followed as close as possible by aircraft actuator system.

2. NONLINEAR H∞ FLIGHT CONTROL WITH MASS AND MOMENT INERTIA UNCERTAINTIES

In the section, we will re-formulate this work of our previous research (Yang *et al*., 2000) by normalizing. Define *U*, *V*, *W*, and *P*, *Q*, *R* be standard notations for linear and angular velocities, respectively; I_{xx} , I_{xz} ,..., etc, be the moments of inertia of the flight vehicle; m_s is the vehicle's mass. F_x , F_y ,

F_z, and L, M, N are the applied forces and moments, which are accessible from the models of gravity, aerodynamics, and thrust, while d_x, d_y, d_z, and d_l, d_m, d_n are the applied forces and moments resulting from the unmodeled aerodynamics or from the unpredictable disturbance such as wind gust. The equations of motion relative to a fixed frame are shown as follows.

$$m_s\dot{U} = m_s\left(-WQ+VR\right)+F_x+d_x \tag{1}$$

$$m_s\dot{V} = m_s\left(-UR+WP\right)+F_y+d_y \tag{2}$$

$$m_s\dot{W} = m_s\left(-VP+UQ\right)+F_z+d_z \tag{3}$$

$$I_{xx}\dot{P} = I_{xz}\left(\dot{R}+PQ\right)+I_{xy}\left(\dot{Q}-PR\right)-I_{yz}\left(R^2-Q^2\right)+\left(I_{yy}-I_{zz}\right)QR+L+d_l \tag{4}$$

$$I_{yy}\dot{Q} = I_{xy}\left(\dot{P}+QR\right)+I_{zy}\left(\dot{R}-PQ\right)-I_{xz}\left(P^2-R^2\right)+\left(I_{zz}-I_{xx}\right)PR+M+d_m \tag{5}$$

$$I_{zz}\dot{R} = I_{yz}\left(\dot{Q}+PR\right)+I_{xz}\left(\dot{P}-QR\right)-I_{yx}\left(Q^2-P^2\right)+\left(I_{xx}-I_{yy}\right)PQ+N+d_n \tag{6}$$

Notice that we do not make the assumption of small deviation to the symbols. The mass moments of inertia matrix I_M and cross-product matrix $S(\omega)$ induced by $\omega = [p\ \ q\ \ r]^{\mathrm{T}}$ are defined as

$$I_M = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix},\quad \mathrm{S}(\omega) = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \tag{7}$$

The trim force u_{Σ_0} and moment u_{Ω_0} can be solved from

$$\sigma = \dot{\sigma} = \omega = \dot{\omega} = d_\sigma = d_\omega = 0\,. \tag{8}$$

Hence, the nonlinear equations of motion with mass and moment inertia uncertainty with respect to the equilibrium point as

$$\dot{\sigma} = -S(\Omega)\sigma + S(\Sigma_0)\omega + \left(m_s+\delta_m\right)^{-1}u_\sigma + \left(m_s+\delta_m\right)^{-1}d_\sigma \tag{9}$$

$$\begin{aligned}\dot{\omega} &= \left(I_M+\delta_I\right)^{-1}S\left(\left(I_M+\delta_I\right)\Omega_0\right)\left(I_M+\delta_I\right)\omega \\ &\quad -\left(I_M+\delta_I\right)^{-1}S(\Omega)\left(I_M+\delta_I\right)\omega + \left(I_M+\delta_I\right)^{-1}u_\omega + \left(I_M+\delta_I\right)^{-1}d_\omega\end{aligned} \tag{10}$$

where δ_m is the varying mass around m_s and δ_I is the varying mass moments of inertia matrix around I_M. State with subscript zero denotes the value at equilibrium point (trim condition), and lower-case symbols denote the deviation from the equilibrium point. To make the computations of the H_∞ control force u_σ and moment u_ω independent of the physical units being used, scaling factors are introduced to normalize (nondimensionalize) the equations of motion. The normalization process employs m_s, $\sqrt{\|I_M\|/m_s}$, and $\sqrt{\|I_M\|/m_s}/U_0$ as the reference mass, reference length r_g, and reference time, respectively. We introduce the following dimensionless variables to normalize the equations of motion:

$$\begin{aligned} &t = (r_g/U_0)\bar{t},\ I_M = \|I_M\|\,\bar{I}_M,\ \sigma = U_0\bar{\sigma},\ \Sigma_0 = U_0\bar{\Sigma}_0 \\ &\omega = \left(U_0/r_g\right)\bar{\omega},\ \Omega_0 = \left(U_0/r_g\right)\bar{\Omega}_0,\ u_\sigma = \left(m_sU_0^2/r_g\right)u_{\bar{\sigma}}, \\ &u_\omega = \left(m_sU_0^2\right)u_{\bar{\omega}},\ \delta_I = \|I_M\|\,\bar{\delta}_I,\ \delta_m = m_s\bar{\delta}_m,\ \rho_{\bar{\sigma}} = \left(m_sU_0^2\right)\rho_\sigma, \\ &\rho_{\bar{\omega}} = \left(m_sU_0^2\right)\rho_\omega,\ d_\sigma = \left(m_sU_0^2/r_g\right)d_{\bar{\sigma}},\ d_\omega = \left(m_sU_0^2\right)d_{\bar{\omega}} \end{aligned}$$

The standard state-space form can be obtained by normalizing (9) and (10) as follows

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u^c \tag{11}$$

where

$$f(x) = \begin{bmatrix} -S(\bar{\Omega}) & S(\bar{\Sigma}_0) \\ 0 & \left(\bar{I}_M+\bar{\delta}_I\right)^{-1}S\left(\left(\bar{I}_M+\bar{\delta}_I\right)\bar{\Omega}_0\right)-\left(\bar{I}_M+\bar{\delta}_I\right)^{-1}S\left(\bar{\Omega}\right)\left(\bar{I}_M+\bar{\delta}_I\right) \end{bmatrix}\begin{bmatrix}\bar{\sigma} \\ \bar{\omega}\end{bmatrix},$$

$$g_1(x) = g_2(x) = \begin{bmatrix} \left(1+\bar{\delta}_m\right)^{-1}I_3 & 0 \\ 0 & \left(\bar{I}_M+\bar{\delta}_I\right)^{-1} \end{bmatrix}.$$

Control command weighting is designed such that nonlinear H_∞ control command can be followed smoothly by flight vehicle's actuator system. Therefore, we specify the output signal z to be

$$z = \begin{bmatrix} h_1(x) \\ W_E u^c \end{bmatrix}, \tag{12}$$

with $W_E = \mathrm{diag}(W_{\bar{\sigma}}\ \ W_{\bar{\omega}})$. The weightings $W_{\bar{\sigma}}$ and $W_{\bar{\omega}}$ are weighting coefficients concerning the trade-off between the tracking performance and the control effort assumed to be $\mathrm{diag}(\mathrm{w}_x\ \ \mathrm{w}_y\ \ \mathrm{w}_z)$ and $\mathrm{diag}(\mathrm{w}_l\ \ \mathrm{w}_m\ \ \mathrm{w}_n)$, respectively.. The ultimate flight control purpose here is to track the velocity command $\bar{\Sigma}_0$ and the body rate command $\bar{\Omega}_0$, and to make the tracking errors $\bar{\sigma}$ and $\bar{\omega}$ as small as possible. To reflect these requirements, we choose the measurement of tracking performance h_1 is

$$h_1(\sigma,\omega) = \left(\frac{1}{2}\rho_{\bar{\sigma}}\bar{\sigma}^T\bar{\sigma} + \frac{1}{2}\rho_{\bar{\omega}}\bar{\omega}^T\bar{I}_M\bar{\omega}\right)^{\frac{1}{2}}. \tag{13}$$

$\rho_{\bar{\sigma}}$ and $\rho_{\bar{\omega}}$ are weighting coefficients to the relative tracking performance between $\bar{\sigma}$ and $\bar{\omega}$. Additionally, the attenuation effort required for arbitrary exogenous disturbance $\bar{d} = [d_{\bar{\sigma}}^T\ \ d_{\bar{\omega}}^T]^T \in L_2$ will be lower than a specified value γ_1. Therefore, the problem of the nonlinear H_∞ flight control design now can be stated as: find the control u^c such that the L_2 gain of the system in (9), (10), and (12) is lower than γ_1, i.e.,

$$\frac{\int_0^\infty z^T z\, dt}{\int_0^\infty \bar{d}^T\bar{d}\, dt} = \frac{\int_0^\infty \left(h_1^2 + \bar{u}^{cT}W_E^2\bar{u}^c\right) dt}{\int_0^\infty \left(\bar{d}_\sigma^T\bar{d}_\sigma + \bar{d}_\omega^T\bar{d}_\omega\right) dt} < \gamma_1^2,\quad \forall \bar{d} \in L_2\,. \tag{14}$$

It can be shown that condition (14) is achieved if there exists a scalar C^1 function $E: R^n \to R^+$ with $E(0) = 0$, satisfying the following HJPDI

$$\left(\frac{\partial E}{\partial x}\right)^T f + \frac{1}{2}\left(\frac{\partial E}{\partial x}\right)^T\left(\frac{1}{\gamma_1^2}g_1g_1{}^T - g_2W_E^{-2}g_2{}^T\right)\left(\frac{\partial E}{\partial x}\right) + \frac{1}{2}h_1^Th_1 < 0 \tag{15}$$

This is a nonlinear first-order second-degree nonlinear partial differential inequality in the unknown function $E(\bar{\sigma},\bar{\omega}) = E(\bar{u},\bar{v},\bar{w},\bar{p},\bar{q},\bar{r})$. If such an E function exists, the desired nonlinear H_∞ controller u^c can be found from

$$u^c(x) = -W_E^{-2}g_2^T(x)\left(\frac{\partial E}{\partial x}\right) \tag{16}$$

The details of proof can be found, for example, from (Isidori *et al.*, 1992 and Van der Schaft *et al.*, 1992). Motivated from the linear result, we search for a possible quadratic solution for the nonlinear control problem in a linear form:

$$E(\bar{\sigma},\bar{\omega}) = \frac{1}{2}\begin{bmatrix}\bar{\sigma}^T & \bar{\omega}^T\end{bmatrix}\begin{bmatrix} K_{\bar{\sigma}}\left(1+\bar{\delta}_m\right)I_3 & 0 \\ 0 & K_{\bar{\omega}}\left(\bar{I}_M + \bar{\delta}_I\right)\end{bmatrix}\begin{bmatrix}\bar{\sigma} \\ \bar{\omega}\end{bmatrix}, \quad (17)$$

where $K_{\bar{\sigma}}$ and $K_{\bar{\omega}}$ are scalar constants to be determined. Conducting the partial differentiations with respect to $\bar{\sigma}$ and $\bar{\omega}$ with the following relations are valid.

$$\omega^T S(\omega)=0,\ \omega^T S(I_M\Omega_0)\omega=0, \text{ and } \sigma^T S(\Omega_0+\omega)\sigma=0 \quad (18)$$

Substituting (11), (13), (17) and (18), into (15), we get a quadratic form of HJPDI for flight control as

$$x^T\begin{bmatrix} \frac{1}{4}\rho_{\bar{\sigma}}I_3 + \left(\frac{1}{2\gamma_1^2}I_3 - \frac{1}{2}W_{\bar{\sigma}}^{-2}\right)K_{\bar{\sigma}}^2 & \frac{1}{2}K_{\bar{\sigma}}\left(1+\bar{\delta}_m\right)S\left(\bar{\Sigma}_0\right) \\ & -\frac{1}{2}K_{\bar{\omega}}S\left(\bar{\Omega}_0\right)\left(\bar{I}_M+\bar{\delta}_I\right) \\ \frac{1}{2}K_{\bar{\sigma}}\left(1+\bar{\delta}_m\right)S^T\left(\bar{\Sigma}_0\right) & -\frac{1}{2}K_{\bar{\omega}}\left(\bar{I}_M+\bar{\delta}_I\right)S^T\left(\bar{\Omega}_0\right) \\ & +\left(\frac{1}{2\gamma_1^2}I_3 - \frac{1}{2}W_{\bar{\omega}}^{-2}\right)K_{\bar{\omega}}^2 I_3 + \frac{1}{4}\rho_{\bar{\omega}}\bar{I}_M \end{bmatrix}x < 0 \quad (19)$$

An explicit (but only sufficient) condition to meet the above inequality is found as

$$K_{\bar{\sigma}} > \sqrt{\frac{\rho_{\bar{\sigma}}}{2\left(1/\omega_{\sigma}^2 - 1/\gamma_1^2\right)}}$$,, k>1 and $w_{\bar{\sigma}} < \gamma_1$ (20)

$$-\frac{K_{\bar{\omega}}}{2}\left(S\left(\bar{\Omega}_0\right)\left(\bar{I}_M+\bar{\delta}_I\right) + \left(\bar{I}_M+\bar{\delta}_I\right)S^T\left(\bar{\Omega}_0\right)\right) + \frac{1}{2}\left(\frac{1}{\gamma_1^2}I_3 - w_{\bar{\omega}}^{-2}\right)K_{\bar{\omega}}^2 I_3$$
$$+\frac{K_{\bar{\sigma}}^2\left(1+\bar{\delta}_m\right)^2 S^T\left(\bar{\Sigma}_0\right)S\left(\bar{\Sigma}_0\right)}{\rho_{\bar{\sigma}}I_3 + 2\left(\frac{1}{\gamma_1^2}I_3 - w_{\bar{\sigma}}^{-2}\right)K_{\bar{\sigma}}^2} + \frac{1}{4}\rho_{\bar{\omega}}\bar{I}_M < 0 \quad (21)$$

where $w_{\bar{\sigma}} = \max\left(w_x, w_y, w_z\right)$ and $w_{\bar{\omega}} = \max\left(w_l, w_m, w_n\right)$. It is noted that uncertainties exist in the above solutions. To remedy this defect, the following procedures are performed. Let the ranges of varying mass and moment of inertia are

$$-\Delta_{m-} \le \delta_m / m_s \le \Delta_{m+} \quad (22)$$

$$-\Delta_{I-} \le \delta_I I_M^{-1} \le \Delta_{I+} \quad (23)$$

If we define $D_+ = 1 + \Delta_{m+}$, $D_- = 1 - \Delta_{m-}$ and $M_+ = I + \Delta_{I+}$, we have $D_- \le 1 + \delta_m / m_s \le D_+$ and $I + \delta_I I_M^{-1} \le M_+$.

If the following inequality is satisfied, inequality (21) will be satisfied.

$$K_{\bar{\omega}}\left\|\bar{\Omega}_0\right\|\left\|\bar{I}_M\right\|\left\|M_+\right\| + \frac{1}{2}\left(\frac{1}{\gamma_1^2}I_3 - w_{\bar{\omega}}^{-2}\right)K_{\bar{\omega}}^2 + \frac{1}{4}\rho_{\bar{\omega}}\left\|\bar{I}_M\right\|$$
$$-\frac{K_{\bar{\sigma}}^2 D_+^2 \bar{\Sigma}_0^T \bar{\Sigma}_0}{\rho_{\bar{\sigma}}I_3 + 2\left(\frac{1}{\gamma_1^2}I_3 - w_{\bar{\sigma}}^{-2}\right)K_{\bar{\sigma}}^2} < 0$$

Then we have

$$K_{\bar{\omega}} > \left(\frac{1}{w_{\bar{\omega}}^2} - \frac{1}{\gamma_1^2}\right)^{-1}\left(\left\|\bar{\Omega}_0\right\|\left\|\bar{I}_M\right\|\left\|M_+\right\| + \sqrt{\left(\left\|\bar{\Omega}_0\right\|\left\|\bar{I}_M\right\|\left\|M_+\right\|\right)^2 + \alpha\left(K_{\bar{\sigma}}\right)}\right) \quad (24)$$

where

$$\alpha\left(K_{\bar{\sigma}}\right) = \frac{1}{2}\left(\frac{1}{w_{\bar{\omega}}^2} - \frac{1}{\gamma_1^2}\right)\left(\rho_{\bar{\omega}}\left\|\bar{I}_M\right\| + \frac{4K_{\bar{\sigma}}^2 D_+ \bar{\Sigma}_0^T \bar{\Sigma}_0}{2\left(\frac{1}{w_{\bar{\sigma}}^2} - \frac{1}{\gamma_1^2}\right)K_{\bar{\sigma}}^2 - \rho_{\bar{\sigma}}}\right)$$

As expected, the allowable ranges of $K_{\bar{\sigma}}$ and $K_{\bar{\omega}}$ are dependent on the trim conditions $\bar{\Sigma}_0$ and $\bar{\Omega}_0$ which implies that the derived control law possesses implicitly the gain-scheduling effect, with controller gain changing with flight conditions. After having obtained the solutions of $K_{\bar{\sigma}}$ and $K_{\bar{\omega}}$, we can compute the desired control commands (forces and moments) from (16) and (17) as following:

$$u_{\sigma}^c = \begin{bmatrix} f_x^c \\ f_y^c \\ f_z^c \end{bmatrix} = -W_{\bar{\sigma}}^{-2}K_{\bar{\sigma}}\begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad u_{\omega}^c = \begin{bmatrix} l^c \\ m^c \\ n^c \end{bmatrix} = -W_{\bar{\omega}}^{-2}K_{\bar{\omega}}\begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad (25)$$

Note that $K_{\bar{\sigma}}$ and $K_{\bar{\omega}}$ are scales, so they will be in dimension by using $K_{\sigma} = m_s U_0 K_{\bar{\sigma}} / r_g$ and $K_{\omega} = m_s U_0 r_g K_{\bar{\omega}}$. However, it is possible that choosing proper values of $K_{\bar{\sigma}}$ and $K_{\bar{\omega}}$ in the solution sets (20) and (24) will obtain an acceptable performance h_1 without consuming a significant control effort u^c. It is worth noting that the linearity assumed here is only limited to the way generating aero data, i.e., linear interpolation between the given discrete aero data points. This linear interpolation of aero data is unavoidable, since real aero data given by wind tunnel tests is discrete. There may exist some fitting errors by using piecewise linear interpolation, but the global aerodynamic model is still nonlinear, and the nonlinear equations of motion in Eqs.(9) and (10) cover all the flight envelope of the vehicle. On the contrary, in the conventional linearized model, aero data are assumed to be fixed at some trim point, and no aero data interpolation is required.

3. CONTROL SURFACES INVERSE ALGORITHM AND ITS APPLICATION RANGE TO STABILITY

The desired force and moment command $u^c = [u_{\sigma}^{cT}\ u_{\omega}^{cT}]^T$ to attenuate the disturbance for the closed-loop six degrees-of-freedom motion system have been obtained from (25). As mentioned earlier, since general flight vehicle has only five or less control surfaces, it is not possible to exactly implement the nonlinear H_∞ command u^c which has six independent components. The question of stability using control surfaces remains to be verified.

Let u^b be the force and moment generated by flight vehicle's control surfaces, and $u^b = \rho_u u^c$, where $\rho_u \le 1$ is called loss rate. Let the control input is u^b now, it is easy to derive that the nonlinear H_∞ flight control system becomes:

$$G_1:\begin{array}{l}\dot{x}=f(x)+g_1(x)d+g_2(x)\rho_u u^c\\ z=\begin{bmatrix}h_1(x)\\ \rho_u W_E u^c\end{bmatrix},\ \rho_u<1\end{array} \tag{26}$$

However, what is the range of ρ_u that the nonlinear H_∞ flight control system using control surfaces still guarantees robustness and stability? Let the new L_2 gain for the system G_1 is γ. Substituting u^c defined in (16) and system defined in (26). It can be shown that condition (14) is achieved if there exists a scalar C^1 function $E:R^n \to R^+$ with $E(0)=0$, satisfying the following HJPDI

$$\begin{aligned}&\left(\frac{\partial E}{\partial x}\right)^T\left(\frac{1}{2\gamma^2}g_1g_1^T+\left(\frac{1}{2}\rho_u^2-\rho_u\right)g_2W_E^{-2}g_2^T\right)\left(\frac{\partial E}{\partial x}\right)\\&+\left(\frac{\partial E}{\partial \bar\sigma}\right)^T\left(-S(\bar\Omega)\bar\sigma+S(\bar\Sigma_0)\bar\varpi\right)+\frac{\rho_{\bar\varpi}}{4}\bar\varpi^T\bar\varpi+\frac{\rho_{\bar\sigma}}{4}\bar\sigma^T\bar\sigma\\&+\left(\frac{\partial E}{\partial \bar\varpi}\right)^T\begin{pmatrix}\left(\bar I_M+\bar\delta_I\right)^{-1}S\left(\left(\bar I_M+\bar\delta_I\right)\bar\Omega_0\right)\\-\left(\bar I_M+\bar\delta_I\right)^{-1}S(\bar\Omega)\left(\bar I_M+\bar\delta_I\right)\end{pmatrix}\bar\varpi<0\end{aligned} \tag{27}$$

To ensure the negativeness of HJPDI (27) for arbitrary $\bar\sigma,\bar\varpi\neq 0$, we get the quadratic form of (27) with the matrix elements defined as

$$x^T\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}x<0$$

$$M_{11}=\frac{1}{4}\rho_{\bar\sigma}I_3+\left(\frac{1}{2\gamma^2}I_3+\left(\frac{1}{2}\rho_u^2-\rho_u\right)W_{\bar\sigma}^{-2}\right)K_{\bar\sigma}^2, \tag{28}$$

$$M_{12}=\frac{1}{2}K_{\bar\sigma}\left(1+\bar\delta_m\right)S\left(\bar\Sigma_0\right),$$

$$\begin{aligned}M_{22}=&-\frac{1}{2}K_{\bar\varpi}S\left(\bar\Omega_0\right)\left(\bar I_M+\bar\delta_I\right)-\frac{1}{2}K_{\bar\varpi}\left(\bar I_M+\bar\delta_I\right)S^T\left(\bar\Omega_0\right)\\&+\left(\frac{1}{2\gamma^2}I_3+\left(\frac{1}{2}\rho_u^2-\rho_u\right)W_{\bar\varpi}^{-2}\right)K_{\bar\varpi}^2I_3+\frac{1}{4}\rho_{\bar\varpi}\bar I_M.\end{aligned} \tag{29}$$

According to (20), we assume

$$K_{\bar\sigma}=k\sqrt{\frac{\rho_{\bar\sigma}}{2\left(1/w_{\bar\sigma}^2-1/\gamma_1^2\right)}},\ k>1 \text{ and } w_{\bar\sigma}<\gamma_1 \tag{30}$$

Substituting (30) into (28), $M_{11}<0$ is satisfied if

$$\gamma>\sqrt{\frac{1}{2\rho_u w_{\bar\sigma}^{-2}-\rho_u^2w_{\bar\sigma}^{-2}-\frac{1}{k^2w_{\bar\sigma}^2}+\frac{1}{k^2\gamma_1^2}}}$$

Since the denominator of the above equation is required to be positive, we can get the range of ρ_u to be

$$1-\sqrt{1-\frac{1}{k^2}+\frac{1}{k^2\gamma_1^2w_{\bar\sigma}^{-2}}}<\rho_u<1+\sqrt{1-\frac{1}{k^2}+\frac{1}{k^2\gamma_1^2w_{\bar\sigma}^{-2}}} \tag{31}$$

The left term can be verified that it is always positive under the condition $w_{\bar\sigma}<\gamma_1$.

Another condition $M_{22}-M_{12}^TM_{11}^{-1}M_{12}<0$ can be proved to be valid, if the following inequality exists:

$$\begin{aligned}&\frac{1}{2}\left(\rho_u^2w_{\bar\varpi}^{-2}-2\rho_uw_{\bar\varpi}^{-2}+\frac{1}{\gamma^2}\right)K_{\bar\varpi}^2+K_{\bar\varpi}\left\|\bar\Omega_0\right\|\left\|\bar I_M\right\|\left\|M_+\right\|+\frac{1}{4}\rho_{\bar\varpi}\left\|\bar I_M\right\|\\&-\frac{K_{\bar\sigma}^2D_+{}^2\bar\Sigma_0{}^T\bar\Sigma_0}{\rho_{\bar\sigma}+2\left(\frac{1}{\gamma^2}-2\rho_uw_{\bar\sigma}^{-2}+\rho_u^2w_{\bar\sigma}^{-2}\right)K_{\bar\sigma}^2}<0\end{aligned} \tag{32}$$

Similarly, having $K_{\bar\varpi}$ shown in (24), we assume

$$K_{\bar\varpi}=\left(\frac{1}{w_{\bar\varpi}^2}-\frac{1}{\gamma_1^2}\right)^{-1}k\left(\left\|\bar\Omega_0\right\|\left\|\bar I_M\right\|\left\|M_+\right\|+\sqrt{\left(\left\|\bar\Omega_0\right\|\left\|\bar I_M\right\|\left\|M_+\right\|\right)^2+\alpha\left(K_{\bar\sigma}\right)}\right) \tag{33}$$

k>1, $w_{\bar\varpi}<\gamma_1$, and

$$\alpha\left(K_{\bar\sigma}\right)=\frac{1}{2}\left(\frac{1}{w_{\bar\varpi}^2}-\frac{1}{\gamma_1^2}\right)\left(\rho_{\bar\varpi}\left\|\bar I_M\right\|+\frac{4K_{\bar\sigma}^2D_+\bar\Sigma_0^T\bar\Sigma_0}{2\left(\frac{1}{w_{\bar\sigma}^2}-\frac{1}{\gamma_1^2}\right)K_{\bar\sigma}^2-\rho_{\bar\sigma}}\right)$$

Substituting $K_{\bar\varpi}$ defined in (33) into (32), we have

$$\begin{aligned}&\left(-\rho_u^2w_{\bar\varpi}^{-2}+2\rho_uw_{\bar\varpi}^{-2}-\frac{1}{\gamma^2}\right)k^2\frac{\left(\left\|\bar\Omega_0\right\|\left\|\bar I_M\right\|\left\|M_+\right\|\right)^2}{\left(\frac{1}{w_{\bar\varpi}^2}-\frac{1}{\gamma_1^2}\right)^2}\\&+\left(-\rho_u^2w_{\bar\varpi}^{-2}+2\rho_uw_{\bar\varpi}^{-2}-\frac{1}{\gamma^2}\right)k^2\frac{\left\|\bar\Omega_0\right\|\left\|\bar I_M\right\|\left\|M_+\right\|\sqrt{\left(\left\|\bar\Omega_0\right\|\left\|\bar I_M\right\|\left\|M_+\right\|\right)^2+\alpha\left(K_{\bar\sigma}\right)}}{\left(\frac{1}{w_{\bar\varpi}^2}-\frac{1}{\gamma_1^2}\right)^2}\\&+\left(-\rho_u^2w_{\bar\varpi}^{-2}+2\rho_uw_{\bar\varpi}^{-2}-\frac{1}{\gamma^2}\right)\frac{k^2}{4}\frac{\rho_{\bar\varpi}\left\|\bar I_M\right\|}{\left(\frac{1}{w_{\bar\varpi}^2}-\frac{1}{\gamma_1^2}\right)}-\frac{1}{4}\rho_{\bar\varpi}\left\|\bar I_M\right\|\\&+\left(-\rho_u^2w_{\bar\varpi}^{-2}+2\rho_uw_{\bar\varpi}^{-2}-\frac{1}{\gamma^2}\right)k^2\frac{\dfrac{K_{\bar\sigma}^2D_+\bar\Sigma_0^T\bar\Sigma_0}{2\left(\frac{1}{w_{\bar\sigma}^2}-\frac{1}{\gamma_1^2}\right)K_{\bar\sigma}^2-\rho_{\bar\sigma}}}{\left(\frac{1}{w_{\bar\varpi}^2}-\frac{1}{\gamma_1^2}\right)}\\&-k\frac{\left\|\bar\Omega_0\right\|^2\left\|\bar I_M\right\|^2\left\|M_+\right\|^2}{\left(\frac{1}{w_{\bar\varpi}^2}-\frac{1}{\gamma_1^2}\right)}+\frac{K_{\bar\sigma}^2D_+\bar\Sigma_0{}^T\bar\Sigma_0}{\rho_{\bar\sigma}+2\left(\frac{1}{\gamma^2}-2\rho_uw_{\bar\sigma}^{-2}+\rho_u^2w_{\bar\sigma}^{-2}\right)K_{\bar\sigma}^2}\\&-k\frac{\left\|\bar\Omega_0\right\|\left\|\bar I_M\right\|\left\|M_+\right\|\sqrt{\left(\left\|\bar\Omega_0\right\|\left\|\bar I_M\right\|\left\|M_+\right\|\right)^2+\alpha\left(K_{\bar\sigma}\right)}}{\left(\frac{1}{w_{\bar\varpi}^2}-\frac{1}{\gamma_1^2}\right)}>0\end{aligned} \tag{34}$$

The following variables are necessary:

$$\alpha_1=2\left(\frac{1}{\gamma_1^2}-\frac{1}{w_{\bar\sigma}^2}\right)K_{\bar\sigma}^2+\rho_{\bar\sigma}$$

$$\alpha_2=2\left(\frac{1}{\gamma^2}-2\rho_uw_{\bar\sigma}^{-2}+\rho_u^2w_{\bar\sigma}^{-2}\right)K_{\bar\sigma}^2+\rho_{\bar\sigma}$$

The range of ρ_u make (34) to be satisfied can be obtained as

(1) $\alpha 1/\alpha 2 \geqq k$

$$1-\sqrt{1-\frac{\alpha_1}{\alpha_2}\frac{1}{k^2}+\frac{\alpha_1}{\alpha_2}\frac{w_{\bar\varpi}^2}{k^2\gamma_1^2}}<\rho_u<1+\sqrt{1-\frac{\alpha_1}{\alpha_2}\frac{1}{k^2}+\frac{\alpha_1}{\alpha_2}\frac{w_{\bar\varpi}^2}{k^2\gamma_1^2}} \tag{35}$$

We have new L_2 gain as follows:

$$\gamma > w_{\bar{\omega}} / \sqrt{-\rho_u^2 + 2\rho_u - \frac{\alpha 1}{\alpha 2}\frac{1}{k^2} + \frac{\alpha 1}{\alpha 2}\frac{w_{\bar{\omega}}^2}{k^2\gamma_1^2}} \tag{36}$$

(2) α1/α2<k

$$1-\sqrt{1-\frac{1}{k}+\frac{1}{w_{\bar{\omega}}^{-2}k\gamma_1^2}} < \rho_u < 1+\sqrt{1-\frac{1}{k}+\frac{1}{w_{\bar{\omega}}^{-2}k\gamma_1^2}} \tag{37}$$

New L_2 gain is

$$\gamma > w_{\bar{\omega}} / \sqrt{-\rho_u^2 + 2\rho_u - \frac{1}{k} + \frac{1}{w_{\bar{\omega}}^{-2}k\gamma_1^2}} \tag{38}$$

The range of loss rate ρ_u guarantee the robustness and stability for system (26) with nonlinear H_∞ control command implemented by control surfaces. It is noticed that the new L_2 gain γ is increased when $\rho_u < 1$ .This fact implies that the disturbance attenuation effect is less than the theoretically predicted value when aerodynamics are taken into account. Although it is unavoidable for any controller design that the performance is degraded caused by finite actuator ability, proper selections of γ_1 and tuning K_s in (35) still retains the robustness properties of the nonlinear H_∞ controller .

It is well known that control surfaces are limited by their time delay and saturation. We have to design a prefilter, K_s, to implement u^c and avoid control surface saturation. The mechanism of determining the best deflection angles of the control surfaces such that u^b can be as close as possible to $K_s u^c$ is called Control Surface Inverse Algorithm (CSIA). The algorithm is based on the minimization of the following command tracking error

$$J_{\text{error}} = \left(K_s u^c - u^b\right)^T \left(K_s u^c - u^b\right), \tag{39}$$

Let u^b be expressed as an abbreviated form:

$$u^b = u^* + A\delta . \tag{40}$$

The optimal control surface deflection δ^{opt} minimizing the command tracking error J_{error} in (39) can be found by Moore-Penrose inverse formula as

$$\delta^{\text{opt}} = (A^T A)^{-1} A^T (K_s u^c - u^*) . \tag{41}$$

If A is invertible, which implies that the flight vehicle has six control surfaces, the tracking error J_{error} can be made exactly equal to zero by using the least-square solution δ^{opt} from (41). If the number of control surface is lower than six, A is not invertible and the nonlinear H_∞ control command can not be tracked exactly by the vehicle's aerodynamic control surfaces. In this case, δ^{opt} represents the best control surface deflection minimizing J_{error}. The algorithm can be referred to (Kung *et al*., 2002). However, the range of prefilter K_s to make CSIA still guarantees robustness and stability will be derived.

Let the control input is u^b now, the nonlinear H_∞ flight control system (26) becomes:

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)d + g_2(x)u^b \\ &= f(x) + g_1(x)d + g_2(x)\left(u^* + A\delta^{\text{opt}}\right) \end{aligned} \tag{42}$$

Substituting δ^{opt} in (41) into (42), we have

$$\begin{aligned} \dot{x} = \ & f(x) + g_1(x)\left(d + \left(I - A(A^TA)^{-1}A^T\right)u^*\right) \\ & + g_2(x)A(A^TA)^{-1}A^T K_s u^c \end{aligned} \tag{43}$$

It is noticed that the property $g_1 = g_2$ from (11) is applied. Let $d_1 = d + \left(I - A(A^TA)^{-1}A^T\right)u^*$, the above equation (43) can be re-written as

$$\dot{x} = f(x) + g_1(x)d_1 + g_2(x)A(A^TA)^{-1}A^T K_s u^c \tag{44}$$

Let prefilter K_s be designed as

$$K_s = k_s W_s \tag{45}$$

where k_s is a constant and W_s is a weighting matrix. It is clear that $A(A^TA)^{-1}A^T W_s$ is equal or less than its maximum eigenvalue and is equal or greater than its minimum eigenvalue. Let the range of k_s be design as follows:

(1) α1/α2≧k

$$\begin{aligned} &\underline{\lambda}\left(A(A^TA)^{-1}A^TW_s\right)^{-1-1}\left(1-\sqrt{1-\frac{\alpha_1}{\alpha_2}\frac{1}{k^2}+\frac{\alpha_1}{\alpha_2}\frac{w_{\bar{\omega}}^2}{k^2\gamma_1^2}}\right) < k_s \\ &< \bar{\lambda}\left(A(A^TA)^{-1}A^TW_s\right)^{-1}\left(1+\sqrt{1-\frac{\alpha_1}{\alpha_2}\frac{1}{k^2}+\frac{\alpha_1}{\alpha_2}\frac{w_{\bar{\omega}}^2}{k^2\gamma_1^2}}\right) \end{aligned} \tag{46}$$

(1) α1/α2<k

$$\begin{aligned} &\underline{\lambda}\left(A(A^TA)^{-1}A^TW_s\right)^{-1}\left(1-\sqrt{1-\frac{1}{k}+\frac{1}{w_{\bar{\omega}}^{-2}k\gamma_1^2}}\right) < k_s \\ &< \bar{\lambda}\left(A(A^TA)^{-1}A^TW_s\right)^{-1}\left(1+\sqrt{1-\frac{1}{k}+\frac{1}{w_{\bar{\omega}}^{-2}k\gamma_1^2}}\right) \end{aligned} \tag{47}$$

Since the control $\rho_u u^c$ is valid for all disturbances $d \in L_2$ with ρ_u in the range of (35) or (37), it is valid for d_1, too. Because the range of k_s is set in (46) or (47), the G_2 system from (44),

$$G_2 : \begin{aligned} &\dot{x} = f(x) + g_1(x)d_1 + g_2(x)A(A^TA)^{-1}A^T K_s u^c, \\ &z = \begin{bmatrix} h_1(x) \\ A(A^TA)^{-1}A^T K_s W_E u^c \end{bmatrix}, \ \rho_u < 1 \end{aligned} \tag{48}$$

will satisfy the following L_2 gain requirement.

$$\frac{\int_0^\infty z^T z \, dt}{\int_0^\infty d_1^T d_1 \, dt} < \gamma^2, \quad \forall d_1 \in L_2 \tag{49}$$

To complement the time delays of control surfaces, it is usually required that the range of k_s is smaller than (46) or (47).

4. CONTROLLER VALIDATION IN LYNX HELICOPTER

In this section the nonlinear H_∞ flight controller and the control surface inverse algorithm developed in the previous sections will be validated by a nonlinear simulator for Lynx helicopter within the Matlab environment. To find the control force and moment from (25), we need the mass and moments of inertia for Lynx helicopter: m_s=*4313.7 kg*, I_{xx}=*2767.1 kg·****m****^{2}*, I_{yy}=*13904.5 kg·****m****^{2}*, I_{zz}=*12208.8 kg·****m****^{2}*, and I_{xz}=*2034.8 kg·****m****^{2}*. The helicopter is hovering and the trim condition is set to $\Sigma_0=[U_0\ V_0\ W_0]^T=[0\ 0\ 0]^T$(ft/sec), and $\Omega_0=[P_0\ Q_0\ R_0]^T=[0\ 0\ 0]^T$ (rad/sec) at altitude=*100 m*. The upper bound of the L_2-gain is selected to γ=2, the weighting coefficients ρ_σ and ρ_ω are all set to 1. Control surface movement is governed by actuator ability. With reference to Takahashi (1994), we take UH-60 to be our actuator model. The main and tail rotor collective pitch are limited between $[6.25^0\ \ 23.25^0]$. The allowable intervals for longitudinal and lateral cyclic pitch are $[-8.7^0\ \ 14^0]$ and $[-7^0\ \ 8^0]$, respectively.

To illustrate that the convergence of the nonlinear H_∞ helicopter control system with actuator constraints is not merely local, we perturb the six DOF nonlinear motion to an initial condition far from trim condition, and then verify its convergence. Initial Perturbed Condition: $\Sigma_0=[U_0\ V_0\ W_0]^T=[20\ 20\ 20]^T$(ft/sec), and $\Omega_0=[P_0\ Q_0\ R_0]^T=[0.5\ 0.5\ 0.5]^T$ (rad/sec). The upper bound of the L_2 gain in (14) is selected to γ_1=2. To think about δ_m/m_s= 1.2, $\delta_I=I_M\times\delta_m/m_s$=1.2, D_+= 1+Δ_{m+}=2.2 , and M_+= I_3+ Δ_{I+}=2.2I_3 , we have C_σ=1.3 , C_ω=1.3 , and maximum eigenvalue=-0.12. The initial condition is very far from the equilibrium states. As in Fig.1, it serves to explain this stability property, where these states converge with steady state error. The steady-state error mainly comes from vertical velocity. It reveals that the control ability in vertical velocity is not very well. The drawback can be overcome by adjusting the prefilter parameters and weighting matrix. On the other hand, the responses still show good convergence for large initial perturbation which implies that theoretically guaranteed properties of the nonlinear H_∞ controller has been somewhat sacrificed during the control law implementation process. Control variable deflection θ_0, θ_{1s}, θ_{1c}, θ_{0T} histories shown in Fig. 2. It can be found that saturations happened on θ_0, θ_{1c}, and θ_{0T}. It means that the control ability will be lost over the uncertainties δ_m/m_s= 1.2 and $\delta_I=I_M\times\ \delta_m/m_s$=1.2.

5. CONCLUSIONS

In this paper, the feasibility of actual implementing nonlinear H_∞ flight control command for general flight vehicle with six degree-of- freedom motions with mass and moment inertia uncertainties is presented. Control surface inverse algorithm to convert the nonlinear H_∞ control law to actual movements of control surface is developed and the application range is derived and proved. The stability of nonlinear H_∞ control is confirmed in the Lynx helicopter simulation. The theoretical results are proved and this paper gives one of useful methods to deal with the actual implementation of nonlinear H_∞ command for general flight vehicles with six degree-of-freedom motions.

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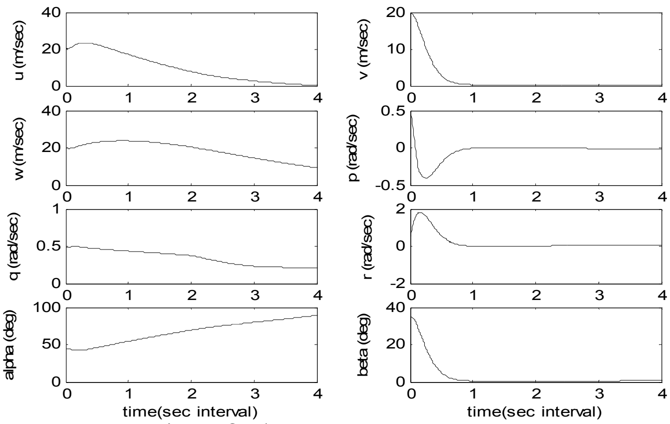

Fig. 1. δm/ms= 1.2 state responses

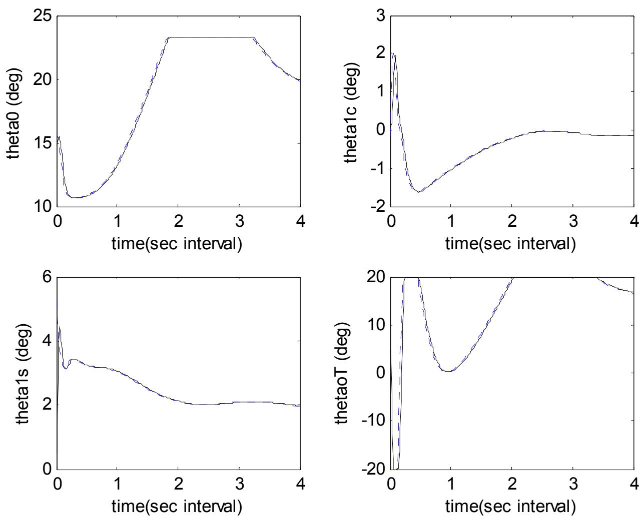

Fig. 2. δm/ms= 1.2 control surfaces responses