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A Lagrangian Method for Model Reduction of Controlled Systems ⋆

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Abstract: This paper presents a method for closed-loop order reduction of linear systems. An approximation is carried out on the Lagrangian or Hamiltonian system that is obtained from the problem to minimize an optimization criterion subject to plant dynamics and system constraints. The resulting Hamiltonian system is reduced in complexity by means of a standard reduction techniques. The merits of the method are illustrated on an example of a distillation process.

Keywords: Model reduction, closed-loop systems, optimization, Lagrangian methods.

1. INTRODUCTION

The advances in modeling and computational tools have lead to the development of sophisticated but computationally intensive models. However, the use of such models in optimal control system design often results in controllers for which the state dimension is at least as large as the number of states of the model. In control system design one therefore generally faces the paradigm that high quality controllers will have a considerable complexity because they are inferred from high quality models. Complex and high order controllers are definitely not preferred in practice. Even when such controllers have desired stability, performance and robustness characteristics, they are generally more difficult to implement in either hardware or software, they are more difficult to understand and maintain, and require substantial on-line computations. On the other hand, simple low order controllers are easy to implement, are less likely to cause failures in software or hardware and are fast. Therefore, the construction of simple low-order controllers for high-order systems has taken a significant interest in the last decades [Wortelboer, 1994, Goddard, 1995, Atwell, 2000, Codrons, 2005].

There exist mainly three ways of obtaining low-order controllers as illustrated in Figure 1. The first method is an indirect one and amounts to reducing the high order plant followed by the controller design (reduce-then-optimize). This procedure is popular in many applications since there exists a vast and extensive amount of literature on model reduction techniques, especially for the class of linear and stable systems [Antoulas, 2005]. In this approach, the state

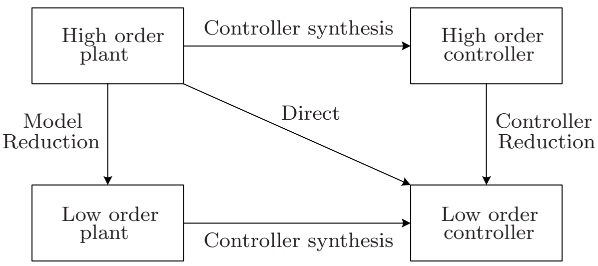

Fig. 1. Strategies for low order controller synthesis

space dimension of the original model is reduced after which a controller is designed based on the approximate model. A major disadvantage of this technique is that the control synthesis procedure lacks all information on the physical characteristics of the plant that is left out or ignored in the reduction scheme. As a consequence, the controller may give poor performance when being implemented on the high order system [Codrons et al., 1999]. One way of circumventing this problem is to use iterative schemes as suggested in [Wortelboer, 1994, Wortelboer et al., 1999, Van den Hof and Schrama, 1995]. Iterative schemes have also been utilized for the synthesis of low order controllers for distributed parameter systems [Fahl, 2000, Hinze and Volkwein, 2004, van Doren et al., 2006] in order to obtain a better representation of the closed-loop system.

The second indirect method, on the other hand, first involves the construction of a controller that is subsequently reduced in complexity (optimize-then-reduce). The idea behind this method is that if a high order controller is optimal with respect to the plant dynamics and desired performance specifications, a small mismatch between the optimal controller and a low order approximation will result in satisfactory performance. Apart from the fact

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that there exist counter examples for this belief, this method is often computationally demanding as it requires the construction of an optimal controller on the basis of the full order plant. Since controller reduction amounts to simplifying one or more components in an interconnected control configuration, the controller reduction problem is, in essence, an approximation problem for interconnected systems where stability, performance and robustness determine the quality of the reduction process. It is for this reason that frequency weighted approximation techniques have been investigated [Anderson and Liu, 1989, Obinata and Anderson, 2001] in which closed loop specifications are translated to reduction schemes.

The third class of reduction techniques are the *direct methods*. In these methods, the parameters that define a low order controller are determined by performing an optimization on the basis of some control criterion. These methods have the advantage that the control objectives are taken into account *explicitly* in the optimization process and that the complexity and structure of the controller is fixed a priori. These methods impose constraints and structure on the controller and use a parameter optimization to obtain an acceptable performance. Probably due to the complexity of a direct reduced order controller synthesis problem, only very few direct methods exist today [Hyland and Richter, 1990, Ly et al., 1985].

It is the purpose of this paper to develop a different view on the problem of model reduction for controlled systems. We consider the adjoint system that is defined by the optimal control problem as the prime object for model reduction. We show that this adjoint system, that is defined by applying variational analysis to the optimization problem, admits a state realization in primal and dual states that can be reduced in complexity. By doing so, we directly infer a reduced order approximation of the *optimally controlled system*. The controller synthesis question to construct a controller that, after interconnection with the plant, establishes the reduced controlled system, then becomes a *realization problem* that we will address in a different paper.

This paper is organized as follows. Section 2 formulates the main optimization problem and provides some background material on Lagrangian dualization theory. In section 3 we specialize the treatment to the control of linear systems in which a quadratic performance criterion is to be minimized. That is, the standard Linear Quadratic Regulator problem is used to demonstrate the reduction strategy of controlled systems that we adopt in this paper. In Section 4, we apply the proposed reduction strategy to an industrial example of a binary distillation process. conclusions are deferred to Section 5.

2. PROBLEM FORMULATION

We consider a general optimal control problem for the time-invariant dynamical system

$$\dot{x} = f(x,u), \qquad x(0) = x_0 \tag{1}$$

where the associated cost is defined by an integral functional of the form

$$J(x,u) = \int_0^{t_1} F(x(t),u(t))\,\mathrm{d}t + E(x(t_1)). \tag{2}$$

Here, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ the input, f and F are given Lipschitz continuous functions on $\mathbb{R}^{n+m}$ that assume values in $\mathbb{R}^n$ and $\mathbb{R}$, respectively. The time at which the response reaches its end-point is t_1. The system and the optimization is subject to a number of constraints. We distinguish inequality and equality constraints of the form

$$g(x,u) \leq 0, \quad h(x,u) = 0 \tag{3}$$

that are supposed to hold for all time instances in the optimization interval $[0,t_1]$. Here, $h : \mathbb{R}^{n+m} \to \mathbb{R}^q$ and $g : \mathbb{R}^{n+m} \to \mathbb{R}^r$ are functions that are twice continuously differentiable and the inequality in (3) is understood to hold componentwise. Following standard terminology, $F(x,u)$ is called the *stage cost* and $E(x)$ is the *end-point weighting*. It is assumed that both F and Φ are non-negative. In addition, it is assumed that the *constraint qualification condition* is satisfied which states that there exist at least one pair (x,u) such that $h(x,u) = 0$ and $g_j(x,u) < 0$ for all components g_j, $j = 1,\ldots,r$ of g.

We aim to minimize J subject to the state evolution (1), the equality and inequality constraints (3). That is, we consider the primal optimization problem

$$\begin{aligned} P_{\mathrm{opt}}: \quad & \text{minimize} && J(x,u) \\ & \text{subject to } && \dot{x} = f(x,u) \\ & && g(x,u) \leq 0 \\ & && h(x,u) = 0 \\ & && x(0) = x_0 \end{aligned} \tag{4}$$

For any pair of vectors (x,u) for which $g(x,u)) \leq 0$ we define the *active constraint set* $\mathcal{A}(x,u) := \{j \mid g_j(x,u) = 0,\ j = 1,\ldots,r\}$.

The nonlinear optimization problem P_{opt} admits a solution via variational analysis. For this, let $L_2([0,t_1])$ denote the Hilbert space of square integrable functions on $[0,t_1]$ equipped with the inner product

$$\langle v,w\rangle := \int_0^{t_1} v(t)^* w(t)\,\mathrm{d}t$$

Define the Lagrangian functional by

$$\begin{aligned} L(x,u,\lambda,\mu,\nu) &:= \langle 1, F(x,u)\rangle + E(x(t_1)) + \\ &+ \langle \lambda, f(x,u) - \dot{x}\rangle + \langle \mu, h(x)\rangle + \langle \nu, g(x)\rangle. \end{aligned} \tag{5}$$

The *Lagrange dual cost* is defined by

$$\ell(\lambda,\mu,\nu) := \inf_{(x,u)} L(x,u,\lambda,\mu,\nu)$$

and is called *feasible* if there exists a triple (λ,μ,ν) for which $\ell(\lambda,\mu,\nu) > -\infty$. The Lagrange dual cost is a concave function of its arguments and satisfies $\ell(\lambda,\mu,\nu) \leq P_{\mathrm{opt}}$ for all functions λ, μ and $\nu \geq 0$ defined on $[0,t_1]$. If we assume that the Lagrange dual cost is feasible, the dual optimization problem is

$$D_{\mathrm{opt}}: \quad \text{maximize}\ \ell(\lambda,\mu,\nu) \tag{6}$$

$$\text{subject to} \quad \nu \geq 0. \tag{7}$$

By construction, the optimal value D_{opt} of the dual optimization problem is a lower bound for P_{opt}, the optimal value of the primal optimization problem, i.e., $D_{\mathrm{opt}} \leq P_{\mathrm{opt}}$.

The generalized Karush-Kuhn-Tucker theorem provides necessary conditions for a local minimum (x^*,u^*) of the primal optimization problem.

Theorem 1. If the cost functional J has a local minimum under the constraints (3) at the regular point (x^*, u^*), then there exist functions λ^*, μ^* and a non-negative function $\nu^* \geq 0$ defined on $[0, t_1]$ with values in $\mathbb{R}^n$, $\mathbb{R}^q$ and $\mathbb{R}^r$, respectively, such that the Lagrangian functional L is stationary at $(x^*, u^*, \lambda^*, \mu^*, \nu^*)$, i.e.,

$$\nabla L(x^*, u^*, \lambda^*, \mu^*, \nu^*) = 0.$$

Moreover, in that case we have that $\nu_i^*(t) g_i(x^*(t), u^*(t)) = 0$ for all $t \in [0, t_1]$ and $i = 1, \dots, r$.

Under suitable convexity conditions on the cost and constraint function a sufficient condition for the existence of a global minimizer for the optimization problem P_{opt} is given as follows.

Theorem 2. Suppose that J, g are convex and h is affine. Assume that the primal optimization (6) satisfies the constraint qualification. Then $D_{\text{opt}} = P_{\text{opt}}$. Moreover, there exist functions λ^*, μ^* and $\nu^* \geq 0$ defined on $[0, t_1]$ such that $D_{\text{opt}} = \ell(\lambda^*, \mu^*, \nu^*)$, i.e., the dual optimization problem admits an optimal solution. In addition, (x^*, u^*) is an optimal solution of the primal optimization problem and $(\lambda^*, \mu^*, \nu^*)$ is an optimal solution of the dual optimization problem, if and only if for all time instances $t \in [0, t_1]$:

(1) $g(x^*, u^*) \leq 0$ and $h(x^*, u^*) = 0$,
(2) $\nu^* \geq 0$ and (x^*, u^*) minimizes $L(x, u, \lambda^*, \mu^*, \nu^*)$ over all $(x, u) \in L_2([0, t_1])$ and
(3) $\nu_j^* g_j(x^*, u^*) = 0$ for all $j = 1, \dots, r$.

Using partial integration, the conditions on the stationary point $(x^*, u^*, \lambda^*, \mu^*, \nu^*)$ of the Lagrangian functional allow a representation as the (unique) solution of the equations

$$\dot{x} = f(x, u) \quad \text{(8a)}$$
$$\dot{\lambda} = -\nabla_x \left[F(x, u) + f(x, u)^\top \lambda + h(x, u)^\top \mu + g(x, u)^\top \nu\right] \quad \text{(8b)}$$
$$0 = \nabla_u \left[F(x, u) + f(x, u)^\top \lambda + h(x, u)^\top \mu + g(x, u)^\top \nu\right] \quad \text{(8c)}$$
$$0 = h(x, u) \quad \text{(8d)}$$
$$0 \leq g(x, u) \quad \text{(8e)}$$
$$0 = \nu_i^\top g_i(x, u) \quad \text{for } i = 1, \dots, r \quad \text{(8f)}$$
$$0 \leq \nu \quad \text{(8g)}$$

where the differential equation is subject to the two-point boundary conditions

$$x(0) = x_0, \quad \lambda(t_1) = \nabla_x E(x(t_1)).$$

We refer to (8) as the *adjoint system* corresponding to the optimization. Note that (8) is an autonomous system in the sense that solutions of (8) only depend on boundary conditions in (8a) and (8b).

The adjoint system (8) represents the optimal controlled system as it contains information about the plant, the optimization criterion and the optimization constraints. It is the purpose of this paper to reduce the complexity of the adjoint system (8) by finding a lower order state space representation for (8). In particular, we aim to reduce the $2n$ dimensional state space of the differential equations (8a)-(8b) by projecting the state variable $\text{col}(x, \lambda)$ on a suitably defined manifold $\mathcal{U}$ of dimension $2k < 2n$.

3. REDUCTION OF LINEAR ADJOINT SYSTEMS

In the remainder of the paper we focus on various reduction strategies for the adjoint system (8). For this, we make the simplifying assumption to consider the convex optimization problem where the system (1) is linear and represented by

$$\Sigma_G : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

The pair (A, B) is assumed to be stabilizable and we denote by G the transfer function corresponding to Σ_G. The cost function is given by (2) with a quadratic stage cost and end-point weighting

$$F(x, u) := \frac{1}{2} \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} C^\top C & C^\top D \\ D^\top C & D^\top D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$
$$E(x) := \frac{1}{2} x^\top(t_1) E x(t_1).$$

That is, $J(x, u) = \frac{1}{2} \int_0^{t_1} y(t)^\top y(t) \, \mathrm{d}t$. Let $Q := C^\top C$, $R = D^\top D$ and $N = D^\top C$ and assume that $R > 0$. Let the end-point weighting $E = P$ be the non-negative definite solution of the algebraic Riccati equation

$$A^\top P + PA - (B^\top P + N)^\top R^{-1} (B^\top P + N) + Q = 0.$$

and suppose that the primal optimization is subject to the affine equality constraint

$$h(x, u) := A_h x + B_h u + C_h = 0. \quad (9)$$

The Lagrangian (5) associated with this optimization is given by

$$L(x, u, \lambda, \mu) := \langle 1, F(x, u) \rangle + \langle \lambda, Ax + Bu - \dot{x} \rangle + \langle \mu, A_h x + B_h u + C_h \rangle + E(x(t_1)).$$

By Theorem 1, an optimal solution $(x^*, u^*, \lambda^*, \mu^*)$ necessarily satisfies the stationarity condition

$$\nabla L(x^*, u^*, \lambda^*, \mu^*) = 0. \quad (10)$$

Theorem 2 promises that (x^*, u^*) solves the primal optimization problem whenever $(x^*, u^*, \lambda^*, \mu^*)$ satisfy the adjoint equations

$$0 = \nabla_\lambda L = Ax + Bu - \dot{x}$$
$$0 = \nabla_x L = Qx + N^\top u + A^\top \lambda + A_h^\top \mu + \dot{\lambda}$$
$$0 = \nabla_u L = Nx + Ru + B^\top \lambda + B_h^\top \mu$$
$$0 = \nabla_\mu L = A_h x + B_h u + C_h$$

with boundary conditions $x(0) = x_0$ and $\lambda(t_1) = Px(t_1)$ (cf. (8)). In matrix form, this is equivalently represented by the system of differential algebraic equations

$$\Sigma_H : \begin{cases} \mathcal{E} \dot{z} = \mathcal{A} z + \mathcal{B} v + \mathcal{F} \\ u^* = \mathcal{C} z \end{cases} \quad (11)$$

where $z = \text{col}(x, \lambda, \mu)$ is the state variable,

$$\mathcal{A} = \begin{pmatrix} A - BR^{-1}N & -BR^{-1}B^\top & -BR^{-1}B_h^\top \\ N^\top R^{-1}N - Q & N^\top R^{-1}B^\top - A^\top & N^\top R^{-1}B_h^\top - A_h^\top \\ A_h - B_h R^{-1}N & -B_h R^{-1}B^\top & -B_h R^{-1}B_h^\top \end{pmatrix}$$
$$\mathcal{E} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} I \\ P \\ 0 \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 \\ 0 \\ C_h \end{pmatrix}$$
$$\mathcal{C} = \begin{pmatrix} -R^{-1}N & -R^{-1}B^\top & -R^{-1}B_h^\top \end{pmatrix}$$

and where the input v in Σ_H is the impulse $v(t) = x_0 \delta(t)$ with δ the Dirac distribution.

In the absence of the constraint (9), the primal optimization problem (6) coincides with the standard linear quadratic regulator problem and the above expression considerably simplifies to the Hamiltonian system

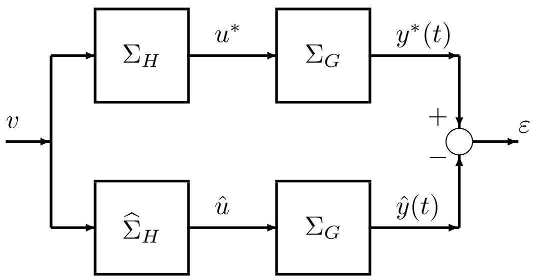


Fig. 2. Reduction of the linear adjoint system

$$\Sigma_H : \begin{cases} \begin{bmatrix} \dot{x}^* \\ \dot{\lambda}^* \end{bmatrix} &= \begin{bmatrix} A - BR^{-1}N & -BR^{-1}B^\top \\ N^\top R^{-1} N - Q & N^\top R^{-1} B^\top - A^\top \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} \\ &+ \begin{bmatrix} I \\ P \end{bmatrix} v(t) \\ u^* &= \begin{bmatrix} -R^{-1}N & -R^{-1}B^\top \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} \end{cases} \tag{12}$$

in which the auxiliary input v is taken to be the impulse function $v(t) = x_0\delta(t)$. In either case, the stationary solution of the Lagrangian is generated as the impulse response of the system Σ_H.

The Hamiltonian system Σ_H will be reduced in complexity by a number of methods. The idea will be to compare the output y^* of the optimally controlled system with the output $\hat{y}$ of the system where the control input is generated by the reduced order Hamiltonian system. See Figure 2. To simplify the exposition and the discussion, we focus on the Hamiltonian system (12). However, with suitable modifications, the reduction techniques discussed below also apply to the DAE system (11).

3.1 Reduction of uncontrollable states

The representation (12) is non-minimal. In fact, if we perform a non-singular state transformation

$$\begin{pmatrix} x \\ \sigma \end{pmatrix} = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

on (12) then one easily shows that $\sigma = 0$ is an uncontrollable state variable. The Hamiltonian system (12) is therefore equivalently represented by the minimal and stable dynamical system

$$\Sigma_H^{\min} : \begin{cases} \dot{x}^* = (A - BR^{-1}(B^\top P + N))x^* + Iv \\ u^* = -R^{-1}(B^\top P + N)x^* \end{cases} \tag{13}$$

that generates the optimal control u^* provided that the input $v(t) = x_0\delta(t)$.

3.2 Modal truncation

In this subsection we reduce (13) to a kth order approximation by performing a modal truncation. Let $\Psi \in \mathbb{R}^{n\times n}$ be a nonsingular matrix such that

$$\Omega := \Psi \left(A - BR^{-1}(B^\top P + N) \right) \Psi^{-1}$$

is in Jordan canonical form

$$\Omega = \mathrm{diag}(\Omega_1, \ldots, \Omega_r)$$

where Ω_j is the jth Jordan block of dimension $\ell_j \times \ell_j$ with ℓ_j the algebraic multiplicity of the jth eigenvalue λ_j of $A - BR^{-1}(B^\top P + N)$. Suppose that the modes are ordered according to

$$\mathrm{Re}(\lambda_r) \leq \cdots \leq \mathrm{Re}(\lambda_2) \leq \mathrm{Re}(\lambda_1) < 0.$$

After transformation, $\Sigma_H^{\min}$ admits a representation in the new coordinate system as follows

$$\Sigma_H^{\min} : \begin{cases} \dot{x}^* = \Omega x^* + \Psi^{-1} v \\ u^* = -R^{-1}(B^\top P + N)\Psi x^* \end{cases}$$

Partition this model according to

$$\begin{pmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} + \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} v$$
$$u^* = (\Upsilon_1 \; \Upsilon_2)\, x^*$$

where $\Omega_{11} = \mathrm{diag}(\Omega_1, \ldots, \Omega_{k_0})$ has dimension $k \times k$ and k is such that $k = \ell_1 + \ldots + \ell_{k_0}$ for some $1 \leq k_0 \leq r$. The order k modal truncation of (13) is then given by

$$\widehat{\Sigma}_H : \begin{cases} \dot{x}_1^* = \Omega_{11} x_1^* + \Gamma_1 v \\ \hat{u} = \Upsilon_1 x_1^* \end{cases} \tag{14}$$

The dynamics of (14) is dominated by the slow modes of $\Sigma_H^{\min}$. The system (14) is both stable and minimal.

3.3 Balanced truncation

As a second method of reduction of the system (13) we consider the method of frequency weighted balancing as described in [Gugercin and Antoulas, 2004, Varga and Anderson, 2003, Sreeram, 2004]. The method is based on the computation of a frequency weighted controllability gramian and a frequency weighted observability gramian associated with a stable linear time invariant system and rational stable frequency weightings on the input and output.

Frequency weighted balanced truncations are particularly interesting for the application here as we wish to minimize the output error $\|y^* - \hat{y}\|$ of the controlled system rather than the error $\|u^* - \hat{u}\|$. See Figure 2. Since $y = Gu$, the output error

$$\|y^* - \hat{y}\|^2 = \|Gu^* - G\hat{u}\|^2 = \|G(u^* - \hat{u})\|^2$$

is equal to a frequency weighted error of the output of the system $\Sigma_H - \widehat{\Sigma}_H$ when excited by the input signal $v(t) = x_0\delta(t)$.

We therefore determine a frequency weighted balanced representation of the system (13) with input and output weight

$$W_{\mathrm{i}} = I; \quad W_{\mathrm{o}} = G$$

and use this representation to truncate the state to its dominant k states. For details of this procedure, see [Gugercin and Antoulas, 2004, Varga and Anderson, 2003, Sreeram, 2004].

4. APPLICATION TO BINARY DISTILLATION

We illustrate the reduction methodology on a model of a binary distillation process. We used a linearized time-invariant model of a stabilized binary distillation column with 41 stages. A detailed description of this originally non-linear model can be found in Skogestad [1997]. A schematic representation of the distillation column with nomenclature is depicted in figure 3. Flow units are in kmol/min, holdups in kmol, and compositions in mole fraction. The model contains two proportional controllers in order to stabilize the levels using the product flows.

Inputs of the model are

$$u = \text{col}(V_B, L_T)$$

and outputs of the model are taken to be the bottom and distillate product compositions

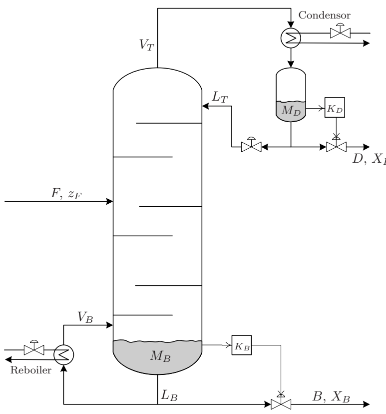
$$y = \text{col}(X_B, X_D).$$

The model is inferred from the total material balance at each of the 41 trays in the column and is described in terms of a state variable of dimension $n = 82$. In this study, we consider only V_B and L_T to exert control over the product compositions X_B and X_D. The resulting plant model is therefore a stable LTI model with 2 inputs, 2 outputs and $n = 82$ states. The stage cost is defined by

$$F(x, u) = \frac{1}{2}[x^\top Qx + u^\top Ru]$$

where $x^\top Qx = X_B^2 + X_D^2 = y^\top y$ and $R = 0.001 \cdot I_2$, with I_2 being the 2×2 identity matrix.

For a reduced order controller of the distillation column, only a frequency weighted balanced truncation was considered. As suggested in the previous section, the plant transfer function has been used as frequency weight in the truncation so as to approximate the optimal controlled output rather than the optimal control input. In Figure 4, the feed-forward results are depicted where the Hamiltonian system was reduced to the order $k = 2$. As can be seen, there is some difference in the control signals but there is virtually no difference between the optimal plant output y^* and its approximation $\hat{y}$.



X_B	Bottom composition
X_D	Distillate composition
V_B	Boilup vapor flow
L_T	Reflux flow
L_B	Bottom liquid flow
V_T	Top vapor flow
M_B	Reboiler holdup
M_D	Condensor holdup
B	Bottom product flow
D	Distillate product flow
K_B	Stabilizing P-controller
K_D	Stabilizing P-controller

Fig. 3. Distillation column

5. CONCLUSIONS

In this paper, a method for closed-loop controller reduction was presented. We consider approximations of the adjoint system that is obtained from a variational analysis on the optimal control. For linear systems with quadratic cost functions, the adjoint system allows a representation as a standard LTI system whose impulse response generates the optimal control trajectories. Modal truncation and balanced truncation were used to obtain approximations of this system. A frequency weighted balanced approximation has been applied to the example of a 82 order binary distillation process that was succesfully reduced to a second order approximation. The controlled outputs of the optimal system and the controlled outputs obtained from the second order approximation of the Hamiltonian system were almost indistinguishable.

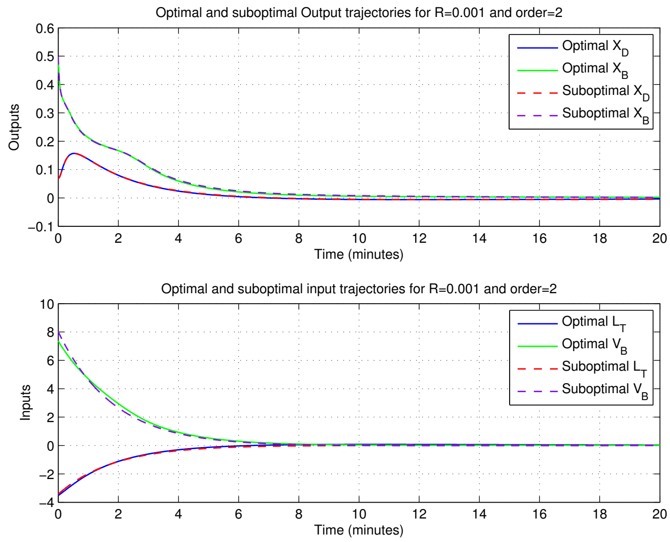


Fig. 4. Approximation results for controlled distillation column

The theory in this paper has ben worked out for linear systems and quadratic cost functions, but the ideas presented here allow generalization to reduction techniques on more general nonlinear adjoint systems such as the one presented in (8). We are currently investigating proper orthogonal decompositions and Galerkin type of projections to reduce the complexity of the adjoint system defined in (8).

Results obtained in this paper might further improve by using frequency weighted Hankel norm approximations to reduce the Hamiltonian system. Optimal Hankel norm approximation is a feasible candidate to reduce the Hamiltonian system as the Hankel norm operator maps past inputs to future outputs. This corresponds to the character of the Hamiltonian system as this system generates optimal trajectories of the controlled system from an input signal that is only non-zero at $t = t_0$. Frequency weighted Hankel norm approximation is expected to have similar benefits as the method of frequency weighted balanced truncation.

The reduced order controllers in this paper were provided as feed-forward control to the plant. Placing a reduced order controller in feed-back with the plant will improve robustness and performance of control. Currently only the optimization problem was constrained by the plant's dynamics. Equality and inequality constraints can be added to the problem to consider low-order (sub-optimal) constrained control.

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