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Rational Approximation of Nonlinear Optimal Control Problems

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Abstract: In this paper rational approximation of solutions to nonlinear optimal control problems is considered. A computational procedure is presented that makes it possible to compute a rational function that approximates the true optimal cost function. It is shown that the rational function has the same series expansion around the origin as the true solution. Finally, two examples are given that compares the new method with the power series approximation, which is a rather well-known method to find approximative solutions.

Keywords: Optimal Control, Computational methods

1. INTRODUCTION

This paper deals with the problem of finding an approximative solution to an optimal feedback control problem for a nonlinear system. The approximation problem is a well studied area with many references, see for example Beard et al. (1998) for a good survey. One of the major methods for finding approximative optimal feedback laws was invented by Al'brekht (1961) and later developed in Lee and Markus (1967), Lukes (1969), and Krener (2001). It is based on a power series solution of the Hamilton-Jacobi-Bellman equation (HJB). The exact optimal solution is in general given by an infinite series. The obvious way to get an implementable approximate solution is to truncate the series after a finite number of terms. However, this approximate solution tends to have bad properties for large values of $|x|$, since these approximations often grow too fast towards infinity.

The method considered in this paper will instead approximate the optimal cost by a rational function. The advantage is that with a rational approximation it is possible to have the exact same power series expansion up to some given order as the power series method would give, which means that the fit with the true optimal solution will be good for small x, while at the same time it is possible to control the rate of growth for large x. A drawback is that it normally comes with a higher computational complexity.

A similar computational method can be found in Vannelli and Vidyasagar (1985), where it is used to estimate the region of attraction for nonlinear systems.

Notation: The notation in this paper is fairly standard. The gradient of V_h, *i.e.*, $\frac{\partial V_h}{\partial x}$ is denoted $V_{h;x}$. For a power series $a(x)$, $a^{[m]}(x)$ will be used to denote the terms of degree m and $a^{m]}(x)$ will denote all terms up to order m.

2. PROBLEM FORMULATION

Consider an optimal control problem

$$\begin{aligned} V(x_0) = \quad & \inf_{u(\cdot)} \int_0^\infty L(x,u)\,dt \\ \text{s.t.} \quad & \dot{x} = F(x,u) \\ & x(0) = x_0 \in \Omega_x \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$ and Ω_0 is a neighborhood of the origin. To simplify the notation, the model is assumed to be control-affine

$$\dot{x} = f(x) + g(x)u \tag{2}$$

and the cost function is assumed to have the structure

$$L(x,u) = l(x) + \frac{1}{2}u^T Ru \tag{3}$$

The derived method will rely on the series expansions of f, g and l, and therefore these functions are assumed to satisfy the following assumption.

Assumption 1. The functions f, g and l are real analytic functions around the origin $x = 0$.

The assumption makes it possible to express these functions as uniformly convergent power series around the origin:

$$f(x) = Ax + f_h(x) \tag{4a}$$
$$g(x) = B + g_h(x) \tag{4b}$$
$$l(x) = \frac{1}{2}x^T\bar{Q}x + l_h(x) \tag{4c}$$

where f_h, g_h and l_h contain the higher order terms of the power series, beginning with terms of order 2, 1 and 3, respectively. To obtain well-defined solution to the ARE, the following assumption is also introduced.

Assumption 2. The linearization of (2), *i.e.*,

$$\dot{x} = Ax + Bu$$

is stabilizable. Furthermore, the matrix Q in

$$l(x) = x^TQx + l_h(x)$$

and the matrix R in (3) are positive semi-definite and positive definite, respectively.

In Lukes (1969), the optimal control problem (1) was solved under rather natural assumptions to obtain $V(x)$ and $u_*(x)$ in a neighborhood of the origin expressed as power series. The optimal solution mostly requires an infinite number of terms to be described and the solution used in practice is therefore truncated. In simulations, different drawbacks with the truncated power series solution have been noticed. First, the optimal return function often tends to grow too fast compared with the optimal. Second, it is rather common that the approximation of the optimal return function turns negative outside a quite small region which the optimal is not. Third, the region in which the obtained feedback law is stabilizing may be small.

The objective in this chapter is therefore to find approximate solutions that in many cases have better properties over a larger region. For that reason, another parametriza-

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tion of the optimal return function is studied, namely rational functions

$$V_r(x) = \frac{R_V(x)}{1 + Q_V(x)} \tag{5}$$

where

$$R_V(x) = R_V^{[2]}(x) + R_{V,h}(x) = \frac{1}{2}x^T P x + R_{V,h}(x) \tag{6a}$$

$$Q_V(x) = Q_V^{[1]}(x) + Q_{V,h}(x) = Tx + Q_{V,h}(x) \tag{6b}$$

and where $R_{V,h}$ and $Q_{V,h}$ are polynomials beginning with orders three and two, respectively.

The advantage with the rational functions is that while being able to match taylor series of the optimal solution up to some desired order, it is possible to specify the growth rate for large $|x|$ by choosing the difference between the order of the numerator and the denominator. In this chapter, the difference in the order between $R_V(x)$ and $Q_V(x)$ will always be chosen as two. The motivation for this choice is that it often gives rather good approximations. However, note that by choosing the coefficients in the polynomials, the difference in order may change which means that different growth rates can be obtained.

3. RATIONAL APPROXIMATION BASED ON OPTIMIZATION

In this section, a method is derived which relies on that the HJB is rewritten as a set of equations, whose solution is parametrized in the denominator coefficients. The advantage with the approach is that arbitrary high order terms of the HJB can be reduced and the rational approximant will still have the same power series up to some desired order.

3.1 Formulation of the Equations

The optimal solution to (1) is given by the HJB. For this class of optimal control problems, the HJB will take the form

$$\begin{aligned} 0 &= H\big(x, V(x)\big) \\ &= l(x) + V_x(x)f(x) - \frac{1}{2}V_x(x)g(x)R^{-1}g(x)^T V_x(x)^T \end{aligned} \tag{7a}$$

$$u(x) = -R^{-1}g(x)^T V_x(x)^T \tag{7b}$$

where the the expression for the optimal feedback law is explicit.

The objective is now to find the V_r that satisfies (7a) up to some given order. For this end, the derivative of V_r with respect to x is needed.

$$V_{r;x}(x) = \frac{V_{r;x,n}}{V_{r;x,d}} = \frac{(1+Q_V)R_{V;x} - R_V Q_{V;x}}{(1+Q_V)^2} \tag{8}$$

If (8) is substituted into (7a), the following equation is obtained

$$0 = \frac{1}{V_{r;x,d}^2}\Big(\big(\frac{1}{2}x^T Q x + l_h\big)V_{r;x,d}^2 + V_{r;x,n}(Ax + f_h)V_{r;x,d} - \frac{1}{2}V_{r;x,n}gR^{-1}g^T V_{r;x,n}^T\Big) \tag{9}$$

where the nominator will be denoted $\tilde{H}\big(x, V_r(x)\big)$. This equation should be satisfied for all x in a neighborhood of the origin, which is equivalent to that $\tilde{H}\big(x, V_r(x)\big) = 0$ in a neighborhood up to the given order. Since different powers of x are independent, all coefficients in $\tilde{H}$ must equal zero.

A more thorough examination of $\tilde{H}$ shows that the coefficients corresponding to the second order terms form the standard ARE

$$0 = A^T P + PA - PBR^{-1}B^T P + Q$$

while the terms of a general order $m \geq 3$ will have the structure

$$\begin{aligned} &R_{V;x}^{[m]}A_c x + M_1 Q_{m-2} - R_V^{[2]}Q_{V;x}^{[m-2]}A_c x = \\ &\qquad \xi\big(R_V^{[m-1]}, \ldots, R_V^{[m-4]}, Q_V^{[m-3]}, \ldots, Q_V^{[m-5]}\big) \end{aligned} \tag{10}$$

where

$$A_c = A - BR^{-1}B^T P$$
$$M_1(x) = x^T\big(P(3A - BR^{-1}B^T) + 2Q\big)x$$

and ξ is a function determined by the functions f, g, l, and R.

To study the solvability of (10) the following lemma is useful.

Lemma 3. Let $P_m(x)$ and $Q_m(x)$ be homogeneous polynomials of degree m and let A_c be a square matrix with eigenvalues strictly in the left half plane. Then an equation of the form

$$P_{m;x}(x)A_c x = Q_m(x) \tag{11}$$

can be solved uniquely for the coefficients in $P_m(x)$.

Proof. See Lyapunov (1992). ■

Based on this lemma, the following result is easily shown.

Lemma 4. Assume that A_c is a Hurwitz matrix. For given values of $R_V^{[2]}$,..,$R_V^{[m-1]}$ and $Q_V^{[1]}$,..,$Q_V^{[m-3]}$, equation (10) is a linear system of equations for the coefficients in $R_V^{[m]}$ and $Q_V^{[m-2]}$. The null space of the associated linear map has a dimension equal to

$$\binom{n+m-3}{n-1} \tag{12}$$

In particular $R_V^{[m]}$ is uniquely determined after an arbitrary choice of Q_{m-2}.

Proof. The size of the null space corresponds to the number of coefficients in Q_V of order $m-2$. The solvability follows from Lemma 3.

To understand the approximating properties of V_r, the following result can be useful.

Lemma 5. Assume that Assumption 1 and 2 are satisfied. Let W be an analytic function such that $W(0) = 0$, $W_x(0) = 0$ and suppose that $H\big(x, W(x)\big)$ has a series expansion beginning with terms of order $m+1$. Then W and V have identical series expansions up to and including terms of order m.

Proof. The optimal return function V has to satisfy (10) with $Q_V = 0$, $R_V = V$. Under Assumptions 1 and 2, it follows that $V^{[2]}, \ldots, V^{[m]}$ are uniquely determined by the requirement that terms of order up to and including m in H are zero. Since the solution is uniquely determined, W must have the same taylor series up to the given order.

From the lemma above, the following useful lemma can be proved.

Lemma 6. Let

$$R_V^{[m]}(x),\ m = 2, \ldots, m_o$$
$$Q_V^{[m]}(x),\ m = 1, \ldots, m_o - 2$$

satisfy (10). Then V_r and V have the same series expansions for terms of orders up to and including m_o.

Proof. The expression for $H\big(x, V_r(x)\big)$ will have the structure

$$H\big(x, V_r(x)\big) = \frac{\tilde{H}\big(x, V_r(x)\big)}{\big(1 + Q_V(x)\big)^4}$$

as was seen in (9). By construction, the terms in $\tilde{H}$ of orders less than or equal to m_o are zero. Since the expansion of the denominator begins with 1, this is true for H as well. The lemma is then a consequence of Lemma 5.

3.2 Choice of Denominator

It is known from Lukes (1969) that the HJB (7) can be solved by a polynomial. The extra degrees of freedom obtained by introducing a denominator in $V(x)$ gave a null space. In Lemma 4, it was shown that for a given Q_V, the terms in R_V will be determined. It means that the denominator can be chosen arbitrarily. Let m be the order of the nominator. Then the number of free parameters, or with other words, the number of coefficients in $Q_V^{m-2]}$, becomes $\binom{m+n-2}{n} - 1$ (can be shown using mathematical induction).

The free parameters can be used for different purposes, such as reducing extra terms in (7a), or to obtain a $V_r(x)$ that does not tend to infinity too fast etc. In the first case, it is good, at least from a reduction point of view, to have a lot of free parameters. However, the obtained minimization problem may become though and it may therefore be advantageous to fix some parameters. In some cases a reasonably good approximation can be obtained even with all parameters chosen as constants. Below, a few different choices of how to choose the free parameters are discussed.

All Parameters Free The most general choice of denominator is of course to let all coefficients in Q_V be free. In this case, all of them can be used to reduce higher order coefficients in $\tilde{H}$, but the obtained optimization problem grows rapidly with the desired order and the number of states.

Denominator with Fixed Highest Order Term Coefficients To reduce the computational complexity, some of the parameters can instead be chosen as constant values. One such choice is to let the highest coefficients be for example $1/(m-2)!$ and let the other coefficients be free. The main motivation for this choice is that the denominator of $V_r(x)$ becomes positive for both large and small $|x|$, since for large $|x|$ the highest order term is dominating while for small $|x|$ the constant term is dominant.

Denominator with All Coefficients Fixed The choice which gives the easiest problem to solve is to let all coefficients in Q_V be fixed. The result is a well-determined system of equations to solve, similar to the case in Lukes (1969), and no optimization is required. In principle, it also means that the obtained problem will be as simple to compute as the ordinary power series method. Despite the simplicity, this choice can sometimes give approximations that are better than the truncated power series as will be seen in Section 4. One choice that may be interesting to test is for example $(x-\alpha)(x+\beta)(1+x^{m-4})$, if the cost function has limits at $x=-\alpha$ and $x=\beta$.

3.3 Minimization of Higher Order Terms in the HJB

If not all coefficients in the denominator are chosen as constants, a minimization problem can be formulated that reduces the coefficients corresponding to terms in the HJB of higher orders than m.

Denote the higher order terms, *i.e.*, terms in $\tilde{H}\big(x, V(x)\big)$ of degree $m+1$ or higher, as $E_m(x)$. That is, if

$$V_{rm}(x) = \frac{R_V^{[2]}(x) + \ldots + R_V^{[m]}(x)}{1 + \ldots + Q_V^{[m-2]}(x)}$$

where $R_V^{[i]}$ and $Q_V^{[i-2]}$, $i = 3, \ldots, m$ satisfy (7a), is substituted into (9), the result is

$$0 = \tilde{H}\big(x, V_{rm}(x)\big) = \underbrace{\text{terms of degree} \geq m+1}_{E_m(x)}$$

The vector with the coefficients of the polynomial $E_m(x)$ will be denoted e_m.

The number of free parameters should be compared with the number of coefficients in $E_m(x)$. The number of terms in $E_m(x)$ can be very large and therefore, $E_m(x)$ is truncated at some additional order m_h. That is, $m+m_h$ is the maximal order of the terms in the HJB that is suppressed.

The parameter excess C_{pe} will then be given by

$$C_{\text{pe}} = \binom{m+n}{n} - \binom{m+m_h+n}{n} \tag{13}$$

If C_{pe} is larger than zero, *i.e.*, if the number of free parameters is larger than the number of coefficients, and if the parameters enter the problem in an appropriate way, it is sometimes possible to zero some of the higher order coefficients exactly using an equation solver. For scalar problems of orders that are not too high, this approach seems to work rather well. However, for larger scalar problems and non-scalar problems, it is quite common that the equation solver requires a huge amount of time or that no solution is returned at all.

Therefore, another approach is used where the higher order coefficients in the HJB are minimized using a numerical optimization routine. The advantage with this approach is that if a set of coefficients exists such that the higher order terms are zeroed, the optimization often finds them. On the other hand, if no such solution exists, for example in a case when the parameter excess C_{pe} is negative, *i.e.*, the number of parameters are fewer than the number of terms in $E_m(x)$, the optimization will still try to give a solution with $|e_m|^2$ as small as possible.

The recursive equation (10) is equivalent to the underdetermined linear system of equations

$$A_m Y_m = b_m \tag{14}$$

where

$$Y_m = (y_{R,3}, y_{Q,1}, y_{R,4}, y_{Q,2}, \ldots, y_{R,m}, y_{Q,m-2})^T$$

$$A_m = \begin{bmatrix} A_{m,1}(y_{R,2}) & 0 & \ldots & 0 \\ 0 & A_{m,2}(y_{R,2}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{m,m}(y_{R,2}) \end{bmatrix},$$

$$b_m = \begin{bmatrix} b_{m,1}(y_{R,2}) \\ b_{m,2}(y_{R,2}, y_{R,3}, y_{Q,1}) \\ \vdots \\ b_{m,m}(y_{R,2}, \ldots, y_{R,m-1}, y_{Q,m-3}) \end{bmatrix}$$

and $y_{R,i}$ and $y_{Q,i}$ are the unknown coefficients in $R_V^{[i]}(x)$ and $Q_V^{[i]}(x)$, respectively. The vector $y_{R,2}$ contains the coefficients in $R_V^{[2]}$ which is P, and is therefore computed using the standard ARE. The matrices $A_{m,i}(\cdot)$ and $b_{m,i}(\cdot)$ are functions determined by the left-hand and right-hand side of (9), respectively.

The optimization problem is then formulated as

$$\begin{aligned} \min_{Y_m} \quad & |e_m(Y_m)|^2 \\ \text{s.t.} \quad & A_m Y_m = b_m \end{aligned}$$

The optimization problem can be solved either as constrained or unconstrained. To motivate this fact, let $i=3$.

Then, if the coefficients in the denominator $y_{Q,1}$ are considered as parameters, it was shown in (10) that only linear equations need to be solved in order to obtain the coefficients in the nominator $y_{R,3}$ expressed in terms of $y_{Q,1}$. Furthermore, in Lemma 4 it was shown that the solution is unique. Repeating this procedure for the higher indexes means that (14) is solved recursively. The coefficients $y_{R,i}$, $i = 3, \dots, m$, will only depend on $y_{Q,j}$, $j \leq i-2$. The result is the unconstrained optimization problem

$$\min_{y_Q} \quad |e_m(y_Q)|^2 \tag{15}$$

where y_Q is the concatenation of $y_{Q,i}$, $i = 1, \dots, m-2$. In this thesis, mostly the unconstrained approach has been used. However, there might be structural benefits with keeping the constraints, since simpler expressions in e_m will be obtained in that case.

This optimization problem (15) is polynomial, and normally it becomes be non-convex. For small m, n, and m_h it is possible to solve the problem globally using for example sum-of-squares relaxation using for example YALMIP, see Löfberg (2004). In this case it is quite important to rewrite the problem as

$$\begin{aligned} \min_{y_Q, Y_{\text{sos}}} \quad & |Y_{\text{sos}}|^2 \\ \text{s.t.} \quad & e_m(y_Q) = Y_{\text{sos}} \end{aligned}$$

where Y_{sos} are extra variables, one for each term in e_m, used to reduce the maximal order of the functions involved.

However, for medium-sized m, n and m_h, the expressions in e_m becomes large and rather involved. In this case, the global methods seem to be too computationally demanding, and it is necessary to search for a local minimum instead. Then the initial conditions become important. The good news is that numerical experience shows that often a local optimum can be found such that the corresponding approximant is good (depending on the choices made in the next section).

Note that the unconstrained problem (15) is a nonlinear least-squares problem which in many solvers, such as the solver in MAPLE, can be utilized to reduce the computation time.

3.4 Design Choices in the Optimization

There are several different design choices for the optimization problem (15). One is the choice of the denominator of V_r, discussed in Section 3.2. Below some other choices are mentioned.

The Order of f, g and l The first design choice is the order of the functions that describe the model and the cost function. It is possible to have arbitrary orders of each of them as long as the orders are larger than or equal to $m-1$. Otherwise, the power series solution up to the desired order m will not be correct as was shown in Lukes (1969). The standard choice in the simulations presented in this thesis, see Section 4, have been to truncate at order $m + m_h$.

Order of E_m Another design parameter is the truncation degree of $E_m(x)$. From (9), it follows that the maximal degree that may show up in $E_m(x)$ is given by

$$\max\left(m_l + 4(m-2), m_f + 4m - 7, 2m_g + 4m - 6\right)$$

where m_f, m_g and m_l are the orders of f, g and l, respectively. This number is often quite large and therefore it is necessary to truncate $E_m(x)$ to obtain a solvable optimization problem. The truncation also implies that higher order terms in $E_m(x)$ are considered irrelevant.

The most basic choice of m_h is to choose the value for which the parameter excess switches from positive to negative. Then, the optimal e_m will be zero, *i.e.*, all higher terms are zeroed. However, in many cases it is possible to obtain better approximations by increasing m_h, which normally will lead to that e_m will not equal zero.

Actually, in some cases one can gain a lot by increasing m_h, without changing m. It means that information from higher order terms are included in the lower order approximant.

Initial Values Since the optimization problem (15) mostly is non-convex, the choice of initial values is important. In this thesis, the default choice is to generate a number of vectors with uniformly distributed random numbers in the interval $[-3, 3]$. The optimization problem (15) is then solved for each of them as initial guess, and the best solution is chosen as the optimum. The number of different sets can be chosen, but the standard choice is two.

3.5 Stability

One of the major objectives for a controller is to stabilize the system. The controller obtained from the method in this chapter can be shown to yield stability at least locally in a neighborhood as shown in the following theorem.

Theorem 7. Consider a nonlinear system in the form (2) that satisfy Assumptions 1 and 2. Let $V_{rm}(x)$ solve (9) up to order m and let the corresponding control law be given by (7b). Then this feedback law will stabilize the system locally in a neighborhood of the origin, and the cost function $V_{rm}(x)$ will be a Lyapunov function for the closed-loop system

$$\dot{x} = f(x) - g(x)R^{-1}g(x)^T V_{rm;x}(x)^T$$

Proof. First note that the cost function $V_{rm}(x)$ can be expanded around zero yielding

$$V_{rm}(x) = \frac{1}{2}x^T P x + V_{rm,h}(x)$$

and the time derivative of $V_{rm}(x)$ using the feedback law (7b) becomes

$$\begin{aligned} \dot{V}_{rm} = V_{rm;x}(f - gR^{-1}g^T V_{rm;x}^T) = \\ - l - \frac{1}{2}V_{rm;x} g R^{-1} g^T V_{rm;x}^T + \frac{E_m}{V_{rm;x,d}^2} \end{aligned}$$

where $V_{rm;x,d}$ is the denominator of $V_{rm;x}$.

The series expansion of the first two terms in the expression above is given by

$$l + \frac{1}{2}V_{rm;x} g R^{-1} g^T V_{rm;x}^T = \frac{1}{2}x^T(Q + PBR^{-1}B^T P)x + \mathcal{O}(x)^3$$

and since $E_m(x)$ contains terms beginning with order $m+1$, it follows that for x in a neighborhood of the origin the optimal return function will satisfy $V_{rm}(x) > 0$ and $\dot{V}_{rm}(x) < 0$. That is, the function $V_m(x)$ will be a Lyapunov function for the closed-loop system and $u = -R^{-1}g(x)^T V_{rm;x}(x)^T$ is a stabilizing control law.

Hence, the controller stabilizes the system locally around the origin, similar as for the power series approximation, see Lukes (1969). However, as for the power series method, no estimate of the region of attraction is obtained by the method. If such a estimate is desired, one has to use some other method, see for example Vannelli and Vidyasagar (1985).

4. EXAMPLES

In this section three examples are presented. The first example is a scalar problem which comes from Navasca (1996). In this example the cost function includes a barrier function on the state. The second example is a multivariable problem. The second one is a physical system, namely a nonlinear phase-locked loop.

4.1 A Scalar Problem

The considered system is given by

$$\dot{x} = (1 + x)u$$

which is a stabilizable system around the origin. The cost function is chosen as

$$l(x) = \ln(1 + x)^2$$

The corresponding optimal control problem can be solved explicitly and the optimal cost function becomes

$$V(x) = \frac{\sqrt{2}}{2} \ln(1 + x)^2$$

while the optimal feedback law is given by

$$u(x) = -\sqrt{2} \ln(1 + x)$$

In the scalar case it is most often possible to solve for extra coefficients in the HJB exactly. This fact has been exploited in this example, where a fifth order rational approximation has been computed. By using the three extra terms in the denominator, three additional terms in the HJB has been zeroed. The obtained solution will therefore have the same taylor series as the optimal solution up to order eight. Actually, the series expansions are the same with three decimals accuracy up to the 14:th degree. The functions f, g and l are truncated after the eighth degree. The result can be seen in Figure 1. The same figure also shows the rational approximation with a denominator where the highest order term is fixed. As can be seen the difference is rather small.

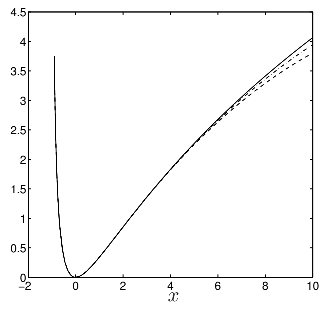

Fig. 1. A comparison between V^* (solid) and two rational approximations. The dash-dotted line corresponds to the approximation with free denominator and the dashed line has fixed highest order term in the denominator.

In Figure 2, a truncated power series solution of order five and a rational approximation with fixed denominator have also been included in the comparison. As can be seen, these solutions are substantially worse than the earlier rational approximations. However, the rational approximation with fixed coefficients are better than the truncated power series solution.

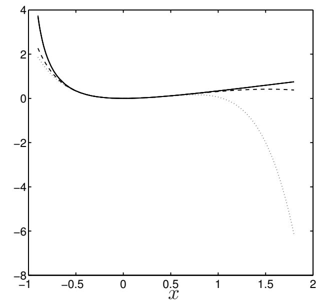

Fig. 2. A comparison between the optimal cost (solid), the three different rational approximations and the truncated power series. The rational approximation with free denominator and with fixed highest degree term are indistinguishable from the optimal solution. The dashed line is the rational with fixed denominator and the dotted line is the power series.

Concerning stability it can also be shown that the rational approximation is substantially better than the power series solution. The region in which the rational approximation with free denominator is stabilizing the system is $x_0 \in [-0.99, 21]$, while the truncated power series solution only stabilizes the system in the region $x_0 \in [-0.99, 0.8]$.

In the last two figures another advantage with the rational approximations is illustrated. Here a higher order approximation of order 8 plus the 6 free parameters in the denominator. For the rational approximation a higher order often give a better approximation in a larger region than a lower one, which happens for this example as seen in Figure 3. However, for truncated power series a higher order most often only yields a better fit with the optimal solution locally around the origin and outside this region an even worse fit is obtained as can be seen in Figure 4.

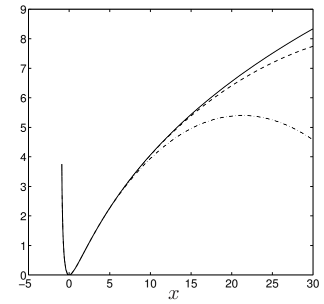

Fig. 3. A comparison between the optimal cost (solid) and two rational approximations of different order. The dashed line is a eighth order approximation and the dashed-dotted line is a fifth order one.

4.2 A Phase Lock Loop

Consider a model for a nonlinear phase lock loop (PLL). The dynamics for the system can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin(x_1) + u$$

The cost function $l(x)$ is chosen as

$$l(x_1, x_2) = \frac{1}{2}x_1^2 + 2x_1x_2 + x_2^2 + x_1\sin(x_1)$$

which makes it possible to find an explicit solution as

$$V(x_1, x_2) = 2(1 - \cos(x_1)) + x_1x_2 + x_2^2$$

For this optimal control problem, the fourth order rational approximation is computed. As the comparison in Figure 5 shows, the rational approximation describes the optimal solution rather well. In this example, the terms of order six and below of the power series of f, g and l are included in $E_m(x)$ and the HJB is also truncated at order six, *i.e.*,

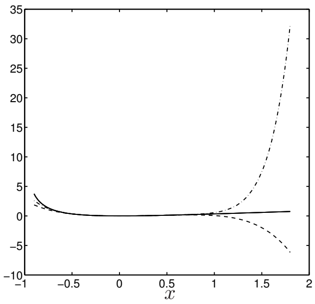

Fig. 4. A comparison between the optimal cost (solid) and four different approximations. Two of them are rational with free denominators but with different order (4 or 8). They are hidden behind the optimal solution. The other approximations are truncated power series, one of order 5 (dashed) and order 8 (dash-dotted).

two orders higher than m. The corresponding value of e_m^2 became $3 \cdot 10^{-13}$.

In the same figure, also a truncated power series solution of order four is presented. The improvement by using the rational approximation is quite large, which is even more clear in Figure 6 where the the error of the rational approximation is compared with the error for the truncated power series solution.

In Figure 7, another comparison of errors is shown. The error that bends upwards and which has the smallest amplitude corresponds to a rational approximation of order 4 with the denominator chosen as

$$Q_V(x_1, x_2) = 1 + \tfrac{1}{6}x_1^2 + \tfrac{1}{6}x_1x_2 + \tfrac{1}{6}x_2^2$$

The other error corresponds to a truncated power series. As can be seen, the rational approximation is still better than the power series but worse than the rational approximation with free denominator (which could be seen in the Figure 6).

Figure 8 shows the error for two higher order approximations. The order of the rational approximation has been increased to six and the truncated power series approximation is of order eight. For the rational approximation, $m_h = 8$ has been used and the functions are also truncated at $m + 8$. It means that the order of rational function is not that high, but information about the model and the cost function up to order 14 is included.

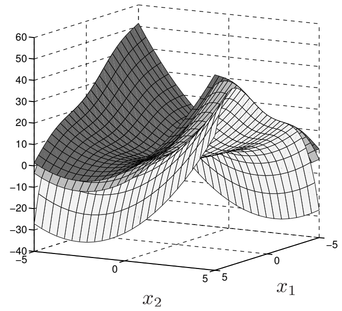

Fig. 5. A comparison between V (dark), the rational approximation (medium dark) and truncated power series (light) the applied to the PLL example.

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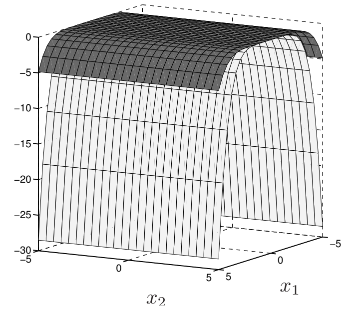

Fig. 6. A comparison between the errors of the rational approximation (dark) and of the truncated power series (light) for the PLL system.

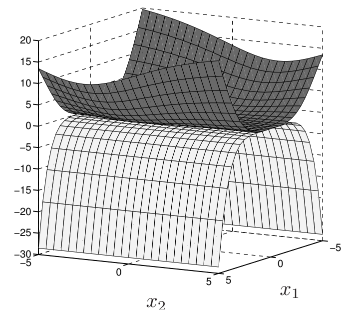

Fig. 7. A comparison between the errors of the rational approximation with fixed denominator (dark) and of the truncated power series (light) for the PLL system.

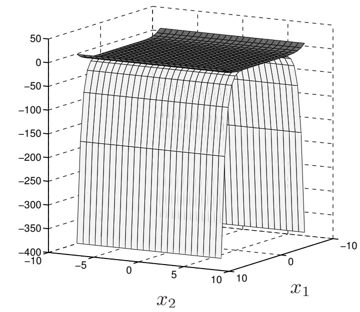

Fig. 8. A comparison between the errors of the rational approximation of order 6 (dark) and of the truncated power series of order 8 (light) for the PLL system.

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