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Stability of A Class of Relay Feedback Systems Arising in Controlled Mechanical Systems with Ideal Coulomb Friction ⋆

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Abstract: Coulomb friction is inevitable in every mechanical system with contact motion. When the mechanical system with Coulomb friction is under feedback control, it can destabilize the system by generating limit cycles. Controlled mechanical systems with ideal Coulomb friction can be viewed as a particular class of relay feedback systems characterized by the zero DC gain property and the positivity of the first Markov parameter. This paper elaborates recent results on sufficient conditions to guarantee the global pointwise stability of such systems. The scope of analysis has been kept broad so that the results apply to systems with multiple inertia elements and multiple Coulomb friction sources. To employ the recent advances in the absolute stability theory, the limiting arguments are adopted to approximate the relay elements to continuous functions. As a result, a new sufficient condition on the global pointwise stability of the systems with multiple Coulomb friction sources is derived by extending the existing result with a single Coulomb friction source when the stiction level is larger than the Coulomb friction level. Also, it has been shown that the describing function criterion is indeed an exact condition when the order of the closed-loop system is 3. Simulation results are presented with a flexible joint mechanism to illustrate the main points.

1. INTRODUCTION

In control of mechanical systems, the Coulomb friction is an important nonlinearity not only as the source of tracking error but also as the cause of instability. This paper focuses on the latter. More specifically, when the mechanical system with the Coulomb friction is under feedback control, the closed loop system may asymptotically converge to a stationary point or generate a nonlinear oscillation called a limit cycle. The Coulomb friction is a complicated phenomenon and its characteristics are not fully understood yet. However, as far as the stability is concerned, it can be reasonably represented by a sign function with an extra condition to model the sticking motion. The sticking motion is a sliding mode which occurs when the mass element temporarily ceases to move while other state variables are still dynamically changing.

The stability of mechanical systems with ideal Coulomb friction has been studied by several researchers (Armstrong & Amin, 1994; Olsson & Åström, 2001; Wouw & Leine, 2004). More recently, it has been shown that the controlled mechanical systems with multiple ideal Coulomb friction sources can be generalized to a class of relay feedback systems (Jeon & Tomizuka, 2008). Stability analysis of relay feedback systems has a long research history. Tsypkin (1984) presented a comprehensive report on the analysis of relay feedback systems using the frequency response technique. Relay feedback recently regained an attention due to Åström & Hägglund (1984). Subsequent studies and further applications of the relay feedback can be found in Johansson *et al.* (1999) and the references therein. Recently, a different methodology called the surface Lyapunov function has been suggested to analyze the global stability of the limit cycles due to relay feedback (Gonçalves *et al.*, 2001). These approaches are mainly concerned with the stability of the limit cycle rather than that of the equilibrium point (or the equilibrium set). Hence, the analysis typically starts by checking the existence of a (locally) stable limit cycle, which involves numerical methods to solve a transcendental equation.

In the relay feedback system drawn from the Coulomb friction, the point of interest is somewhat different from that of the conventional relay feedback studies. First of all, the relay element originating from the Coulomb friction is an undesirable part of the plant. Therefore, the main purpose is to design the controller to avoid the limit cycle generated by this inevitable relay element. Secondly, the magnitude of the relay (or equivalently, the Coulomb friction level) is relatively small compared to other interaction forces. This requires that, in most cases, the stability conditions need to hold globally. For any type of relay feedback systems, a necessary and sufficient condition to guarantee global asymptotic stability is yet to come. Among a few feasible approaches is the absolute stability theory. Most recent results can be found in Mancera & Safonov (2005) and Rantzer (2001).

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This paper presents new stability conditions for mechanical systems with Coulomb friction directly drawn from the recent advances in the absolute stability theory. The paper is organized as follows. Section 2 explains the background knowledge. The main analytical results are described in Section 3. Simulation results follow in Section 4 and, finally, the conclusions are summarized in Section 5.

2. PRELIMINARY RESULTS

2.1 Background

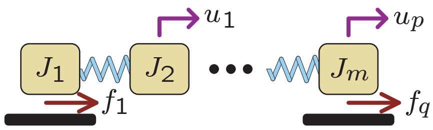


Fig. 1. A controlled mechanical system with multiple Coulomb friction forces

Figure 1 describes the schematic of the general mechanical system considered in this paper. J_i's ($i = 1, \ldots, m$) denote inertia elements which are interconnected to each other and u_j ($j = 1, \ldots, p$) is the control action applied to the corresponding inertia element. Independently from these control actions, we assume that there exist $q(\leq m)$ Coulomb friction forces denoted by f_k ($k = 1, \ldots q$). As shown in Fig. 1, we assume that each component of u and f can only be applied to the corresponding inertia element.

A single ideal Coulomb friction model acting on a mechanical system can be written as

$$f(v) = \begin{cases} -f_c\mathbf{sgn}(v) & \text{if } v \neq 0 \\ -u^t & \text{if } v = 0 \text{ and } |u^t| \leq f_s \\ -f_s\mathbf{sgn}(u^t) & \text{otherwise} \end{cases} \quad (1)$$

where v is the velocity at the contact serface, f_c is the magnitude of Coulomb friction and $f_s(\geq f_c)$ is the stiction level. Parameter r will be used to denote the ratio of the stiction force to the Coulomb friction force, i.e., $r = f_s/f_c \geq 1$, and will be called as the *detachment friction ratio*. The total force u^t includes all the interaction forces except for the Coulomb friction as well as the control input u applied to the corresponding inertia.

The state space realizations of the plant $G_P(s)$ and the controller $G_K(s)$ are denoted by

$$G_P(s) = \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & 0 \end{array}\right], \quad G_K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]. \quad (2)$$

$x_p(t) \in \mathbf{R}^{n_p \times 1}$ and $x_k(t) \in \mathbf{R}^{n_k \times 1}$ are state vectors for $G_P(s)$ and $G_K(s)$ respectively, where n_p and n_k denote the corresponding system orders. The order of the closed loop system is denoted by n, i.e., $n = n_p + n_k$. y is the measurement output vector of the plant with its order denoted by n_y and $v \in \mathbf{R}^q$ is the velocity vector associated with all the Coulomb friction sources. Then the system in Fig. 1 can be represented by a block diagram shown in Fig. 2 where $u = [u_1 \cdots, u_p]^T$ and $f = [f_1, \cdots, f_q]^T$. Accordingly, the input and the output matrices of the plant can be broken up into two parts as follows.

$$B_P = [B_u \; B_f], \quad C_P = \begin{bmatrix} C_y \\ C_v \end{bmatrix} \quad (3)$$

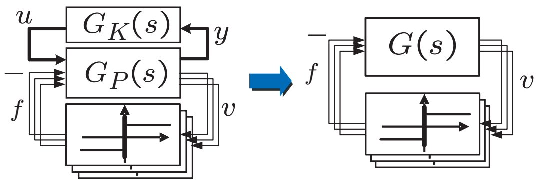


Fig. 2. Block diagram of the closed loop system

The closed-loop system from the friction input to the velocity output is denoted by the $q \times q$ transfer function matrix $G(s)$. The minimal realization of $G(s)$ will be represented by closed-loop system matrices denoted by (A, B, C). More specifically,

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_P + B_u D_K C_y & B_u C_K & B_f \\ B_K C_y & A_K & 0 \\ \hline C_v & 0 & 0 \end{array}\right]. \quad (4)$$

A is a closed-loop system matrix so it must be Hurwitz. For simplicity, we will assume that there is no pole-zero cancelation between G_P and G_K and also that A has distinct eigenvalues.

2.2 Representation as a Relay Feedback System

The following Proposition presented by Jeon & Tomizuka (2008) characterizes the relay feedback system considered in this paper by capturing the essential features of $G(s)$.

Proposition 1. Assume that the detachment friction ratio $r_i = 1$ for all $i = 1, \ldots, q$. Then the overall interconnected system in Fig. 2 can be written as the following ideal relay feedback system without hysteresis.

$$\dot{x} = Ax - B\mathbf{sgn}(Cx) \quad (5)$$

where $x^T = [x_p^T \; x_k^T]$, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times q}$, and $C \in \mathbf{R}^{q\times n}$. Furthermore (A, B, C) satisfies the following conditions.

$$c_i b_i > 0, \; c_i b_j = 0 \;\; \forall i, \, j \in \{1, \ldots, q\}, \; i \neq j \quad (6a)$$

$$CA^{-1}B = 0 \quad (6b)$$

where c_i and b_j denote the i^{th} row of C and the j^{th} column of B respectively.

Proof. The proof follows by choosing the state variables of the plant as $x_p = [\theta_1 \; \dot{\theta}_1 \; \theta_2 \; \dot{\theta}_2 \; \cdots \; \theta_m \; \dot{\theta}_m]^T$ where θ_i is the position of the i^{th} inertia element. Refer to Jeon & Tomizuka (2008) for more details. □

Proposition 1 leads to a number of important observations.

1. The system in (5) is a switched affine system with a set of q switching hyperplanes
$$\mathcal{E}_i := \{x | c_i x = 0\}, \quad i \in \{1, \ldots, q\} \quad (7)$$
which are orthogonal to each other.
2. The equilibrium set of (5) is a polytope in $\mathbf{R}^q$
$$\{x_e\} := \left\{ \sum_{i=1}^{q} \eta_i A^{-1} b_i \;\middle|\; |\eta_i| \leq 1 \right\} \quad (8)$$
and it lies on $\mathcal{E}_i$ for all i.
3. There exists a sliding mode in $\mathcal{E}_i$ represented as
$$S_i := \{x \,|\, x \in \mathcal{E}_i, \;\; |c_i A x| \leq c_i b_i\}, \;\; i \in \{1, \ldots, q\}. \quad (9)$$
The trajectory of $x(t)$ on these sliding modes is governed by the Filippov solution (Filippov, 1988)
$$\dot{x} = \mathcal{P}_\sigma A x - B_{\sigma^c}\mathbf{sgn}(C_{\sigma^c} x) \quad (10)$$

where $\mathcal{P}_\sigma$ is a projection matrix defined by

$$\mathcal{P}_\sigma := I - \sum_{i\in\sigma} \frac{b_i c_i}{c_i b_i} = I - B_\sigma (C_\sigma B_\sigma)^{-1} C_\sigma. \quad (11)$$

$\sigma(t) \subseteq \{1, ..., q\}$ denotes the index set at time t such that $c_i x(t) = 0$, $\forall i \in \sigma(t)$ and $\sigma^c(t) = \{1, ..., q\} - \sigma(t)$. As subscripts, they indicate the submatrices of B (or C) that are formed by collecting the columns (or the rows) indexed by the corresponding set. For example, if $\sigma(t) = \{1, 3\}$, then $B_\sigma = [b_1\ b_3]$. The corresponding matrix will vanish if the set (σ or σ^c) is empty.

4. For $r_i > 1$, the sliding region S_i in (9) simply extends to the corresponding switch plane, i.e.,

$$\tilde{S}_i := \{x \,|\, x \in \mathcal{E}_i,\ |c_i A x| \le r_i c_i b_i\} \quad (12)$$

and the system dynamics can be modified as

$$\dot{x} = \mathcal{P}_\sigma A x - B_{\sigma^c} \mathbf{sgn}\big(C_{\sigma^c} x\big) \ \text{ for } x(t) \in \bigcap_{i\in\sigma(t)} \tilde{S}_i \quad (13)$$

5. Denoting the (i, j) entry of $G(s)$ as $G_{ij}(s)$, $G_{ii}(s)$ is relative degree one and $G_{ij}(s)$ has at least one zero at the origin for all i, j. Also, the roots of the numerator of $G_{ii}(s)$ belong to the spectrum of $\mathcal{P}_\sigma A$ for $\sigma = \{i\}$.
6. It is enough to consider that $CB = I$ for the stability analysis since $\mathbf{sgn}(c_i x) = \mathbf{sgn}(k c_i x)$ for any $k > 0$.

3. GLOBAL STABILITY CONDITIONS

3.1 Stability Definition for Equilibrium Set

This paper is intended to elaborate global stability conditions for the system in (13) by exploring available theoretical results. Especially, we are concerned in the switched systems with equilibrium sets, so the stability will be termed in the context of the *pointwise global stability*.

Definition 1. (Yakubovich *et al.*, 2004) The equilibrium set $\{x_e\}$ is *pointwise globally stable*, if it is globally asymptotically stable and every solution tends to a stationary vector in $\{x_e\}$ as $t \to +\infty$.

One example of systems that are asymptotically stable but not pointwise stable is the single mass system with an ideal PID (proportional-integral-derivative) controller when the detachment friction ratio $r = 1$ (Jeon & Tomizuka, 2008).

3.2 Stability Results from Switched System Model

The stability of switched systems such as (10) or (13) is often studied by looking at the dynamics on each partition space separately and searching for useful relations at state transitions (Gonçalves *et al.*, 2001; Liberzon, 2003). The following lemma in Jeon & Tomizuka (2008) explains how the instability of any sliding mode is related to the global stability of the whole system. In this paper, the *stable sliding mode* of a transfer function is defined such that all the roots of its numerator lie in the left-half plane except for the single zero at the origin.

Lemma 1. If the $\mathcal{P}_\sigma A$ in (13) has an unstable eigenvalue for any nonempty σ (i.e., any nonempty element of the power set of $\{1, ..., q\}$), then $\{x_e\}$ is not Lyapunov stable.

Proof. The existence of such σ, say σ_u, means that $\bigcap_{i\in\sigma_u} \tilde{S}_i$ is the unstable region in the state space. The proof follows from the fact that this unstable region always overlaps with the small neighborhood of $\{x_e\}$. □

In switched affine systems, the vector field is a piecewise continuous function. Therefore, the piecewise quadratic Lyapunov function is often considered as a suitable Lyapunov function. Using such a piecewise quadratic Lyapunov function, it was shown by Jeon & Tomizuka (2008) that the pointwise global stability can be guaranteed for the system in (10) by the existence of an appropriate multiplier transfer function matrix of the PI (proportional-integral) type. In fact, a complimentary condition using the PD (proportional-derivative) type multiplier had been proposed by Barabanov (See Yakubovich *et al.* (2004)). These two conditions can be combined as follows.

Theorem 1. If there exists a $T = \mathbf{diag}(\tau_1, \cdots, \tau_q) \succeq 0$ such that either of the following two conditions is met,

$$(I + sT)G(s) \text{ is positive real (PR)} \quad (14a)$$

$$\left(I + \frac{1}{s}T\right) G(s) \text{ is strictly positive real (SPR)} \quad (14b)$$

then $\{x_e\}$ of (10) is pointwise globally stable.

Proof. The condition (14a) can be found in Yakubovich *et al.* (2004). Refer to Jeon & Tomizuka (2008) for the proof of the condition (14b). □

In fact, the condition (14a) is not restricted to (10) but can also be applied to (13) (i.e., for $r_i > 0$). However, the condition (14b) is valid only for (10). The conditions in (14) are derived from the switched affine system models. In fact, they are encompassed by a larger class of stability conditions known as the *stability multipliers* which came from a different approach called the absolute stability.

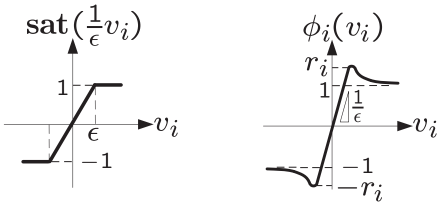
3.3 Stability Results from Continuous Approximations

The idea of using the stability multipliers has long been studied in the realm of the absolute stability theory. The absolute stability problem is limited to systems with a unique equilibrium point. Hence, it cannot be directly applied to the systems in (10) or (13). One way to get around this problem is to approximate the ideal relay to the saturation function as shown in Fig. 3(a) and let ϵ approach 0. As suggested by Rantzer (2001), the case with the non-unity detachment friction ratio ($r_i > 1$) can also be replaced by a continuous function ϕ_i for the i^{th} Coulomb friction source as shown in Fig. 3(b) with $\epsilon \to 0$:

$$\phi_i(v_i) \begin{cases} = v_i/\epsilon & \text{if } |v_i| \le \epsilon r_i \\ \in [1, r_i]\,\mathbf{sgn}(v_i) & \text{otherwise} \end{cases} \quad (15)$$

For later purpose, ϕ_i is assumed to be Lipschitz. Refer to Tsypkin (1984) and Rantzer (2001) for more details on the justification of these approximations. Also, by the limiting arguments, it is easy to check that the system equilibrium changes from the origin to the set $\{x_e\}$ as ϵ tends to zero.

As mentioned above, the general absolute stability conditions are explained in the name of the stability multipliers. Initially, it was formulated by Borockett & Willems (1965) and Zames & Falb (1968) for the SISO (single input and single output) nonlinearity. The main idea is to find a suitable multiplier to render both the linear part and the nonlinear part passive so that the global stability directly



(a) $r_i = 1$ (b) $r_i > 1$

Fig. 3. Approximated nonlinear functions

follows from the passivity theorem. The stability multipliers recently drew a new attention in an effort to generalize them to MIMO (multiple input and multiple output) nonlinearities. Mancera & Safonov (2005) presented the largest class of stability multipliers for the repeated MIMO monotone nonlinearities. These conditions are, however, derived for the monotone nonlinearities. Hence, they are valid only for the standard relay element with $r_i = 1$ as shown in (10). This drawback has been removed by Rantzer (2001) who used the continuous approximation function shown in Fig. 3(b) and applied the stability theorem formulated by the integral quadratic constraint (IQC) (Megretsky & Rantzer, 1997). However, Rantzer (2001) only considered the case with a single Coulomb friction force. The following Theorem is the main result of this paper, which is a direct extension of the result by Rantzer (2001) to multiple Coulomb friction sources.

Theorem 2. Suppose each Coulomb friction is represented by $f_i(t) = \phi_i(v_i(t))$ with the Lipschitz function ϕ defined in (15). Consider a $q \times q$ transfer function matrix $H(s)$, the inverse Laplace transform of which, is represented by $h(t)$ and its (i, j) entry denoted by $h_{ij}(t)$, i.e.,

$$H(s) = \int_{\infty}^{\infty} e^{-ts} \begin{bmatrix} h_{11}(t) & \cdots & h_{1q}(t) \\ \vdots & \ddots & \vdots \\ h_{q1}(t) & \cdots & h_{qq}(t) \end{bmatrix} dt \tag{16}$$

Also, consider a symmetric constant matrix $\Gamma \in \mathbf{R}^{q \times q}$ with its (i, j) entry denoted by Γ_{ij}. If there exists a multiplier $M(s) := \Gamma + H(s)$ such that, for all $i \in \{1, ..., q\}$,

$$\frac{\Gamma_{ii}}{r_{max}} - \sum_{j=1,\, j \neq i}^{q} |\Gamma_{ij}| \geq \max\left\{ \sum_{k=1}^{q} \|h_{ik}\|_{\mathcal{L}_1}, \sum_{k=1}^{q} \|h_{ki}\|_{\mathcal{L}_1} \right\} \tag{17a}$$

$$M(s)G(s) \text{ is positive real (PR)} \tag{17b}$$

where $r_{max} = \max\{r_1, ..., r_q\}$, then the equilibrium set $\{x_e\}$ of the system in (13) is pointwise globally stable.

As is the case for the SISO nonlinearity (Rantzer, 2001), the proof of Theorem 2 hinges on the following result established by Megretsky & Rantzer (1997).

Theorem 3. Let $G(s) = C(sI - A)^{-1}B \in \mathcal{RH}_{\infty}^{\ell \times m}$ (i.e., stable and real rational) and $\phi : \mathbf{R}^{\ell} \to \mathbf{R}^m$ is a bounded causal operator. The feedback system $\dot{x} = Ax - B\phi(Cx)$ is exponentially stable if there exists a measurable Hermitian valued function $\Pi(j\omega) : j\mathbf{R} \to \mathbf{C}^{(\ell+m)\times(\ell+m)}$ such that

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{f}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{f}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{18a}$$

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \succ 0, \forall \omega \in \mathbf{R} \tag{18b}$$

where $f = \phi(v)$ with any square integrable f and v. The superscript * denotes the conjugate transpose and $\bullet \succ 0$ means that the matrix is positive definite.

Proof of Theorem 2. The condition (17b) implies (18b) with the choice of

$$\Pi(j\omega) = \begin{bmatrix} 0 & \Gamma + H^*(j\omega) \\ \Gamma + H(j\omega) & \epsilon I \end{bmatrix} \text{ for } \epsilon > 0. \tag{19}$$

So we only need to show that (18a) also holds. Denoting $\bar{h}(t)$ as the time reversal of $h(t)$ (i.e., $\bar{h}(t) = h(-t)$), the i^{th} entry of the convolution $(\bar{h} * f)_i(t)$ is bounded as

$$|(\bar{h} * f)_i(t)| \leq r_{max} \sum_{j=1}^{q} \|h_{ij}(t)\|_{\mathcal{L}_1}. \tag{20}$$

Also, note that $|f_i(t)| \geq 1$ when $v_i(t) \neq \epsilon f_i(t)$ (see Fig. 3(b)). Hence, we can show using (17a) that

$$(v(t) - \epsilon f(t))^T \left(\Gamma f(t) + (\bar{h} * f)(t)\right) \geq 0. \tag{21}$$

By definition,

$$\|H_{ij}(s)\|_{\infty} \leq \|h_{ij}(t)\|_{\mathcal{L}_1} \tag{22}$$

which implies $\Gamma + \frac{1}{2}(H(j\omega) + H(j\omega)^*) \succ 0, \forall \omega \in \mathbf{R}$. Therefore,

$$\begin{aligned} &\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{f}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{f}(j\omega) \end{bmatrix} d\omega \\ &\geq \int_{-\infty}^{\infty} \mathbf{Re}\left\{ \hat{v}(j\omega)^* \left(\Gamma + H(j\omega)^*\right) \hat{f}(j\omega) \right\} d\omega \\ &\geq \int_{-\infty}^{\infty} \mathbf{Re}\left\{ \left(\hat{v}(j\omega) - \epsilon \hat{f}(j\omega) \right)^* \left(\Gamma + H(j\omega)^*\right) \hat{f}(j\omega) \right\} d\omega \\ &\geq 2\pi \int_{-\infty}^{\infty} (v(t) - \epsilon f(t))^T \left(\Gamma f(t) + (\bar{h} * f)(t)\right) dt \geq 0 \end{aligned}$$

and the condition (18a) is satisfied. □

Note that the multiplier $M(s)$ is PR by itself.

3.4 Relation to Hurwitz Condition: SISO Case

In the early stages of the absolute stability problem, a great deal of effort was devoted to formulating the absolute stability condition in terms of the Hurwitz condition of the linear system part. This led to such famous conjectures as the Aizerman's conjecture and the Kalman's conjecture. Although both of them turned out to be wrong in general, they are proven to be true at least up to a certain order. The Aizerman's conjecture is true for $n \leq 2$ and the Kalman's conjecture is true for $n \leq 3$ (Lozano *et al.*, 2000). In fact, the lowest possible order of $G(s)$ with a non-trivial sliding mode is $n = 3$. For $n = 3$, it is easy to check that the Nyquist plot of $G(s)$ does not intersect the negative real axis if its sliding mode is stable (Jeon & Tomizuka, 2008). This is equivalent to the Hurwitz condition of $A - kBC$ for all $k \geq 0$, which satisfies the Kalman's conjecture for the sector-bounded nonlinearity with its slope $k \in (0, \infty)$. Therefore, by combining the limiting argument and the validity of the Kalman's conjecture for the 3^{rd} order system, we can draw the following conclusion.

Lemma 2. When $n = 3$, the equilibrium set $\{x_e\}$ of the system in (10) is pointwise globally stable if $G(s)$ has stable sliding mode.

Note that the stable sliding mode is also essential to guarantee the (local) pointwise stability (Tsypkin, 1984). This Lemma only holds for the case with the unity detachment friction ratio (i.e., $r = 1$) since the slope of the nonlinearity must belong to the sector $(0, \infty)$. In the stability analysis of systems with nonlinearities, the approximate methods such as the describing function can be erroneous not only in predicting the limit cycles but also in assuring the stability. Refer to Engelberg (2002) and Lozano *et al.* (2000) for the respective counterexamples. However, Lemma 2 implies that the describing function is indeed an exact condition to guarantee the stability of the system in (10) with the lowest possible order to hold the non-trivial sliding mode (i.e., as long as $n = 3$).

4. EXAMPLE

To confirm the main result in the previous section, simulation studies are performed considering a single link flexible joint mechanism shown in Fig. 4. It is a two-mass-spring system with a transmission gear. The plant parameters are obtained from the experimental setup studied in Jeon & Tomizuka (2008) and they are listed in Table 1 with their denotations. The system equations are written as

$$J_m\ddot{\theta}_m + b_m\dot{\theta}_m = -\frac{k_j}{N}\left(\frac{\theta_m}{N} - \theta_\ell\right) + u + f_m(\dot{\theta}_m) \quad (23a)$$

$$J_\ell\ddot{\theta}_\ell + b_\ell\dot{\theta}_\ell = k_j\left(\frac{\theta_m}{N} - \theta_\ell\right) + f_\ell(\dot{\theta}_\ell). \quad (23b)$$

The subscript m denotes the motor side quantities and the subscript ℓ the load side quantities. θ and $\dot{\theta}$ represent position and velocity, respectively.

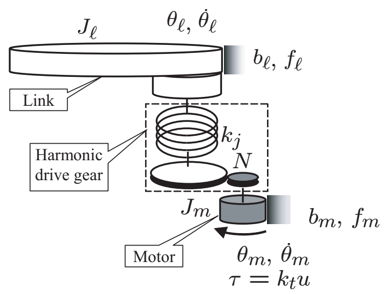


Fig. 4. Schematic of a single link flexible joint mechanism

Table 1. System parameters

Para.	Denotation	Value	Unit
N	gear ratio	50	
J_m	motor inertia	1.087×10^{-4}	[kgm^2]
J_ℓ	load inertia	1.019	[kgm^2]
k_j	spring const.	28000	[Nm/rad]
k_t	motor torque const.	1.309×10^{-1}	[Nm/V]
b_m	motor damping	0.9×10^{-3}	[Nms/rad]
b_ℓ	load damping	1.5	[Nms/rad]
f_m^c	motor Coulomb fric. level	0.0618	[Nm]
f_ℓ^c	load Coulomb fric. level	2.06	[Nm]
r_m	motor detach. fric. ratio	1.5	
r_ℓ	load detach. fric. ratio	1.5	

Consider that the motor position and the motor velocity are measured, i.e., $y = \left[\theta_m\ \dot{\theta}_m\right]^T = C_y x_p(t)$ and that the system is under state feedback control with an

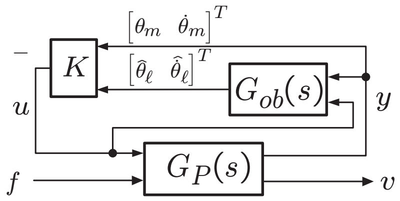


Fig. 5. Observer-based state feedback control

asymptotic observer as shown in Fig. 5. After choosing $x_p = [\theta_m\ \dot{\theta}_m\ \theta_\ell\ \dot{\theta}_\ell]^T$ as the plant state vector, a reduced order observer is designed to estimate θ_ℓ and $\dot{\theta}_\ell$. Observer is designed to have the closed-loop poles at 45 Hz with 0.45 damping ratio. With the observer fixed, three different state feedback gains, K_1, K_2 and K_3, are selected to see the effect of Coulomb friction. Table 2 shows the desired closed-loop poles for the reduced-order observer and controllers. K_1, K_2 and K_3 place the desired closed-loop poles (i.e. the eigenvalues of $A_P - B_P K$) at (7, 10) Hz, (13, 14) Hz and (15, 16) Hz respectively with 0.707 damping ratio. The closed-loop transfer function matrices from f to v under feedback gains, K_1, K_2, and K_3, will be denoted by $G_1(s)$, $G_2(s)$, and $G_3(s)$ respectively.

Table 2. Closed-loop poles for $G_{ob}(s)$ and K

Design target		Desired closed-loop pole locations
Observer	$G_{ob}(s)$	$-127.2 \pm 225.5j$
	K_1	$-31.10 \pm 31.10j$, $-44.43 \pm 44.43j$
Controller	K_2	$-57.76 \pm 57.76j$, $-62.20 \pm 62.20j$
	K_3	$-66.64 \pm 66.64j$, $-71.09 \pm 71.09j$

Note that there exist two separate Coulomb friction sources (i.e., $q = 2$), one at the motor side and the other at the load side. The motor side Coulomb friction will be indexed by 1 and the load side by 2. To check the stability of the closed-loop system, we can start with the necessary condition given in Lemma 1. For $G_2(s)$ and $G_3(s)$, it can be easily checked that the system matrix $\mathcal{P}_\sigma A$ is stable for any index set $\sigma \subseteq \{1, 2\}$ (See (13)). However, the feedback gain K_1 causes $\mathcal{P}_\sigma A$ to have two unstable eigenvalues at $20.54\pm114.60j$ for $\sigma = 2$. So, we can conclude from Lemma 1 that $G_1(s)$ is not Lyapunov stable. It is also confirmed by the simulation results which revealed a limit cycle as shown in Fig. 6. Upper plot is the motor velocity and the lower plot is the load velocity. The sticking motion is observed in the limit cycle of the motor velocity.

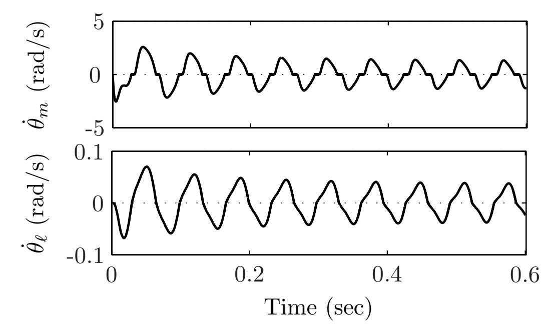


Fig. 6. Closed-loop velocity responses under K_1

The next step is to check if any result in the previous section can determine the stability of $G_2(s)$ and $G_3(s)$. Either by applying the Kalman-Yakubovich-Popov (KYP) lemma or by checking the eigenvalues of $\mathbf{Re}G_3(j\omega)$, we can verify that $G_3(j\omega)$ is PR and its global stability is guaranteed. What is left is whether $G_2(s)$ is stable or not. We can first check the conditions in Theorem 1. Using the KYP lemma, (14a) and (14b) can be formulated as the feasibility problem of the semi-definite programming (SDP). Using the SDP solver SeDuMi (Sturm, 1998), both (14a) and (14b) are shown to be infeasible for $G_2(s)$.

Unlike other conditions, it is not straightforward to check if there exists a multiplier to satisfy the condition in Theorem 2. Since $G_3(s)$ is PR, we can choose a multiplier $M(s)$ as

$$M(s) = G_3(s)G_2(s)^{-1}. \tag{24}$$

After subsequent pole-zero cancelations, $M(s)$ becomes

$$M(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \tag{25}$$

where

$$H_{21}(s) \cong \frac{-8327.2(s+127.83)(s^2+12.848s+8312.9)}{\mathbf{det}(sI - A_P + B_P K_3)(s^2+66.035s+15263)}$$

$$H_{11}(s) = \frac{NJ_\ell}{k_j}\left(s^2 + \frac{b_\ell}{J_\ell}s + \frac{k_j}{J_\ell}\right)H_{21}(s),\ H_{12}(s) = H_{22}(s) = 0.$$

Then, taking the inverse Laplace transforms of $H_{ij}(s)$, the corresponding $\mathcal{L}_1$ norms of $h_{ij}(t)$ are calculated as

$$\|h_{11}(t)\|_{\mathcal{L}_1} \cong 0.5510, \quad \|h_{21}(t)\|_{\mathcal{L}_1} \cong 0.0146 \tag{27}$$

According to Theorem 2, the global stability is guaranteed for the systems with Coulomb friction with the detachment friction ratio $r_{max} \leq 1/\left(\|h_{11}(t)\|_{\mathcal{L}_1} + \|h_{21}(t)\|_{\mathcal{L}_1}\right) = 1.768$. So, the stability of $G_2(s)$ is validated, and is also confirmed by the simulation results in Fig. 7.

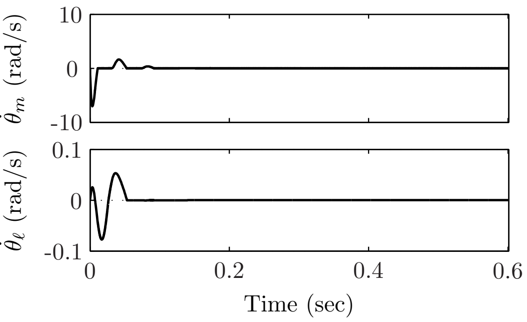


Fig. 7. Closed-loop velocity responses under K_2

5. CONCLUDING REMARKS

In this paper, the controlled mechanical systems with ideal Coulomb friction were studied as a particular class of relay feedback systems. In contrast to the typical relay feedback studies, the stability conditions need to hold globally in the mechanical systems with Coulomb friction. Compared to the conventional results, the analysis emphasized the multiple Coulomb friction sources with non-unity detachment friction ratio. In addition to the recently found stability conditions, a new MIMO stability condition was introduced by extending the existing condition for the single Coulomb friction source. The describing function criterion was shown to be an exact condition when the order of the system is 3 (i.e., the smallest order to hold the non-trivial sliding mode). The validity of main stability conditions were illustrated with simulation results implementing a single link flexible joint mechanism. Stability multiplier method may still be conservative for the relay feedback systems in the sense that it holds for a wide range of sector-bounded nonlinearities. Also, the systematic way to check the existence of such multiplier is a challenge.

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