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Discrete-time dynamic feedback linearization of a VTOL using observed states

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Abstract: This paper addresses the trajectory tracking problem for a remotely controlled quadrotor vertical take off and landing aircraft (VTOL), under the restriction that only the inertial coordinates are available for measurement. The problem is solved in two steps: first, a discrete-time local exponential observer is designed which allows estimating the roll and pitching angles as well as all the velocities of the VTOL; Second, a discrete-time dynamic linearizing controller is proposed and the VTOL actual states variables are replaced by their corresponding estimates. It is shown that a kind of separation principle holds, in the sense that exponential convergence to the prescribed trajectory is preserved. Real-time experiments show that the proposed observer-controller scheme exhibits good performance.

1. INTRODUCTION

The VTOL is a kind of unmanned aerial vehicle. It has been studied by many researchers, and different control strategies have been proposed over the past twenty years, see e.g. Hauser et al. [1992]. However only few of them have been tested experimentally. One of the main obstacles to perform real-time experiments is that the control law needs to feedback all the VTOL states, but not all of them are available for measuring. Due to this obstacle many researchers only have tested their results through simulations. The control strategies for VTOL can be divided in continuous-time and discrete-time designs. Almost all of the strategies have been proposed in continuous-time, but few of them have been implemented in real-time experiments. For instance, the continuous-time control strategies given by Kendoul et al. [2006], Park et al. [2005], Samir et al. [2004], Tayebi and McGilvray [2004] have been tested in real-time experiments, but two aspects are not taken into account in the formal analysis: a) all of them have used a zero order hold, because they used a digital computer to implement the control law; b) the velocities are estimated by using an approximate derivative from the measured position, and a low pass filter is used to diminish the inherent noise. These aspects are convenient for testing the strategies by choosing an appropriate sampling period, but the convergence is not formally ensured. On the other hand, the low pass filter introduces new dynamics, and again the convergence is not formally ensured. The issues a) and b) above can be analyzed from two points of view: 1) by using a type of observer to estimate those states that are not available for measuring, such that the convergence to the actual states is guaranteed; 2) by analyzing the control scheme in a discrete-time approach.

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The control of nonlinear systems when not all states are available for feedback has been studied by many researchers. In discrete-time, theoretical results can be found in Lin and Byrnes [1994]. This work gives necessary and sufficient conditions for existence of a local exponential observer, and can be seen as the counter part of continuous-time case given by Xia and Gao [1988]. Although the theoretical results are available for discrete-time nonlinear systems, there are few formal results for VTOL. For instance Guisser et al. [2006] and Lozano et al. [2006]. The former is an interesting work, but the scheme is not tested in real-time experiments. The latter uses a kind of discrete-time observer for angular dynamics, but it is not clear how the inertial velocities have been estimated.

This paper contributes in three directions to real-time control of a VTOL: 1) sufficient conditions are given for existence of a local exponential observer. It needs only the inertial coordinates to estimate the roll and pitch angles, and all velocities; 2) It is shown that the mathematical model of VTOL is linearizable by regular feedback after a proper dynamic extension. It is designed as if all states were measured; then the actual states are replaced by their estimates. After that, sufficient conditions are given to satisfy a kind of separation principle in the sense that convergence is ensured; 3) the strategy is tested in real-time. For the coordinates measuring, the positioning system reported in Rejón and Aranda-Bricaire [2007] is used. The experimental VTOL prototype used in this work is the Draganflyer produced by Draganfly Innovations Inc. shown in fig. 1.

This paper is organized in five sections. In section 2, the model and basic properties in discrete-time are given. In section 3, the main results are presented; namely, the local exponential observer and the linearizing control law are designed, and a kind of separation principle is derived. In section 4, some real-time experiments are presented. Finally in section 5 some conclusions are given.

Fig. 1. The vertical take off and landing aircraft (http://www.rctoys.com).

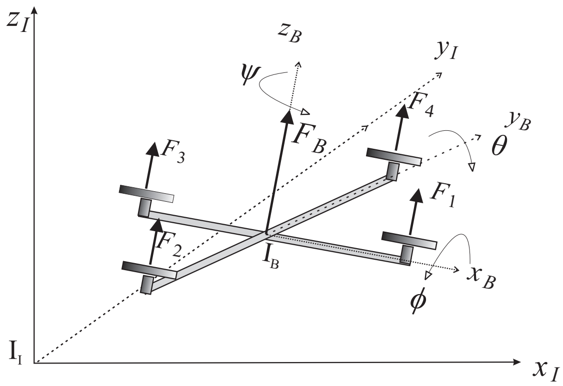

Fig. 2. Schematic free-body diagram of four-rotor mini-helicopter.

2. THE MODEL AND BASIC PROPERTIES.

2.1 Continuous-time model.

A schematic free-body diagram of VTOL is given in fig. 2. The dynamic model is quite standard, the reader can see e. g. Castillo et al. [2007], Madani and Benallegue [2006], Samir et al. [2004], Tayebi and McGilvray [2004]. Is this paper the yaw angle ψ is assumed to be zero and the model includes the remote-control parameters. Then the simplified model is given by

$$\begin{aligned} \ddot{x} &= \tfrac{u_1}{m}\sin\theta\cos\phi \\ \ddot{\theta} &= -\tfrac{\alpha_1}{I_y}\dot{\theta} - \tfrac{\alpha_2}{I_y}\theta + \tfrac{\alpha_3}{I_y}u_3 \\ \ddot{y} &= -\tfrac{u_1}{m}\sin\phi \\ \ddot{\phi} &= -\tfrac{\beta_1}{I_x}\dot{\phi} - \tfrac{\beta_2}{I_x}\phi + \tfrac{\beta_3}{I_x}u_2 \\ \ddot{z} &= \tfrac{u_1}{m}\cos\theta\cos\phi - g \end{aligned} \tag{1}$$

where u_i, $i = 1, 2, 3$ are the control inputs, m, I_x and I_y are the mass and moments of inertia of the VTOL. $(x,\, y,\, z)$ are the cartesian coordinates and θ, ϕ are the pitch and roll angles respectively. The parameters α_i, β_i, $i = 1, 2, 3$ scale remote-control input voltage and the input force. The constant $g = 9.81$ m/s^2 is the gravity acceleration. The inputs satisfy the following equations

$$u_1 = \sum_{i=1}^{4} F_i,\ u_2 = l\,(F_3 - F_1)\,,\ u_3 = l\,(F_4 - F_2)\,,$$

where F_i is the thrust force produced by i-rotor and l is the distance from the rotors to the center of mass of the aircraft. Since the yaw angle is zero, the four-rotor craft has five degrees of freedom; however only three of them are actuated. Let us define the states variables

$$\begin{aligned} &x_1 = x,\ x_2 = \dot{x},\ x_3 = \theta,\ x_4 = \dot{\theta}, \\ &x_5 = y,\ x_6 = \dot{y},\ x_7 = \phi,\ x_8 = \dot{\phi}, \\ &x_9 = z,\ x_{10} = \dot{z}, \end{aligned}$$

and the vector

$$\chi = [x_1,\, x_2,\, x_3,\, x_4,\, x_5,\, x_6,\, x_7,\, x_8,\, x_9,\, x_{10}]^{\mathsf{T}}\,.$$

Then the model (1) takes the form

$$\dot{\chi} = F_c\,(\chi, u) = f(\chi) + g(\chi)u \tag{2}$$

where $u = [u_1,\, u_2,\, u_3]^{\mathsf{T}}$,

$$\begin{aligned} f(\chi) = \Big[&x_2,\, 0,\, x_4,\, -\tfrac{\alpha_1}{I_y}x_4 - \tfrac{\alpha_2}{I_y}x_3,\, x_6 \\ &0,\, x_8,\, -\tfrac{\beta_1}{I_x}x_8 - \tfrac{\beta_2}{I_x}x_7,\, x_{10},\, g \Big]^{\mathsf{T}}, \end{aligned} \tag{3}$$

$$\begin{aligned} g(\chi) &= [\, g_1(\chi)\ g_2(\chi)\ g_3(\chi)\,]\,, \\ g_1(\chi) &= \left[0\ \tfrac{c_7 s_3}{m}\ 0\ 0\ 0\ \tfrac{-s_7}{m}\ 0\ 0\ 0\ \tfrac{c_7 c_3}{m} \right]^{\mathsf{T}}, \\ g_2(\chi) &= \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ \tfrac{\beta_3}{I_x}\ 0\ 0 \right]^{\mathsf{T}}, \\ g_3(\chi) &= \left[0\ 0\ 0\ \tfrac{\alpha_3}{I_y}\ 0\ 0\ 0\ 0\ 0\ 0 \right]^{\mathsf{T}}, \end{aligned} \tag{4}$$

where c_i means $\cos x_i$ y s_i means $\sin x_i$.

Taking $F_c\,(\chi_o, u_o) = 0$ we can find the equilibria of the VTOL. Making some algebra, the equilibria are given by

$$\begin{aligned} x_o &= [x_{1o},\, 0,\, 0,\, 0,\, x_{5o},\, 0,\, 0,\, 0,\, x_{9o},\, 0]^{\mathsf{T}}\,, \\ u_o &= [mg,\, 0,\, 0]^{\mathsf{T}}\,. \end{aligned} \tag{5}$$

In others words, any point in the space becomes an equilibrium point if the roll and pitch angles, and all velocities vanish.

2.2 Discrete-time model.

Consider a discrete-time system

$$x\,((k+1)\,T) = F_d\,(x\,(kT)\,, u\,(kT))\,, \tag{6}$$

where $T > 0$ is the sampling period. In discrete-time analysis, the input u is considered to be constant $\forall t \in [kT, (k+1)T)$, $k \in \mathbb{Z}_+$. In the analysis, the abridged notation $x^+ = x\,((k+1)\,T)$, $x = (kT)$ is used. Then the discrete-time model (6), can be written as $x^+ = F_d\,(x, u)$.

Using the Euler approximation for continuous-time model (2), the discrete-time model of the VTOL is given by

$$\chi^+ = F_d\,(\chi, u) = \chi + T\,[f(\chi) + g(\chi)u]\,, \tag{7}$$

where $f(\chi)$ and $g(\chi)$ are derived by (3), (4).

From condition $F_d\,(\chi_o, u_o) = \chi_o$, we can find the equilibria of VTOL. Making some algebra $x_{2o} = x_{4o} = x_{6o} = x_{8o} = x_{10o} = 0$ is obtained, and as consequence $x_{3o} = x_{7o} = 0$ also is obtained. Therefore the equilibria of the discrete-time model of the four-rotor aircraft are the same as (5).

2.3 Basic properties.

Let us define the output function as those states that are considered as measurable. In this application we are considering as measurable states only the cartesian coordinates. Then the output is given by

$$\left.\begin{aligned} \gamma_1 &= h_1\,(\chi) = x_1, \\ \gamma_2 &= h_2\,(\chi) = x_5, \\ \gamma_3 &= h_3\,(\chi) = x_9. \end{aligned}\right\} \tag{8}$$

The output can be expressed as

$$\gamma = C\chi,$$

where

$$C = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \end{bmatrix}.$$

According to Lin and Byrnes [1994], if the first order approximation of model (7) has a detectable (A, C) pair, then there exists a discrete-time local exponential observer for discrete-time nonlinear system (7).

The first order approximation is given by

$$\begin{aligned} \chi^+ &= A_d\chi + B_d u, \\ \gamma &= C\chi, \end{aligned} \tag{9}$$

where

$$A_d = \left[\frac{\partial F_d(\chi, u)}{\partial \chi}\right]_{\chi_o, u_o} \in \mathbb{R}^{10\times 10},$$

$$B_d = \left[\frac{\partial F_d(\chi, u)}{\partial u}\right]_{\chi_o, u_o} \in \mathbb{R}^{10\times 3},$$

$$A_d = \begin{bmatrix} 1 & T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & Tg & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{T\alpha_2}{I_y} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -Tg & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{T\beta_2}{I_x} & a_{88} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&\frac{T}{m} \\ 0&0&0&0&0&0&0&\frac{T\beta_3}{I_x}&0&0 \\ 0&0&0&\frac{T\alpha_3}{I_y}&0&0&0&0&0&0 \end{bmatrix}^{\mathsf{T}},$$

$a_{44} = \frac{T(I_y-\alpha_1)}{I_y}$, $a_{88} = \frac{T(I_x-\beta_1)}{I_x}$ are parameters.

It is not difficult to check that the first order approximation (9) satisfies the full rank observability condition $rank(\mathcal{O}) = 10$. This means that there exists a local exponential observer for discrete-time model of the VTOL.

In the next section a local exponential observer will be explicitly designed.

3. MAIN RESULTS.

In this section a local exponential observer is designed, it needs only the cartesian coordinates to estimate the roll and pitch angles, and all velocities. In order to design a linearizing control law, a dynamic extension is introduced. The dynamic extension consists on adding two sampling delays in the thrust control. The dynamic control law is designed as if all states were measurable. Then the VTOL states are replaced by their estimates. Finally, a kind of separation principle is proven, in the sense that the closed-loop system converges to desired trajectory.

3.1 Local exponential observer design.

From the theoretical results given by Lin and Byrnes [1994]. If the discrete-time observer is not of exponential type, then the separation principle can not be ensured. A counter-example in which the separation principle is not satisfied can be found in Sundarapandian [2005]. Therefore we aim to design a discrete-time local exponential observer for VTOL. The discrete-time observer needs to satisfy two conditions:

A) If $\chi_o = \hat{\chi}_o$, then $\chi = \hat{\chi}$, $\forall k \in \mathbb{Z}_+$, for all admissible u,

B) Define the error observer by $e = \chi - \hat{\chi}$. There exists an open neighborhood U of the origin of such that

$$\|e\| \triangleq \|\chi - \hat{\chi}\| \le M a^k \|\chi_o - \hat{\chi}_o\|,$$

$\forall k \in \mathbb{Z}_+$ and $e_o \in U$, for some positive constants M and $0 < a < 1$.

If the observer is defined as a copy of model (7) plus output error injection, then the condition A is satisfied. In the sequel, additional conditions are derived to comply with condition B.

To begin with, note that the model (7) can be rewritten as

$$\begin{aligned} \chi^+ &= \chi + T\left[f(\chi) + g(\chi)u\right] \\ &= A_d\chi + B_d\left[u - u_o\right] + R_d(\chi, u_1), \end{aligned}$$

where $R_d(\chi, u_1) = \chi + T\left[f(\chi) + g(\chi)u\right] - A_d\chi - B_d\left[u - u_o\right]$,

$$R_d(\chi, u_1) = \left[0,\, R_{d_2}(\cdot),\, 0,\, 0,\, 0,\, R_{d_6}(\cdot),\, 0,\, 0,\, 0,\, R_{d_{10}}(\cdot)\right]^{\mathsf{T}},$$

$$R_{d_2}(\cdot) = T\left(\tfrac{u_1}{m}s_3c_7 - gx_3\right),$$

$$R_{d_6}(\cdot) = T\left(-\tfrac{u_1}{m}s_7 + gx_7\right),$$

$$R_{d_{10}}(\cdot) = T\tfrac{u_1}{m}\left(c_3c_7 - 1\right).$$

Note that $R_d(\chi, u_1)$ is smooth on χ and u_1, it implies that

$$\|R_a(\chi, u_1)\| \le Tg\sqrt{x_3^2 + x_7^2} \le Tg\|\chi\|. \tag{10}$$

Let us define the observer as

$$\begin{aligned} \hat{\chi}^+ &= \hat{\chi} + T\left[f(\hat{\chi}) + g(\hat{\chi})u\right] + L_d(\gamma - \hat{\gamma}), \\ &= A_d\hat{\chi} + B_d\left[u - u_o\right] + R_d(\hat{\chi}, u_1) + L_dC(\chi - \hat{\chi}). \end{aligned} \tag{11}$$

Recall that $e = \chi - \hat{\chi}$, then $e^+ = \chi^+ - \hat{\chi}^+$ is given by

$$e^+ = F_{de}(\cdot) = (A_d - L_dC)\,e + \Gamma_d(\chi, \hat{\chi}, u_1), \tag{12}$$

where

$$\Gamma_d(\chi, \hat{\chi}, u_1) = R_d(\chi, u_1) - R_d(\hat{\chi}, u_1).$$

In the next Proposition the main result of this subsection is given.

Proposition 1. Consider the discrete-time VTOL system (7) and the family of discrete-time nonlinear systems (11), parameterized by the matrix $L_d \in \mathbb{R}^{10\times 3}$. Then there exists a matrix $L_d \in \mathbb{R}^{10\times 3}$ such that the discrete-time nonlinear system (11) is a local exponential observer for discrete-time VTOL.

Proof. As a first step, we need to find the equilibrium point of observer (12). From the condition $F_{de}(\cdot) = e_o$ and $\hat{\chi} = \chi - e$, we can find the equilibria for error dynamics (12). Note that the point $e_o = 0$ is a solution for this equation. Also, note that in a neighborhood (e_o, χ, u_{1o}), the Jacobian matrix of error expression (12) is given by

$$\left[\frac{\partial F_{de}(\cdot)}{\partial e}\right]_{(e_o, \chi, u_{1o})} = (A_d - L_dC) - I,$$

where $I \in \mathbb{R}^{10\times 10}$ is the identity matrix. Since the first order approximation is observable, then there exists $L_d \in \mathbb{R}^{10\times 3}$ such that the matrix $A_d - L_dC$ has all its eigenvalues strictly into the unit circle. Therefore, the matrix $(A_d - L_d C) - I$ is nonsingular. By Inverse Function Theorem the point $e_o = 0$ is an isolated equilibrium point. Moreover in the same neighborhood, the nonlinear terms given by $\Gamma_d(\cdot)$ satisfy

$$\left\| \Gamma_d(\cdot) \right\| \le Tg\sqrt{e_3^2 + e_7^2} \le Tg \left\| e \right\|. \quad (13)$$

In order to prove that the discrete-time model satisfies condition B, take into account that the matrix $A_d - L_d C$ has all its eigenvalues strictly into the unit circle. Then there exist a discrete-time Lyapunov function $V = e^\intercal P_1 e$ that satisfies $\lambda_{d_{11}} \|e\|^2 \le V \le \lambda_{d_{12}} \|e\|^2$, where $P_1 = P_1^\intercal > 0$, $\lambda_{d_{11}} = \lambda_{\min}(P_1)$, $\lambda_{d_{12}} = \lambda_{\max}(P_1)$ and P_1 is solution of the discrete-time Lyapunov equation

$$(A_d - L_d C)^\intercal P_1 (A_d - L_d C) - P_1 = -Q_1,$$

for any $Q_1 = Q_1^\intercal > 0$. Let us compute the forward difference of V along the trajectories of observed error expression (12), taking into account $-\lambda_{d_{13}} \|e\|^2 \ge -e^\intercal Q_1 e \ge -\lambda_{d_{14}} \|e\|^2$, where $\lambda_{d_{13}} = \lambda_{\min}(Q_1)$, $\lambda_{d_{14}} = \lambda_{\max}(Q_1)$ and making some algebra as in Rejón and Aranda-Bricaire [2006], we obtain

$$\begin{aligned} \triangle V &\le -\lambda_{d_{13}} \|e\|^2 + 2\lambda_{d_{12}} Tg \|e\|^2 + \lambda_{d_{12}} (Tg)^2 \|e\|^2 \\ &= -\left(\lambda_{d_{13}} - 2\lambda_{d_{12}} Tg - \lambda_{d_{12}} (Tg)^2\right) \|e\|^2 . \end{aligned}$$

If

$$\lambda_{d_{13}} - 2\lambda_{d_{12}} Tg - \lambda_{d_{12}} (Tg)^2 > 0, \quad (14)$$

is satisfied then by Theorem 28 given in pp. 267 Vidyasagar [1993] for discrete-time systems, it follows that the observer error converges locally exponentially to zero. Therefore the system (11) is a local exponential observer for the discrete-time model of the VTOL. ■

3.2 Dynamic control law in discrete-time.

Consider the discrete-time VTOL model (7) and its output given by (8).

From the method to obtain a static feedback linearizing controller given by Aranda-Bricaire et al. [1996] and Monaco and Normand-Cyrot [1987], it is possible to check that with two forward shifts the output γ^{+2} is affected by the control u_1. Also it is possible to check that decoupling matrix $D(\chi, u) = \frac{\partial \gamma^{+2}(\chi, u)}{\partial u}$ is singular. Therefore, a linearizing control law by static feedback can not be obtained. This problem can be solved using a suitable dynamic extension. This dynamic extension permits to obtain a linearizing control law, such that the closed loop system in new coordinates is linear. The dynamic extension is given by

$$\begin{aligned} u_1 &= \xi_1, \\ \xi_1^+ &= \xi_2, \\ \xi_2^+ &= \bar{u}_1, \end{aligned} \quad (15)$$

where $\bar{u}_1$ is a new input. Define the vectors

$$\begin{aligned} \xi &= [\xi_1, \, \xi_2]^\intercal, \\ X &= [\chi^\intercal, \, \xi^\intercal]^\intercal, \\ \bar{u} &= [\bar{u}_1, \, \bar{u}_2, \, \bar{u}_3]^\intercal = [\bar{u}_1, \, u_2, \, u_3]^\intercal. \end{aligned}$$

By using X and $\bar{u}$ the augmented discrete-time model has the form

$$X^+ = F_d(X, \bar{u}) = \bar{f}(X) + \bar{g}(X)\bar{u}, \quad (16)$$

where

$$\bar{f}(X) = \begin{bmatrix} \chi + T\left[f(\chi) + g_1(\chi)\xi_1\right] \\ \xi_2 \\ 0 \end{bmatrix},$$

$$\bar{g}(X) = \begin{bmatrix} 0 & Tg_2(\chi) & Tg_3(\chi) \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Applying again forward shifts to the outputs (8), the following set of expressions is obtained

$$\gamma^{+4}(X, \bar{u}) = b(X) + a(X, \bar{u}) \quad (17)$$

where

$$\begin{aligned} a(X, \bar{u}) &= [a_1(\cdot), \, a_2(\cdot), \, a_3(\cdot)]^\intercal, \\ b(X) &= [b_1(X), \, b_2(X), \, b_3(X)]^\intercal, \\ b_1(X) &= x_1 + 4Tx_2 + 3T^2 \tfrac{\xi_1}{m} s_3 c_7 + 2T^2 \tfrac{\xi_2}{m} s_3^+ c_7^+, \\ b_2(X) &= x_5 + 4Tx_6 - 3T^2 \tfrac{\xi_1}{m} s_7 - 2T^2 \tfrac{\xi_2}{m} s_7^+, \\ b_3(X) &= x_9 + 4Tx_{10} + 3T^2 \left(\tfrac{\xi_1}{m} c_3 c_7 - g\right) \\ &\quad + 2T^2 \left(\tfrac{\xi_2}{m} c_3^+ c_7^+ - g\right) - T^2 g, \\ a_1(X, \bar{u}) &= T^2 \tfrac{\bar{u}_1}{m} s_3^{+2} c_7^{+2}, \; a_2(X, \bar{u}) = -T^2 \tfrac{\bar{u}_1}{m} s_7^{+2}, \\ a_3(X, \bar{u}) &= T^2 \tfrac{\bar{u}_1}{m} c_3^{+2} c_7^{+2}, \\ s_i^{+2} &= \sin x_i^{+2}, c_i^{+2} = \cos x_i^{+2}, \\ x_3^{+2} &= \left(1 - T^2 \tfrac{\alpha_2}{I_y}\right) x_3 + T\left(2 - T\tfrac{\alpha_1}{I_y}\right) x_4 + T^2 \tfrac{\alpha_3}{I_y} \bar{u}_3, \\ x_7^{+2} &= \left(1 - T^2 \tfrac{\beta_2}{I_x}\right) x_7 + T\left(2 - T\tfrac{\beta_1}{I_x}\right) x_8 + T^2 \tfrac{\beta_3}{I_x} \bar{u}_2. \end{aligned}$$

The model (16) has a decoupling matrix given by

$$D(X, \bar{u}) = \frac{\partial \gamma^4(X, \bar{u})}{\partial \bar{u}} = \frac{\partial a(X, \bar{u})}{\partial \bar{u}},$$

where

$$|D(X, \bar{u})| = T^{10} \frac{\alpha_3 \beta_3}{m^3 I_x I_y} (\bar{u}_1)^2 \cos x_7^{+2}.$$

If $\bar{u}_1 \neq 0$ and $-\frac{\pi}{2} < x_7^{+2} < \frac{\pi}{2}$ are satisfied, then the decoupling matrix $D(X, \bar{u})$ is not singular.

Note that the dimension of the discrete-time model (16) is $n = 12$. This model has a relative vector degree $r = [4, 4, 4]^\intercal$ and it satisfies $\sum_{i=1}^{3} r_i = 12 = n$. Hence the system (16) is linearizable by regular static state feedback.

For convenience, the design of the linearizing control law is broken into two steps:

Step 1) Since $D(X, \bar{u}) = \frac{\partial a(X, \bar{u})}{\partial \bar{u}}$ is nonsingular and by using the Implicit Function Theorem, the expression

$$a(X, \bar{u}) = v, \quad (18)$$

where $v \in \mathbb{R}^3$, has a solution given by

$$\bar{u} = a^{-1}(X, v). \quad (19)$$

In explicit form

$$\begin{aligned} \bar{u}_1 &= \tfrac{m}{T^2} \sqrt{v_1^2 + v_2^2 + v_3^2}, \\ \bar{u}_2 &= \tfrac{I_y}{T^2 \alpha_3} \arctan\left(\tfrac{v_1}{v_3}\right) - \tfrac{I_y}{T^2 \alpha_3} \left(1 - \tfrac{T^2 \alpha_2}{I_y}\right) x_3 \\ &\quad - \tfrac{I_y}{T \alpha_3} \left(2 - \tfrac{T \alpha_1}{I_y}\right) x_4, \\ \bar{u}_3 &= \tfrac{I_x}{T^2 \beta_3} \arcsin\left(-\tfrac{v_2}{\sqrt{v_1^2 + v_2^2 + v_3^2}}\right) \\ &\quad - \tfrac{I_x}{T^2 \beta_3} \left(1 - \tfrac{T^2 \beta_2}{I_x}\right) x_7 - \tfrac{I_x}{T \beta_3} \left(2 - \tfrac{T \beta_1}{I_x}\right) x_8. \end{aligned}$$

Step 2) Substituting (18) into (17), the next expression is obtained

$$\gamma^{+r}(X,\bar{u}) = b(X) + v = w, \tag{20}$$

where $w \in \mathbb{R}^3$ is an external control given by

$$w_j = y_{d_j}^{+r_j} - \sum_{i=1}^{r_j} c_{ji}\left(\gamma_j^{+(r_j-1)} - y_{d_j}^{+(r_j-1)}\right),$$

where y_{d_j} are bounded enough desired trajectories, c_{ji} are parameters such that form polynomials with eigenvalues strictly into the unit circle. System (20) is linearized by

$$v_j = -b_j(X) + w_j.$$

Finally, the exact linearizing control law is given by

$$\begin{aligned}\bar{u} &= a^{-1}(X,v),\\ v_j &= -b_j(X) + w_j.\end{aligned} \tag{21}$$

Define new coordinates by

$$e_{ji} = h_j^{+(r_j-1)}(X) - y_{d_j}^{+(r_j-1)}, \tag{22}$$

where $i = 1,\cdots,r_j$, $j = 1,2,3$. Let us define the vectors

$$\begin{aligned}Y &= \left[y_{d_1},\cdots,y_{d_1}^{+3},\, y_{d_2},\cdots,y_{d_2}^{+3},\, y_{d_3},\cdots,y_{d_3}^{+3}\right]^\intercal,\\ e_s &= \left[e_{11},\ \cdots,e_{14},\, e_{21},\ \cdots,e_{24},\ e_{31},\ \cdots,\ e_{34}\right]^\intercal.\end{aligned}$$

The closed-loop system (17)-(21) in new coordinates (22) has the form

$$e_s^+ = A_s e_s, \tag{23}$$

where

$$A_s = \operatorname{diag}\left[A_{s_j}\right],\ j = 1,2,3,$$

$$A_{s_j} = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ c_{j1} & c_{j2} & c_{j3} & c_{j4}\end{bmatrix}.$$

Since the coefficients c_{ji} form polynomials which eigenvalues are strictly into the unit circle, then matrix A_{s_j} has all its eigenvalues strictly into the unit circle. Since A_s is a block diagonal, then A_s matrix also has all its eigenvalues strictly into the unit circle. Then there exists a Lyapunov function $V = e_s^\intercal P_2 e_s$ that satisfies

$$\lambda_{d_{21}}\|e_s\|^2 \le V \le \lambda_{d_{22}}\|e_s\|^2$$

where $P_2 = P_2^\intercal > 0$, $\lambda_{d_{21}} = \lambda_{\min}(P_2)$ and $\lambda_{d_{22}} = \lambda_{\max}(P_2)$, and is a solution of the discrete-time Lyapunov equation

$$A_s^\intercal P_2 A_s - P_2 = -Q_2,$$

for any $Q_2 = Q_2^\intercal > 0$. Let us compute the forward difference of V along of errors (23)

$$\triangle V = \left(e_s^+\right)^\intercal P_2 e_s^+ - e_s^\intercal P_2 e_s = -e_s^\intercal Q_2 e_s.$$

From Theorem 28 given in pp. 267 Vidyasagar [1993], it follows that all errors of the closed-loop system (17)-(21) converge locally exponential to zero. As a consequence the closed-loop system converges to the desired trajectory.

3.3 Separation principle.

Before giving the main result. Take into account the following two considerations for functions related with discrete-time VTOL:

a) Since the augmented system is smooth enough on X and $\bar{u}$, it follows that $\left\|a(X,\bar{u}) - a\left(\hat{X},\bar{u}\right)\right\| \le \rho_s\left\|X - \hat{X}\right\|$, $\rho_s > 0$, $X,\hat{X} \in U \subset \mathbb{R}^{12}$.

b) The Variables ξ_1, ξ_2 are available to feedback. Therefore $\hat{X} = [\,\hat{\chi}^\intercal\ \xi^\intercal\,]^\intercal$.

The main result is given in the next Theorem.

Theorem 2. Consider the discrete-time augmented system (16), the local exponential observer (11), and the control law that uses observed states

$$\begin{aligned}\bar{u} &= a^{-1}\left(\hat{X},\hat{v}\right),\\ \hat{v}_j &= -b_j\left(\hat{X}\right) + \hat{w}_j,\\ \hat{w}_j &= y_{d_j}^{+r_j} - \sum_{i=1}^{r_j} c_{ji}\hat{e}_{ji}\\ \hat{e}_{ji} &= \left(\hat{\gamma}_j^{+(r_j-1)} - y_{d_j}^{+(r_j-1)}\right)\end{aligned} \tag{24}$$

where j = 1,2,3, the matrix $A_d - L_d C$ has all its eigenvalues strictly into the unit circle, the c_{ji} coefficients form polynomials whose eigenvalues are strictly into the unit circle, the desired trajectory Y is bounded enough and satisfies $\|Y\| \le \delta_Y$, $\delta_Y > 0$. Then the closed loop system (16)-(24)-(11) converges locally exponentially to desired trajectory.

Proof. In order to prove a kind of separation principle, take into account that the control law $\bar{u}$ can be rewritten as $\hat{v} = a\left(\hat{X},\bar{u}\right)$. Defining $\Gamma_{d_s}(\cdot) = a\left(\hat{X},\bar{u}\right) - a(X,\bar{u})$, we can obtain

$$\begin{aligned}\hat{v} &= a(X,\bar{u}) + \left[a\left(\hat{X},\bar{u}\right) - a(X,\bar{u})\right]\\ &= v + \Gamma_{d_s}(\cdot).\end{aligned}$$

By substituting $\hat{v}$ into (17) the next expression is obtained

$$\begin{aligned}\gamma^{+4} &= b(X) + \hat{v}\\ &= b(X) + v + \Gamma_{d_s}(\cdot).\end{aligned} \tag{25}$$

From considerations, a) and b) above we can see

$$\begin{aligned}\left\|\Gamma_{d_s}(\cdot)\right\| &= \left\|a\left(\hat{X},\bar{u}\right) - a(X,\bar{u})\right\| \le \rho_s\left\|X - \hat{X}\right\|\\ &= \rho_s\left\|\chi - \hat{\chi}\right\|.\end{aligned}$$

Since the observer (11) is locally exponential, then there exists $\delta_e > 0$ such that $\|e\| \le \|\chi_o - \hat{\chi}_o\| \le \delta_e$. Therefore

$$\left\|\Gamma_{d_s}(\cdot)\right\| \le \rho_s\|e\| \le \rho_s\delta_e. \tag{26}$$

Note that the closed-loop system (17)-(24) can be seen as a perturbed system, with vanishing perturbation. Using the change of coordinates (22), the closed-loop system $\gamma^{+4} = b(X) + v$ can be expressed as $e_s^+ = A_s e_s$, therefore

$$e_s^+ = A_s e_s + \Gamma_{d_s}(\cdot). \tag{27}$$

In order to prove that all errors of the closed-loop system (17)-(24)-(11) converge to zero, let us recall that A_s and $A_d - L_d C$ are matrices whose eigenvalues are into the unit circle. Then there exists a discrete-time Lyapunov function $V = e^\intercal P_1 e + e_s^\intercal P_2 e_s$, where $P_i = P_i^\intercal > 0$, $\lambda_{d_{i1}} = \lambda_{\min}(P_i)$, $\lambda_{d_{i2}} = \lambda_{\max}(P_i)$ and P_i, $i = 1,2$, are solutions for Lyapunov equations

$$\begin{aligned}(A_d - L_d C)^\intercal P_1 (A_d - L_d C) - P_1 &= -Q_1,\\ A_s^\intercal P_2 A_s - P_2 &= -Q_2,\end{aligned}$$

Table 1. The quad rotor VTOL parameters and L_d matrix.

Parameters	$L_d \times 0.01$		
$I_x = I_y = 0.005 \text{ Nm}^2$	0.124	0	0
$m = 0.48$ kg	1.637	0	0
$g = 9.81 \text{ m/s}^2$	0.714	0	0
$T = 0.01$ s	3.547	0	0
$\zeta = 0.9$	0	0.124	0
$\alpha_1 = \beta_1 = 0.00585$	0	1.637	0
$\alpha_2 = \beta_2 = 0.00290$	0	-0.714	0
$\alpha_3 = \beta_3 = 0.0160$	0	-3.547	0
$\omega_n = 1.3$ rad	0	0	0.069
	0	0	0.697

for any $Q_i = Q_i^\intercal > 0$. Let us compute the forward difference of V along of the trajectories errors (12) and (27),

$$\begin{aligned}\triangle V = & -e^\intercal Q_1 e + 2e^\intercal (A_d - L_d C)^\intercal P_1 \Gamma_d + \Gamma_d^\intercal P_1 \Gamma_d \\ & - e_s^\intercal Q_2 e_s + 2e_s^\intercal A_s^\intercal P_2 \Gamma_{d_s} + \Gamma_{d_s}^\intercal P_2 \Gamma_{d_s}.\end{aligned} \quad (28)$$

Taking into account $-\lambda_{d_{13}} \|e\|^2 \geq -e^\intercal Q_1 e \geq -\lambda_{d_{14}} \|e\|^2$, $-\lambda_{d_{23}} \|e_s\|^2 \geq -e_s^\intercal Q_2 e_s \geq -\lambda_{d_{24}} \|e_s\|^2$, where $\lambda_{d_{i3}} = \lambda_{\min}(Q_i)$, $\lambda_{d_{i4}} = \lambda_{\max}(Q_i)$. Using the bound (13) and (26), the next expression is obtained

$$\begin{aligned}\triangle V \leq & -\lambda_{d_{13}} \|e\|^2 + 2\lambda_{d_{12}} Tg \|e\|^2 + \lambda_{d_{12}} (Tg)^2 \|e\|^2 \\ & - \lambda_{d_{23}} \|e_s\|^2 + 2\lambda_{d_{22}} \rho_s \|e_s\| \|e\| + \lambda_{d_{22}} \rho_s^2 \|e\|^2 \\ \leq & - \left(\lambda_{d_{13}} - 2\lambda_{d_{12}} Tg - \lambda_{d_{12}} (Tg)^2\right) \|e\|^2 \\ & - \left(\lambda_{d_{23}} - \lambda_{d_{22}} \rho_s^2\right) \|e_s\|^2 + 2\lambda_{d_{22}} \rho_s \|e_s\| \|e\|.\end{aligned}$$

Defining the matrix

$$S_d = \begin{bmatrix} \lambda_{d_{13}} - 2\lambda_{d_{12}} Tg - \lambda_{d_{12}} (Tg)^2 & -2\lambda_{d_{22}} \rho_s \\ 0 & \lambda_{d_{23}} - \lambda_{d_{22}} \rho_s^2 \end{bmatrix},$$

we can to rewrite $\triangle V$ as

$$\triangle V < -\left[\, \|e\| \;\; \|e_s\| \,\right] \left[M_d S_d + S_d^\intercal M_d \right] \begin{bmatrix} \|e\| \\ \|e_s\| \end{bmatrix} \quad (29)$$

If the local exponential observer satisfies the condition $\lambda_{d_{13}} - 2\lambda_{d_{12}} Tg - \lambda_{d_{12}} (Tg)^2 > 0$ and the dynamic control law satisfies the condition

$$\lambda_{d_{23}} - \lambda_{d_{22}} \rho_s^2 > 0, \quad (30)$$

then matrix S_d is positive definite. By Lemma 9.7 given in pp. 360 Khalil [2002] for M-matrices, then there exists a positive diagonal matrix M_d such that $M_d S_d + S_d^\intercal M_d > 0$. Let us consider $m_{ii} > 0$ and solving the matrix inequality

$$4m_{22} \frac{\left(\lambda_{d_{13}} - 2\lambda_{d_{12}} Tg - \lambda_{d_{12}} (Tg)^2\right)}{\left(\lambda_{d_{23}} - \lambda_{d_{22}} \rho_s^2\right)} > m_{11} > 0,$$

and by choosing

$$m_{22} = \frac{1}{2} \text{ and } m_{11} = \frac{\left(\lambda_{d_{13}} - 2\lambda_{d_{12}} Tg - \lambda_{d_{12}} (Tg)^2\right)}{\left(\lambda_{d_{23}} - \lambda_{d_{22}} \rho_s^2\right)},$$

then $M_d S_d + S_d^\intercal M_d > 0$ is satisfied. By Theorem 28 given in pp. 267 Vidyasagar [1993] for discrete-time systems, all errors of the closed-loop system (17)-(24)-(11) converge locally exponentially to zero. ■

4. SIMULATION AND REAL-TIME EXPERIMENT

For real-time experiments a remotely-controlled prototype of a VTOL produced by Draganfly Innovations Inc. is used, it is shown in fig. 1. For the coordinates measurement a positioning system based on ultrasonic signals and Newton method was used. In the Fig. 3 the general scheme is shown. More details about the positioning system design can be found in Rejón and Aranda-Bricaire [2007]. The coordinates are delivered by the positioning system and are used by the local exponential observer to estimate the roll and pitch angles, and all velocities. Then, the control law is computed using the observed states.

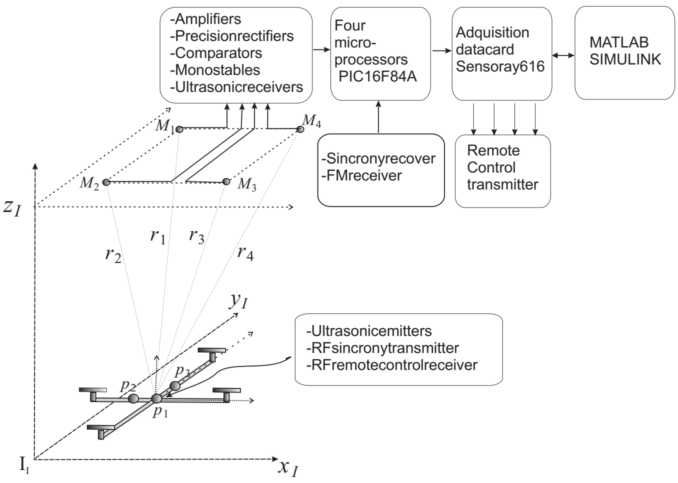

Fig. 3. The positioning system scheme based on ultrasonic signals for VTOL.

The parameters that were used in simulations and real-time experiments are given in Table 1. The discrete-time eigenvalues were calculated by using:

$$\lambda_d = \exp(T\lambda(3\omega_n)),$$

where

$$\lambda(3\omega_n) = [\lambda_1,\, \lambda_1^*,\, \lambda_1,\, \lambda_1^*,\, \lambda_2,\, \lambda_2^*,\, \lambda_2,\, \lambda_2^*,\, \lambda_1,\, \lambda_1^*],$$

$$\begin{aligned}\lambda_1 &= -\xi\omega_n + i\omega_n\sqrt{1-\xi^2},\ \lambda_2 = 0.95\lambda_1, \\ \lambda_1^* &= -\xi\omega_n - i\omega_n\sqrt{1-\xi^2},\ \lambda_2^* = 0.95\lambda_1^*.\end{aligned}$$

The matrix L_d is given in the Table 1. In the simulation, the initial condition for VTOL model, discrete-time local exponential observer and desired trajectory were respectively

$$\begin{aligned}\chi_o &= [\,0.35\ 0\ 0\ 0\ 0.35\ 0\ 0\ 0\ 0.45\ 0\,]^\intercal \text{ m}, \\ \hat{\chi}_o &= [\,0.30\ 0\ 0\ 0\ 0.30\ 0\ 0\ 0\ 0.40\ 0\,]^\intercal \text{ m}, \\ \xi &= [\,mg\ mg\,]^\intercal, \\ y_d &= [\,0.35\ 0\ 0\ 0\ 0.35\ 0\ 0\ 0\ y_{3d}(kT)\ 0\,]^\intercal \text{ m},\end{aligned}$$

where $y_{3d}(kT) = 0.45 + 0.05\sin(0.1kT)$. The discrete-time simulation of control scheme is shown in the fig. 4. In this simulation white noise was added to the states considered as measurable. We can see that the states converges to desired trajectories and the observer errors are in the neighborhood of zero. The real-time experiment is shown in the fig. 5, the experiment was made in a controlled laboratory environment. Note that the desired trajectory is in the neighborhood of the desired trajectory and the observer errors also are in the neighborhood of zero.

5. CONCLUSION

The trajectory tracking problem by dynamic output feedback for remotely controlled quad-rotor VTOL has been

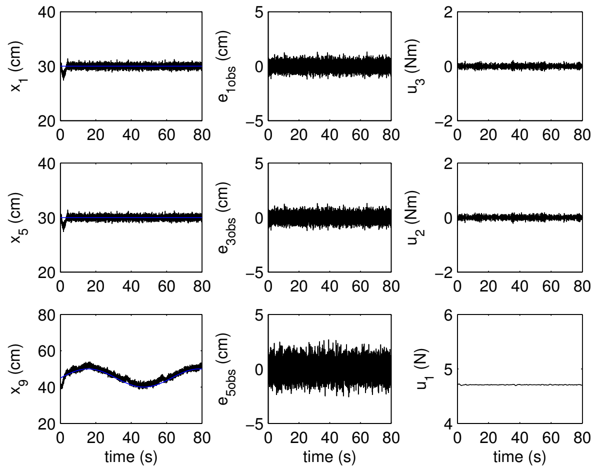

Fig. 4. Simulation of discrete-time control scheme.

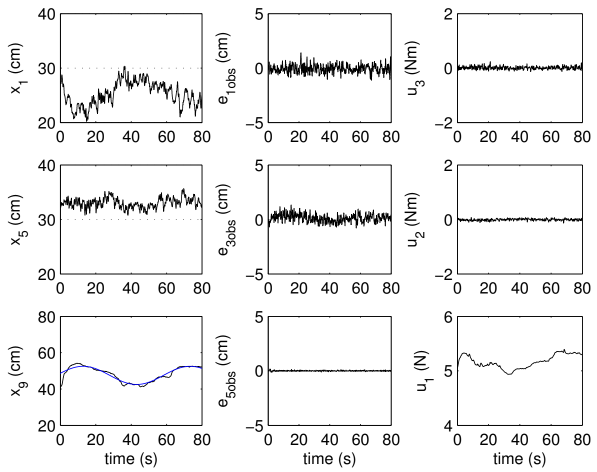

Fig. 5. Real-time experiment of discrete-time control scheme.

addressed. The proposed solution is based on a combination of a Luenberger-type observer and a dynamic feedback linearization scheme. Sufficient conditions have been stated such that a kind of separation principle holds. The discrete-time setting in which the results are derived allows to take into account explicitly the sampling period. Real-time experiments are provided so as to validate the observer-controller scheme. One possible extension of this work would be the inclusion of the yaw angle into the analysis.

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