Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

# Method for Analysis of Synchronization Applied to Supermarket Refrigeration System

**Rafael Wisniewski*** **Lars F. S. Larsen***

* *Advanced Engineering, Danfoss A/S, DK-6430 Nordborg, Denmark (e-mail: wisniewski@danfoss.com, lars.larsen@danfoss.com).*

**Abstract:** Synchronisation arises in dynamical systems that are composed of a number of interconnected subsystems. In the paper synchronization in a particular application - a supermarket refrigeration system - is studied. The temperature control in supermarket display cases is typically maintained by a number of distributed hysteresis controllers. Synchronization is then manifested by the opening and closing actions of expansion valves at the same time. Synchronization is interpreted in this paper as a limit cycle in a state space created by transitions among piecewise-affine dynamical systems. Stability of the resultant limit cycle is examined by a Poincaré like map. We show that the synchronisation takes place if the corresponding Poincaré map is stable.

## 1. INTRODUCTION

The temperature control in a supermarket refrigeration system is typically maintained by a number of distributed hysteresis controllers. A problem that often arises in this control setup is synchronization. It is manifested by the opening and closing actions of all the valves at almost the same time. Consequently, the compressors periodically have to work more intensely, which results in low efficiency and increased wear.
Synchronization - or more generally - state agreement, arises in dynamical systems that are composed of a number of interconnected subsystems Lin et al. (2005b). State agreement means that the states of the subsystems are all equal. It is known in a variety of applications, e.g. biochemical systems Gunawardena (2003), rendezvous in multi-agent systems Lin et al. (2005a), consensus in computer science Moases and Rajsbaum (2002) and Tabuada and Pappas (2002).

Synchronization in a supermarket refrigeration system will be interpreted as a stable limit cycle in a state space created by transitions among piecewise-affine dynamical systems. It will be shown how piecewise-affine systems living on polyhedra are "glued" together to form a single dynamical system defined on a coherent state space. Further by enforcing a certain transversality condition on the constituent systems it will be shown that the stability of the resultant limit cycle can be examined by a Poincaré map. The Poincaré map has been used before in Hiskens (2001) and Hiskens and Reddy (2007) for stability of limit cycles arising in switched systems. The method developed in this paper will be applied for analysis of synchronization in a refrigeration system consisting of two display cases and a compressor unit.

The paper is organized in the following way. Section 2 describes the control system architecture in a supermarket refrigeration system and explains the origin of synchronization. In Section 3 a non-linear model is presented and subsequently a simplified piecewise-affine model is derived. Section 4 puts forward a method for gluing state spaces of a switched system together. The method is then applied to the supermarket refrigeration system. In Section 5 the synchronization i.e. the resultant limit cycle on the glued space is analyzed. Section 7 concludes the paper.

## 2. SYSTEM DESCRIPTION

In a supermarket many of the goods need to be refrigerated to ensure preservation for consumption. These goods are normally placed in open refrigerated display cases that are located in the sales area for self service.
A simplified supermarket refrigeration circuit is shown in Fig. 1. Compressors comprises the heart of the system. In majority of supermarkets, the compressors are connected in parallel. The compressors supply the flow of refrigerant by compressing the low pressure refrigerant, which is drained from the display cases trough the suction manifold. The compressors keep a certain constant pressure in the suction manifold, thus ensure the desired evaporation temperature. From the compressors, the refrigerant flows

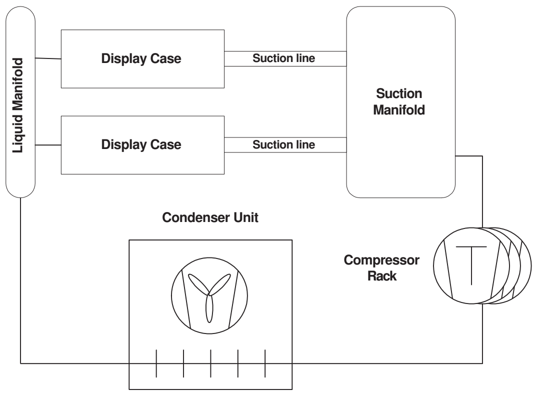


Fig. 1. A simplified layout of a typical supermarket refrigeration system.

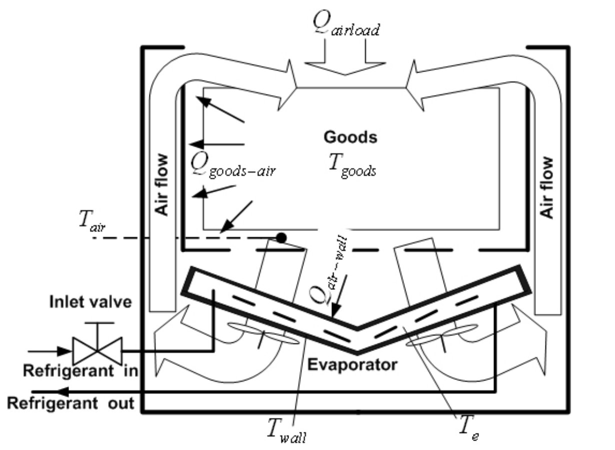


Fig. 2. Cross section of a refrigerated display case.

to the condenser and then on to the liquid manifold. The evaporators inside the display cases are connected in parallel to an expansion valve and further to the liquid manifold. The outlets of the evaporators lead to the suction manifold thus closing the circuit.
Fig. 2 shows a cross section of an open display case. The refrigerant is fed into the evaporator located at the bottom of the display case. Here the refrigerant evaporates while absorbing heat from the surrounding air circulating through the evaporator. The resulting air flow creates an air-curtain at the front of the display case. The air-curtain is colder than the goods; this leads to a heat transfer from the goods $\dot{Q}_{goods-air}$ and - as a side effect - from the surrounding $\dot{Q}_{load}$ to the air-curtain. Inside each display case a temperature sensor is mounted, which measures the air temperature close to the goods. This measurement serves in the control loop as an indirect measure of the goods' temperature. Furthermore, an on/off inlet valve is located at the refrigerant inlet to the evaporator. It is used to control the temperature in the display case.

### 2.1 Traditional Control

The control systems used in today's supermarket refrigeration systems are decentralized. Each of the display cases is equipped with a temperature controller and a superheat controller that governs the filling level of the evaporator. The compressor rack is equipped with a suction pressure controller, and the condenser - with a condenser pressure controller. Furthermore, various supervisory controllers may be used to adjust the set-points. In this paper, we will only consider the display case temperature and the compressor controllers. The display case temperature is controlled by a hysteresis controller that opens and closes the inlet valve. This means that when the temperature $T_{air}$ reaches a certain upper temperature bound the valve is opened and $T_{air}$ decreases until the lower temperature bound is reached and the valve is closed again.
In the supermarket many of the display cases are of alike design and they are working under uniform conditions. As a result, the inlet valves of the display cases are switched with very similar switching frequencies. The valves have a tendency to synchronize leading to periodic high and low flow of evaporated refrigerant into the suction manifold thus creating large variations in the suction pressure.
Turning on and off the compressors in the compressor rack controls the suction pressure. To avoid excessive compressor switching, a dead band around the reference of the suction pressure is commonly used. When the pressure exceeds the upper bound, one or more additional compressors are turned on to reduce the pressure, and vice versa when the pressure falls below the lower bound. In this way, moderate changes in the suction pressure do not initiate a compressor switching. Nevertheless, pronounced synchronization effects lead to frequent compressor switchings causing large fluctuations in the suction pressure and a high wear of the compressors. Fig. 3 shows a simulation illustrating the effects of synchronization.

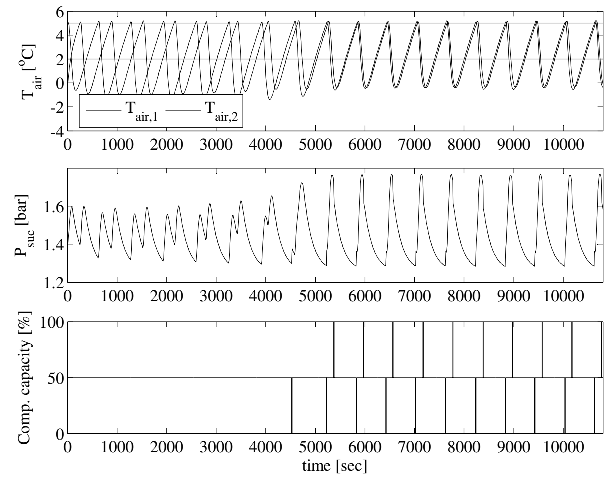


Fig. 3. Effects of synchronization using the traditional control

## 3. MODELING

The model of the supermarket refrigeration system is composed of individual models of the display cases, the suction manifold, the compressor rack, the condensing unit. The main emphasis in this paper is laid on the suction manifold and the display cases such that the dynamics relevant for controlling the compressors and display cases are captured. The compressor dynamic is typically much faster than the rest of system and hence not included.

### 3.1 Non-linear hybrid model

The nonlinear hybrid model presented here is a summary of the model presented in Larsen et al. (2007).
The state of the combined model of a supermarket refrigeration system consists of the suction pressure $P_{suc}$, and for each display case four states - the goods temperature $T_{good}$, the air temperature $T_{air}$, the wall temperature (of the evaporator) $T_{wall}$, and the mass of refrigerant in the evaporator $M_r$. The input to the model is the volume flow produced by the compressors $\dot{V}_{comp}$, and the state of the inlet valve (closed or opened, $\delta \in \{0,1\}$). The system is affected by two disturbances - the heat load from the surroundings $\dot{Q}_{load}$, and $\dot{m}_{r,const}$ which is a constant mass flow into the manifold giving rise to un-modeled refrigerated entities.

$$\frac{dT_{goods,i}}{dt} = -\frac{\dot{Q}_{goods-air,i}}{M_{goods,i}\ Cp_{goods,i}} \tag{1}$$

$$\frac{dT_{wall,i}}{dt} = \frac{\dot{Q}_{air-wall,i} - \dot{Q}_{e,i}}{M_{wall,i}\ Cp_{wall,i}} \tag{2}$$

$$\frac{dT_{air,i}}{dt} = \frac{\dot{Q}_{goods-air,i} + \dot{Q}_{load,i} - \dot{Q}_{air-wall,i}}{M_{air}\ Cp_{air,i}} \tag{3}$$

$$\frac{dM_r}{dt} = \begin{cases} \frac{M_{r,max,i} - M_{r,i}}{\tau_{fill,i}} & \text{if } \delta_i = 1 \\ -\frac{\dot{Q}_{e,i}}{\Delta h_{lg}} & \text{if } \delta_i = 0 \wedge M_{r,i} > 0 \\ 0 & \text{if } \delta_i = 0 \wedge M_{r,i} = 0 \end{cases} \tag{4}$$

$$\frac{dP_{suc}}{dt} = \frac{\dot{m}_{in-suc} + \dot{m}_{r,const} - \dot{V}_{comp} \cdot \rho_{suc}}{V_{suc} \cdot \frac{d\rho_{suc}}{dP_{suc}}} \tag{5}$$

where $i$ denotes the $i$'th display case and $\dot{Q}$ denotes a heat flux and the subscript the media in between which the heat is flowing. Furthermore

$$\dot{Q}_{goods-air,i} = UA_{goods-air,i}(T_{goods,i} - T_{air,i}) \tag{6}$$

$$\dot{Q}_{air-wall,i} = UA_{air-wall,i}(T_{air,i} - T_{wall,i}) \tag{7}$$

$$\dot{Q}_{e,i} = UA_{wall-ref,i}(M_{r,i})(T_{wall,i} - T_e) \tag{8}$$

$$UA_{wall-ref,i}(M_{r,i}) = UA_{wall-ref,max,i}\frac{M_{r,i}}{M_{r,max,i}} \tag{9}$$

$$m_{s\dot{u}c,in} = \sum_{i=1}^{n} \frac{\dot{Q}_{e,i}}{\Delta h_{lg}} \tag{10}$$

$$\dot{V}_{comp} = \sum_{i=1}^{q} comp_i \cdot \frac{1}{100} \cdot \eta_{vol} \cdot V_{sl} \tag{11}$$

where $n$ is the number of display cases and $q$ is the number of discrete compressor entities. $UA$ is the overall heat transfer coefficient with the subscript denoting the media between which the heat is transferred. $M$ denotes the mass, and $Cp$ the heat capacity, where the subscript indicates the media.
The model contains some non-linear refrigerant specific functions of the suction pressure:

- $\Delta h_{lg}$, which is the enthalpy difference across the evaporator
- $\rho_{suc}$, which is the density of the refrigerant
- $\frac{d\rho_{suc}}{dP_{suc}}$, which is the pressure derivative of the refrigerant density
- $T_e$, which is the evaporation temperature

In Larsen et al. (2007) a detailed description of these functions is given, and in Appendix A the values of the system parameters are provided.

As seen in Eq. (4) the system has a hybrid nature as the inputs to the system are discrete - opening/closing of inlet valves and start/stop of the compressors in the compressor rig.

### *3.2 Simplified model*

In order to obtain an adequate set of equations for analyzing synchronization in accordance with Section 5, the system equations (1) to (11) are additionally simplified to a second order (for each display case) affine switched system.

The simplification relies on the following assumptions:

(1) The heat capacity of the goods is large, thus the temperature of the goods in a display case is constant and equal $T_{g0}$.
(2) The heat capacity of the air is small.
(3) The evaporator is instantly filled (emptied) when the inlet valve is opened (closed).
(4) The mass flow out of the display case when the valve is open is constant and equal $\dot{m}_0$.
(5) The evaporation temperature $T_e$ and the density $\rho_{suc}$ of the refrigerant in the suction manifold are affine functions of suction pressure $P_{suc}$,
$$T_e = a_T P_{suc} + b_T \text{ and } \rho_{suc} = a_\rho P_{suc} + b_\rho.$$
See eq.(A.1) and (A.2) for the parameters.
(6) The gradient $\frac{d\rho_{suc}}{dP_{suc}} \equiv \frac{d\rho_{suc0}}{dP_{suc0}}$ is constant.
(7) The compressor delivers a constant volume flow $\dot{V}_{comp}$.
(8) The heat load $\dot{Q}_{load}$ on the display cases is constant.

Based on these assumption the dynamics of the air temperature $T_{air,i}$ in the $i$th display case is described by following set of equations

$$\frac{dT_{air,i}}{dt} = \frac{\dot{Q}_{goods-air,i} + \dot{Q}_{load,i} - \delta_i \dot{Q}_{e,max,i}}{\left(1 + \frac{UA_{goods-air,i}}{UA_{air-wall,i}}\right) M_{wall,i} Cp_{wall,i}} \quad \text{with} \tag{12}$$

$$T_{wall,i} = T_{air,i} - \frac{\dot{Q}_{goods-air,i} + \dot{Q}_{load,i}}{UA_{air-wall,i}}, \tag{13}$$

$$\dot{Q}_{goods-air,i} = UA_{goods-air,i}(T_{g0,i} - T_{air,i}), \tag{14}$$

$$\dot{Q}_{e,max,i} = UA_{wall-ref,max,i}(T_{wall,i} - a_T P_{suc} - b_T), \tag{15}$$

where $\delta_i \in \{0, 1\}$, $\delta_i = 1$ indicates that the inlet valve to the $i$th display case is open.

The suction manifold dynamics is governed by the expression

$$\frac{dP_{suc}}{dt} = \frac{\sum_i^n \delta_i \dot{m}_{0,i} + \dot{m}_{r,const} - \dot{V}_{comp}(a_\rho P_{suc} + b_\rho)}{V_{suc} \cdot \frac{d\rho_{suc0}}{dP_{suc0}}}. \tag{16}$$

Hereby the non-linear hybrid system has been reduced to a second order affine system with discrete inputs.

## 4. SPACE GLUING

Synchronization in a supermarket refrigeration system will be interpreted as a limit cycle in a state space created by gluing certain polyhedra together. The gluing algorithm is defined by transitions among piecewise-affine dynamical systems. Stability of the resultant limit cycle is examined by the Poincaré map. The next section applies this method for examining synchronization in a refrigeration system consisting of two display cases and a compressor unit.
To motivate this approach, we will start by studying the simulation of the simplified model consisting of 2 display cases, depicted in Fig. 4. The lower graph shows that the two display cases start working synchronously after

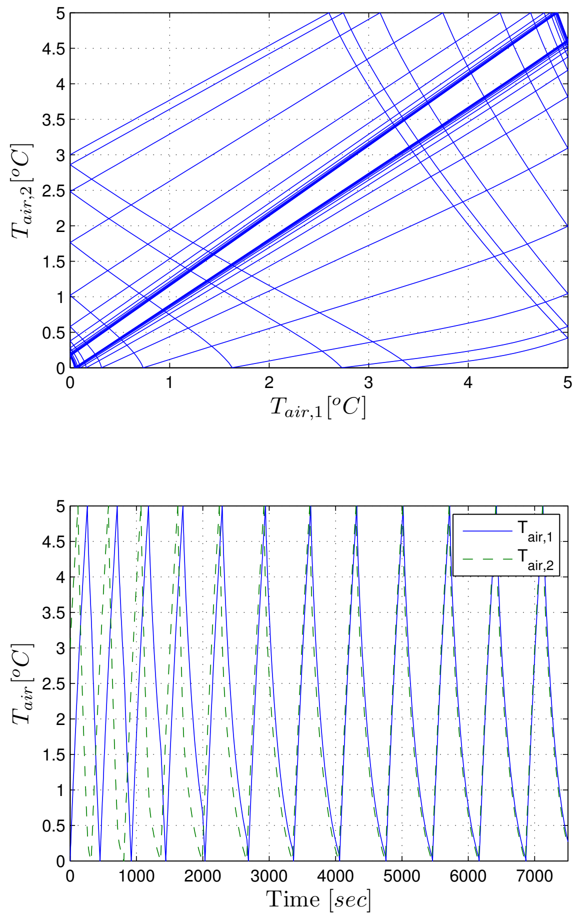


Fig. 4. Simulation of the simplified model, showing the synchronization as a stable limit cycle.

3000 *sec*. It is seen in the upper part of Fig. 4 that the synchronization is manifested by a closed orbit - a limit cycle - near the diagonal line $\varrho = \{(T_{air,1}, T_{air,2})|\ T_{air,1} = T_{air,2}\}$. Perfect synchronisation of the two display cases takes place when the stable limit cycle coincides with $\varrho$.

Being less specific for a while, let us consider an autonomous switched system, consisting of a finite family $\mathcal{F}$ of dynamical systems each living on a polyhedron $P$. The transition between two systems takes place autonomously on the crossing of a system trajectory with a face $F$ of the corresponding polyhedron. A limit cycle - arises if after several transitions the trajectory of the switched system - a composition $\Phi$ of the flow maps - returns to the same point, say $p$. Suppose that $p$ is on a facet [1] $F$ of a polyhedron in $\mathcal{F}$ and the transversality condition:

- each vector field of $\mathcal{F}$ points in (out) of the corresponding polyhedron on the in-coming facet, whereas out (in) on the out-coming facet;

is fulfilled, then the fixed point $p$ of the map $\Phi$ is locally stable if all eigenvalues of differential $D\Phi(p)$ belong to the open unit disk. The mentioned transversality condition basically rules out the possibility of chattering or gazing along a face and allows the use standard methods known from smooth (non-hybrid) systems, for a detailed discussion the reader is referred to Wisniewski and Larsen (2008).

[1] a facet is a face of co-dimension 1

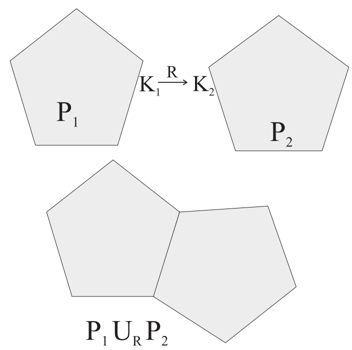


Fig. 5. The polyhedron $P_1$ is glued to the polyhedron $P_2$ by identifying points of the facet $K_1$ with the points of the facet $K_2$.

Before we discuss how to analyze the map $\Phi$, we will describe the idea of gluing the spaces on which the affine systems live on together. This defines a complex - a high dimensional mosaic of the polyhedra. We start by considering a transition between a smooth vector field $\xi_1 : P_1 \to \mathbb{R}^n$ and $\xi_2 : P_2 \to \mathbb{R}^n$ on a facet $F_1 \in P_1^{n-1}$. We assume that a reset map $R : F_1 \to F_2$, with $F_2$ a facet of $P_2$, is a diffeomorphism. Now we can glue the polyhedron $P_1$ and $P_2$ together by identifying $F_1$ with $F_2$ via the reset map $R : F_1 \to F_2$

$$P_1 \sqcup_R P_2 \equiv P_1 \sqcup P_2 / \sim,$$

where the equivalence relation $\sim$ identifies $x \in K_1$ with $R(x) \in K_2$, and $\sqcup$ stands for the disjoint union, $P_1 \sqcup P_2 = P_1 \times \{1\} \cup P_2 \times \{2\}$. In other words, a neighborhood of $F_1$ (which is now identified with $F_2$) in the set $P_1 \sqcup_R P_2$ is the union of an open neighborhood of $F_1$ in $P_1$ and an open neighborhood of $F_2$ in $P_2$. The situation is depicted in Fig. 5. We assume the transversality condition is fulfilled and define the flow map as a composition of the flow maps of the individual systems comprising the switched system. The space of a switched system is a disjoint union of polyhedra, whose facets are identified by the reset maps

$$X = \bigsqcup_{P \in \mathcal{F}} P / \sim$$

$\sim$ is the equivalence relation that for each reset map $R : F \to F' \in \mathcal{R}$ identifies $x \in F$ with $R(x) \in F'$, where $F, F'$ are facets of $P, P'$ both in $\mathcal{F}$, respectively; see Fig. 5.

We turn to the hysteresis controlled refrigeration system described in Section 3, it is an autonomous switched system consisting of four dynamical systems $\xi_i : B \to \mathbb{R}^2$, $i = 1, .., 4$, one for each combination of the positions (on/off) of the inlet valves $\delta_1$, $\delta_2$. $B = [\underline{T_{air}}, \overline{T_{air}}] \times [\underline{T_{air}}, \overline{T_{air}}] \times \mathbb{R}$ is the polyhedron $B = P_1 = P_2 = P_3 = P_4$ that the flow lives on, and the vector fields $\xi_i$ are given by Eq. (12). The reset maps are in this case simply the identity maps on the 4 facets of $B$.
The state-space of the refrigeration system is then defined by gluing the polyhedra $P_1, P_2, P_3$ and $P_4$ together along the facets specified by the transitions. The result is a single state space homeomorphic to a band as in Fig. 6.

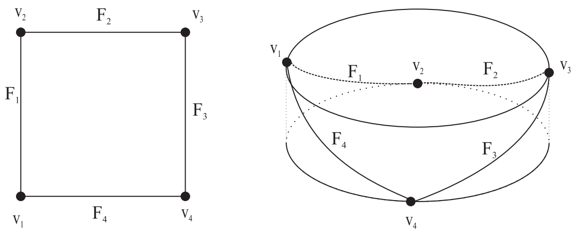


Fig. 6. The state space of the refrigeration system the product of a band with $\mathbb{R}$. In all Figures $\mathbb{R}$ is suppressed

## 5. SYNCHRONIZATION ANALYSIS

In the previous section we have motivated the approach of creating a single coherent state space by gluing the polyhedra $P_i$, $i = 1, ..., 4$ together. We have stated without proof that the glued space is homeomorphic to a band. In this section we will utilize this property to study a Poincarè map. The basic idea is to analyze stability of a candidate for a limit cycle found by e.g. simulation. A possible candidate could be perceived if simulation shows that the trajectory $\gamma$ of the switched system after several transitions returns to a neighborhood of the starting point $p$ at the the face of a polyhedron in $\mathcal{F}$. We study a composition $\Phi$ of the flow maps comprising the trajectory $\gamma$. If all the eigenvalues of differential $D\Phi(p)$ belong to an open unit disk the limit cycle is stable. We will here describe a method for finding differential $D\Phi(p)$ without explicitly computing the Poincarè map.

We start by stating following proposition which gives us an important intermediate result to eventually compute differential $D\Phi(p)$.

*Proposition 1.* Let $P$ be a polyhedron, $\xi : P \to \mathbb{R}^n$ be a vector field. Let $K_1, K_2 \in P^{n-1}$ and their supporting hyperplanes [2] be $H_i = \{x \in \mathbb{R}^n | \ \langle x, N_i \rangle = \alpha_i\}$ $(i = 1, 2)$. Suppose there are $\tau > 0$ and $p \in K_1$ such that $q \equiv \phi^\xi(p, \tau) \in K_2$ (i.e. $\tau$ is the time between 2 successive transitions). Then there are an open neighborhood $U$ of $p$ in $H_1$ and a smooth map $h : U \to \mathbb{R}$ such that

(1) $h(p) = \tau$,
(2) $\phi^\xi(x, h(x)) \in H_2$ for any $x \in U$,
(3) and $\phi^\xi \cdot (\mathrm{id}, h)$ is a diffeomorphism from $U$ onto an open set of $H_2$.

Furthermore, the differential of $Dh$ at the point $p$ is

$$Dh(p) = -\frac{1}{\langle \xi(q), N_2 \rangle} N_2^{\mathrm{T}} \ D\phi_\tau^\xi\big|_{H_1}(p), \tag{17}$$

where $\phi_t^\xi\Big|_{H_1}$ is the restriction of the flow map $\phi_\tau^\xi$ to $H_1$. □

Proof of the proposition follows directly form Sec. 3.1 in Palis and de Melo (1982). Note that the differential of the flow map $\phi_\tau^\xi$ in Eq. (17) is to be calculated with respect to the coordinates of the hyperplane $H_1$

In particular if $\xi$ is affine, $\xi(x) = Ax + b$ then the differential according to proposition 2 becomes

$$Dh(p) = -\frac{1}{N_2^{\mathrm{T}}(Ae^{A\tau}p + b)} N_2^{\mathrm{T}} e^{A\tau} B, \tag{18}$$

[2] $H$ is a supporting hyperplane for a facet $F$ if and only if $F \subset H$.

where the isomorphism $L : \mathbb{R}^{n-1} \to H_1$ is given by $L(x) = Bx + p$.

We study the map

$$\Psi \equiv \phi^\xi \circ (j, h) : U \to H_2,$$

where $U \subset H_1$ and $j : U \to \mathbb{R}^n$ stands for the inclusion (for a point $p \in U$, $j(p)$ is the same point but seen in $\mathbb{R}^n$), takes the starting point of the flow line on $F_1$ and maps it to the end point on $F_2$. A subsequent map takes the end point on $F_2$ and maps it to a point on $F_3$.

Since $H_i$ $(i = 1, 2)$ is a hyperplane, there is an affine map $L_i : \mathbb{R}^n \to \mathbb{R}^n$, say $L_i(x) = B_i x + c_i$, such that $L_i(H_i) = \{0\} \times \mathbb{R}^{n-1}$. We represent the map $\Psi$ in the local coordinates

$$\Psi' \equiv \tilde{\pi} \circ L_2 \circ \Psi \circ L_1^{-1} \circ \tilde{i},$$

where $\tilde{i} : \mathbb{R}^{n-1} \to \mathbb{R}^n$, given by $(x_1, ..., x_{n-1}) \mapsto (0, x_1, ..., x_{n-1})$ is the inclusion, and $\tilde{\pi} : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $(x_1, x_2, ..., x_n) \mapsto (x_2, ..., x_n)$ is the projection.

We apply the chain rule to calculate the differential of $\Psi'$ at a point $p$

$$D\Psi'(p) = \tilde{\pi} B_2 \left( D\phi_\tau^\xi(p) + \xi(q) Dh(p) \right) B_1^{-1} \tilde{i},$$

where $q = \phi_\tau^\xi(p)$, which according to proposition 2 is equivalent to

$$D\Psi'(p) = \tilde{\pi} B_2 \left( \mathrm{id} - \frac{\xi(q) N_2^{\mathrm{T}}}{\xi^{\mathrm{T}}(q) N_2} \right) D\phi_\tau^\xi(p) B_1^{-1} \tilde{i}, \tag{19}$$

where id stands for the identity matrix.
Hereby we are able to compute $D\Phi(p)$ by iterative use of Eq. (19).

## 6. APPLICATION ON SIMPLE SUPERMARKET REFRIGERATION SYSTEM

In this section we apply the method developed in Sections 4 and 5 for analysis of synchronization in a supermarket refrigeration system, i.e. we study stability of the limit cycle depicted in Fig. 4.
As mentioned in Section 4 the state space of a supermarket refrigeration system is $B = [\underline{T_{air}}, \overline{T_{air}}] \times [\underline{T_{air}}, \overline{T_{air}}] \times \mathbb{R}$, where in the particular example studied in this section $\underline{T_{air}} = 0$ and $\overline{T_{air}} = 5$. The polyhedron $B$ is defined as the intersection of the four half-spaces

$$H_{(\delta_1,\delta_2)}^+ \equiv \{x \in \mathbb{R}^3 | \ \langle N_{(\delta_1,\delta_2)}, x \rangle \geq \alpha_{(\delta_1,\delta_2)}\},$$

with $(\delta_1, \delta_2 \in \{0, 1\})$, where $N_{(0,0)} = [1\ 0\ 0]^{\mathrm{T}}$, $N_{(0,1)} = [0\ -1\ 0]^{\mathrm{T}}$, $N_{(1,1)} = [1\ 0\ 0]^{\mathrm{T}}$, $N_{(1,0)} = [0\ 1\ 0]^{\mathrm{T}}$, $\alpha_{(0,0)} = \alpha_{(1,0)} = \underline{T_{air}}$ and $\alpha_{(0,1)} = \alpha_{(1,1)} = \overline{T_{air}}$.

The supermarket refrigeration system consists of four affine dynamical systems

$$\xi_{(\delta_1,\delta_2)} : B \to \mathbb{R}^3, \ x \mapsto A_{(\delta_1,\delta_2)} x + a_{(\delta_1,\delta_2)},$$

given by Eq. (12).

The limit cycle is defined by the points $\left(p_{(\delta_1,\delta_2)}, T_{(\delta_1,\delta_2)}\right) \in H_{(\delta_1,\delta_2)} \times \mathbb{R}_+$ with

$$p_{(0,1)} = \phi^{\xi_{(0,0)}}_{T_{(0,0)}}(p_{(0,0)}) = [4.90\ 5.00\ 0.10]^{\mathrm{T}},\ T_{(0,0)} = 252.57;$$
$$p_{(1,1)} = \phi^{\xi_{(0,1)}}_{T_{(0,1)}}(p_{(0,1)}) = [5.00\ 4.63\ 0.33]^{\mathrm{T}},\ T_{(0,1)} = 6.67;$$
$$p_{(1,0)} = \phi^{\xi_{(1,1)}}_{T_{(1,1)}}(p_{(1,1)}) = [0.04\ 0.00\ 1.21]^{\mathrm{T}},\ T_{(1,1)} = 438.90;$$
$$p_{(0,0)} = \phi^{\xi_{(1,0)}}_{T_{(1,0)}}(p_{(1,0)}) = [0.00\ 0.16\ 0.99],\ T_{(1,0)} = 6.67,$$

which has been found by simulation of the refrigeration system.

A Poincaré map is the following composition

$$\Phi \equiv \psi_{(1,0)} \circ \psi_{(1,1)} \circ \psi_{(0,1)} \circ \psi_{(0,0)},$$

where $\psi_{(\delta_1,\delta_2)} \equiv \phi^{\xi_{(\delta_1,\delta_2)}} \circ (j_{(\delta_1,\delta_2)}, h_{(\delta_1,\delta_2)})$ and the maps $j_{(\delta_1,\delta_2)}$ and $h_{(\delta_1,\delta_2)}$ are defined in Proposition 2. The derivative of $\Phi$ at $p_{(0,0)}$ is

$$D\Phi(p_{(0,0)}) = \quad (20)$$
$$D\psi_{(1,0)}(p_{(1,0)})D\psi_{(1,1)}(p_{(1,1)})D\psi_{(0,1)}(p_{(0,1)})D\psi_{(0,0)}(p_{(0,0)}),$$

where $D\psi_{(\delta_1,\delta_2)}$ is given by Eq. (19), for instance

$$D\psi_{(0,0)}(p_{(0,0)}) =$$
$$\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{\xi_{(0,0)}(p_{(0,1)})N^{\mathrm{T}}_{(0,1)})}{\xi^{\mathrm{T}}_{(0,0)}(p_{(0,1)})N_{(0,1)})}\right)e^{A_{(0,0)}T_{(0,0)}}\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}.$$

Now evaluating each $D\psi_{(\delta_1,\delta_2)}$ for a proper combinations of $\delta_1, \delta_2 \in \{0,1\}$ and the system parameters in Appendix A the eigenvalues of $D\Phi(p_{(0,0)})$ are computed,

$$eig\left(D\Phi(p_{(0,0)})\right) = \{0.66, 0.00\}.$$

Each eigenvalue belongs to the open unit disk, hence the closed orbit with transitions at the points $p_{(0,1)}$, $p_{(1,1)}$, $p_{(1,0)}$ and $p_{(0,0)}$ is a stable limit cycle.

## 7. CONCLUSION

A method for analyzing limit cycles arising in autonomous switched systems was proposed. It was shown how to glue the state spaces of the switched systems to get a single coherent space. It was demonstrated that by relatively simple computation, stability of the Poincaré map is determined. The method applies generally to switched non-linear systems as long as the transversality condition is fulfilled. However, explicit solution was provided for switched affine systems.

The method was applied for a simplified model of a supermarkets refrigeration consisting of two display cases and a compressor unit. The approach developed in this paper shows that a limit cycle generated by interaction of distributed hysteresis controllers is stable, thus synchronisation in the supermarket refrigeration system takes place.

## REFERENCES

J. Gunawardena. *Chemical reaction network theory for in-silico biologist, lecture notes.* Bauer Center for Research, Harvard University, Cambrigde, MA, USA, 2003.

I.A. Hiskens. Stability of hybrid limit cycles: application to the compass gait biped robot. In *Proceedings of the 40th IEEE Conference on Decission and Control*, volume 1, pages 774–779. IEEE, 2001.

I.A. Hiskens and P.B. Reddy. Switching-induced stable limit cycles. *Nonlinear Dynamics*, 50:575–585, 2007.

L. F. S. Larsen, R. Izadi-Zamanabadi, R. Wisniewski, and C. Sonntag. Supermarket refrigeration systems - a benchmark for the optimal control of hybrid systems. Technical report, *HYCON*, http://astwww.bci.uni-dortmund.de/hycon4b/wprelated/sr.pdf, 2007.

J. Lin, A. S. Morse, and B. D. O. Anderson. The multi-agent rendezvous problem - part 1 the synchronous case. *Submitted to SIAM Journal on Control and Optimization*, 2005a.

Z. Lin, B. Francis, and M. Maggiore. State agreement problem for coupled nonlinear systems with time-varying interaction. *Submitted to SIAM Journal on Control and Optimization*, 2005b.

Y. Moases and S. Rajsbaum. A layered analysis of consensus. *SIAM Journal on Computing*, 31:989–1021, 2002.

J. Palis and W. de Melo. *Geometric Theory of Dynamical Systems.* Springer-Verlag, 1982.

M. J. Skovrup. *Thermodynamic and Thermophysical Properties of Refrigerants - Software package in Borland Delphi, Version 3.00.* Department of Energy Engineering, Technical University of Denmark, DK, `http://www.et.web.mek.dtu.dk/WinDali/Files/RefEqns_3.10.zip`, 2000.

P. Tabuada and G.J. Pappas. Discrete synchronization of hybrid systems. In *Proceedings of the 41st IEEE Conference on Decision and Control*, December 2002.

R. Wisniewski and L.F.S. Larsen. Synchronisation analysis and limit cycles in switched systems. *Submitted to Journal of Control*, 2008.

## Appendix A. REFRIGERANT PROPERTIES AND SIMULATION PARAMETERS

A number of refrigerant depended properties is used in the model described in Section 3. These properties can be either computed by using the freeware software package "RefEqns" for Matlab (Skovrup (2000)) or estimated as follows. The refrigerant R134a is employed. The first order approximations of the evaporation temperature $T_e$ and the density $\rho_{suc}$ in the vicinity of $P_{suc0} = 1.5bar$ are

$$T_e = 16.2 \cdot P_{suc} - 41.9, \quad (A.1)$$
$$\rho_{suc} = 4.6 \cdot P_{suc} + 0.4. \quad (A.2)$$

In the model described in Section 3 the following parameters are used:

| | | | | | |
|---|---|---|---|---|---|
| **Display cases** | | | | | |
| $UA_{wall-ref,max}$ | 500 | $\frac{J}{s \cdot K}$ | $M_{air}$ | 50 | $kg$ |
| $M_{goods}$ | 200 | $kg$ | $C_{p,air}$ | 1000 | $\frac{J}{kg \cdot K}$ |
| $C_{p,goods}$ | 1000 | $\frac{J}{kg \cdot K}$ | $M_{r,max}$ | 1 | $kg$ |
| $UA_{goods-air}$ | 300 | $\frac{J}{s \cdot K}$ | $\tau_{fill}$ | 40 | $s$ |
| $M_{wall}$ | 260 | $kg$ | $T_{SH}$ | 10 | $K$ |
| $C_{p,wall}$ | 385 | $\frac{J}{kg \cdot K}$ | $T_{g0}$ | 3.5 | $^oC$ |
| $UA_{air-wall}$ | 500 | $\frac{J}{s \cdot K}$ | $\dot{m}_0$ | 1.0 | $kg/s$ |
| | | | $\frac{d\rho_{suc0}}{dP_{suc0}}$ | 4.6 | $\frac{kg}{m^3 bar}$ |
| *The same parameters are used for all display cases.* | | | | | |
| **Compressor** | | | | | |
| $V_{sl}$ | 0.08 | $\frac{m^3}{s}$ | $\eta_{vol}$ | 0.81 | – |
| **Suction manifold** | | | | | |
| $V_{suc}$ | 5.00 | $m^3$ | | | |
| **Air temperature control** | | | | | |
| $\underline{T}_{air}$ | 0.00 | $^oC$ | $\overline{T}_{air}$ | 5.00 | $^oC$ |