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Stability and Stabilization of Retarded Fractional Delay Differential Systems ⋆

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Abstract: The main objective of this paper is twofold. First, we devise a stability test for determining whether a linear retarded fractional delay differential system has no characteristic roots in a specified right half of the complex plane. Second, we develop a practical method for computing the abscissa of stability (defined as the largest of the real parts of the characteristic roots) for this type of system. The method is based on a known technique which makes repeated use of a stability test, thereby avoiding the calculation of all the roots. The stability test and the method for computing the abscissa of stability provide useful computational tools for the design of fractional differential systems using the method of inequalities as well as other numerical optimization approaches. Numerical examples are also given.

Keywords: fractional system; retarded delay system; stability; abscissa of stability; numerical stabilization; the method of inequalities.

1. INTRODUCTION

In recent years, a lot of research interest has been given to fractional systems. Many physical systems have their mathematical models described by fractional differential equations [see, e.g., Podlubny, 1999a,b, and the references therein].

In the design of dynamical and control systems by the method of inequalities [Zakian and Al-Naib, 1973, Zakian, 1979a, 1987, 2005], as well as parameter optimization methods [e.g., Fleming, 1985, Kuhn and Schmidt, 1987, Burke et al., 2006], one usually needs a computational tool for checking the system's stability and determining a design parameter that stabilizes the system. This is because numerical search algorithms are in general able to seek a design solution only if they start from a *stability point* (i.e., a point for which all the associated performance measures are finite or defined). Once such a point is obtained, the algorithms search for a solution inside the space of all stability points.

Consider a linear retarded fractional delay differential system (RFDDS) whose characteristic function $f(s)$ is described by

$$f(s) \triangleq P_0(s) + \sum_{k=1}^{n} P_k(s)e^{-\gamma_k s} + \sum_{k=1}^{\tilde{n}} \tilde{P}_k(s)e^{-u_k(s)}, \quad (1)$$

where $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n$, the polynomials $P_k(s)$ and $\tilde{P}_k(s)$ are of the form $\sum_{j=0}^{l_k} a_j s^{\alpha_j}$ with $\alpha_j \geq 0$ and $\deg P_0 > \deg P_k$ for all k, $u_k(s)$ are of the form $\sum_{j=1}^{m_k} b_j s^{\delta_j}$ with $0 < \delta_j \leq 1$ and $b_j \geq 0$, and none of u_k assumes the form αs. Obviously, $f(s)$ in (1) is very general. The characteristic functions of a rational system (which is a polynomial) and a retarded delay differential system (which is a quasipolynomial) are special cases of (1).

Bonnet and Partington [2000, 2001] have shown that a linear RFDDS is BIBO stable if and only if its characteristic function $f(s)$ has all the zeros with negative real parts. In this connection, Hwang and Cheng [2006] develop a numerical stability test for determining whether all the zeros of $f(s)$ have negative real parts. In conjunction with the method of inequalities, Hwang and Cheng's test provides a useful tool for checking the stability of the RFDDS. So far, however, none has considered developing a numerical procedure for stabilizing the RFDDS. This is the main subject to be investigated in this paper.

Let α denote the abscissa of stability of the characteristic function $f(s)$, which is defined by

$$\alpha \triangleq \max\{\mathrm{Re}(s) : f(s) = 0\}. \quad (2)$$

Evidently, the RFDDS is BIBO stable if and only if

$$\alpha < 0. \quad (3)$$

Following the method of inequalities, it is readily appreciated that inequality (3) is a useful criterion for stabilizing the RFDDS.

Once a practical method for evaluating α is available, the problem of finding a stability point in the design process can be solved in a straightforward manner by iterative numerical methods. This approach has been used successfully by many people [e.g., Arunsawatwong, 1996, Burke et al., 2006, Zakian and Al-Naib, 1973, Zakian, 1979a, 1987, 2005, and the references therein].

This paper has two objectives. First, we devise a stability test for checking whether $f(s)$ has no zeros in a specified right half of the complex plane by extending the procedure in Hwang and Cheng [2006]. (See Section 2 for the defin-

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ition of stability used in this paper.) Second, we devise a practical method for computing the abscissa of stability of $f(s)$ in (1). The method is based on a known technique by Zakian [1979b], which makes repeated use of the stability test.

The problem of computing the abscissa of stability was first considered by Zakian [1979b] in connection with rational systems. Zakian [1979b]'s technique exploits the fact that the abscissa of stability only involves the zero furthest to the right in the complex plane. The idea is useful especially when $f(s)$ has infinitely many zeros, because it is impossible to compute all of them. Moreover, by modifying this technique, Arunsawatwong [1996] devised a method for computing the abscissa of stability of a quasipolynomial.

The structure of the paper is as follows. In Section 2, we develop a new stability test for checking whether $f(s)$ has no zeros in a given right half of the complex plane. Section 3 establishes the practical method for computing the abscissa of stability of $f(s)$. In Section 4, numerical examples are given so as to show the effectiveness of the methods developed in this paper and to demonstrate the use of the abscissa of stability in stabilizing an RFDDS by the method of inequalities. In Section 5, the conclusions are given.

2. STABILITY TEST

This section develops a numerical procedure for checking whether the characteristic function $f(s)$ in (1) has no zeros in a specified right half of the complex plane.

To this end, the following definition is useful. Let $H(\rho)$ denote the right half plane given by

$$H(\rho) \triangleq \{s \in \mathbb{C} : \mathrm{Re}(s) \geq \rho\},$$

where $\rho \in \mathbb{R}$ is given. The function $f(s)$ is said to be *stable with respect to* $H(\rho)$ (or $H(\rho)$-stable) if none of its zeros lies in $H(\rho)$. Following this, it is readily appreciated that a linear RFDDS is BIBO stable if and only if $f(s)$ is $H(0)$-stable. Clearly, Hwang and Cheng [2006]'s procedure is used only for testing the $H(0)$-stability of $f(s)$.

In developing the stability tests in this work and in Hwang and Cheng [2006], the key idea used is the well-known Cauchy's residue theorem [see, e.g., Brown and Churchill, 2003], which states as follows.

Cauchy's Residue Theorem *Let $F(s)$ be analytic within and on a simple closed contour Γ except for finitely many points $s_1, s_2, \ldots, s_n$ lying in the interior of Γ. Then*

$$\int_\Gamma F(s)\, ds = \mathbf{i}2\pi \sum_{i=1}^{n} \mathrm{Res}(s_i),$$

where the integral is taken in the positive direction and $\mathrm{Res}(s_i)$ *denotes the residue of $F(s)$ at the points s_i.*

By defining

$$F(s) \triangleq 1/f(s), \tag{4}$$

we can see that $f(s)$, in general, has no zeros within Γ if and only if the integral $\int_\Gamma F(s)\, ds = 0$. However, it may happen that the sum of the residues of $F(s)$ at all poles is equal to zero. For this reason, Hwang and Cheng [2006] suggest replacing the contour integral $\int_\Gamma F(s)\, ds$ with the following integral:

$$J \triangleq \int_\Gamma \frac{F(s)}{(s + h_1 + \mathbf{i}h_2)^k\, F(\mathbf{i}h_2)} ds \tag{5}$$

where $k \geq 1$ is a specified integer, and h_1 and h_2 are randomly chosen parameters so that the point $(-h_1 - \mathbf{i}h_2)$ must lie outside Γ. Accordingly, we can see that $f(s)$ has no zeros within Γ if and only if $J = 0$ for any $(-h_1 - \mathbf{i}h_2)$ outside Γ.

The term $(s + h_1 + \mathbf{i}h_2)^k$ in the integral (5) prevents the sum of the residues from being zero when the function $f(s)$ has zeros inside Γ. Note in passing that the idea of using the term $(s + h_1 + \mathbf{i}h_2)^k$ is due to Tuan and Duc [2000]. See more discussion on the important roles of k, h_1 and h_2 in Section 2.1.

Since the branch cut of the function $f(s)$ in (1) consists of the negative real axis including the origin, we divide the stability test into two cases: (i) $\rho \leq 0$ and (ii) $\rho > 0$. Each case uses a different contour so that the integration path does not cross the branch cut.

Now we are ready to consider the stability test.

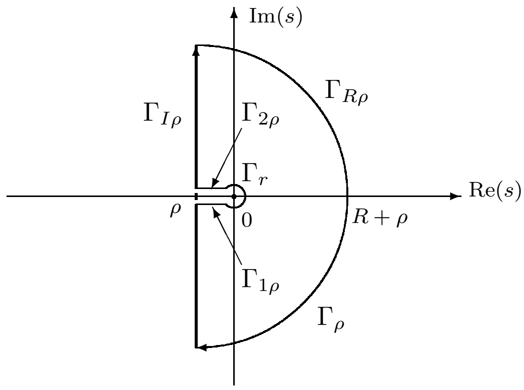

Fig. 1. The contour Γ_ρ for $\rho \leq 0$.

2.1 Case I: $\rho \leq 0$

For $\rho \leq 0$, we define the integration contour Γ_ρ as follows.

$$\begin{aligned}
\Gamma_\rho \;&\triangleq \Gamma_{I\rho} \cup \Gamma_{R\rho} \cup \Gamma_{1\rho} \cup \Gamma_{2\rho} \cup \Gamma_r, \\
\Gamma_{I\rho} \;&= \{s = \rho + \mathbf{i}\,\omega : \omega \in [-R, -a] \cup [a, R]\}, \\
\Gamma_{R\rho} \;&= \left\{s = \rho + R\, e^{\mathbf{i}\theta} : -\pi/2 \leq \theta \leq \pi/2\right\}, \\
\Gamma_{1\rho} \;&= \left\{s = x - \mathbf{i}a : \rho \leq x \leq -\sqrt{r^2 - a^2}\right\}, \\
\Gamma_{2\rho} \;&= \left\{s = x + \mathbf{i}a : \rho \leq x \leq -\sqrt{r^2 - a^2}\right\}, \\
\Gamma_r \;&= \left\{s = r\, e^{\mathbf{i}\theta} : 0 \leq |\theta| \leq \pi - \arcsin(a/r)\right\},
\end{aligned} \tag{6}$$

where $R \to \infty$, $r > a > 0$ and $r \to 0$. See Fig. 1. Next define the contour integral

$$J(\rho) = \int_{\Gamma_\rho} \frac{F(s)}{(s + h_1 + \mathbf{i}h_2)^k\, F(\rho + \mathbf{i}h_2)} ds. \tag{7}$$

We can easily deduce that, for $\rho < 0$, $f(s)$ is $H(\rho)$-stable if and only if $f(s)$ has no zero in the real interval $[\rho, 0]$ and $J(\rho) = 0$ for any $(-h_1 - \mathbf{i}h_2)$ outside Γ.

The procedure for implementing the stability test is as follows. First, check whether $f(s)$ has no zero in $[\rho, 0]$, which can be done conveniently and speedily by efficient numerical search algorithms. If no zero is found, we then compute $J(\rho)$ defined in (7). If $J(\rho)$ is sufficiently close to zero, we conclude that $f(s)$ is $H(\rho)$-stable.

Now it remains only to explain how to compute the integral $J(\rho)$. It is easy to show that, with the term $(s+h_1+\mathbf{i}h_2)^k$, the integral along the path $\Gamma_{R\rho}$ converges to zero as $R\to\infty$. Since $f(s)$ has no zero at the origin (otherwise, it would be detected earlier), we can also show that the integral along Γ_r converges to zero as $r\to 0$.

Hence, it follows from the above and from (7) that

$$J(\rho) = J_{I\rho} + J_{1\rho} + J_{2\rho} \tag{8}$$

where $J_{I\rho}$, $J_{1\rho}$ and $J_{2\rho}$ denote the contour integrals over the paths $\Gamma_{I\rho}$, $\Gamma_{1\rho}$ and $\Gamma_{2\rho}$, respectively.

Now consider $J_{I\rho}$. As $r\to 0$ and $R\to\infty$, we obtain

$$J_{I\rho} = \int_{\rho-\mathbf{i}\infty}^{\rho+\mathbf{i}\infty} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\,F(\rho+\mathbf{i}h_2)}ds. \tag{9}$$

Because $J_{I\rho}$ in (9) is an improper integral, after the change of variable $s=\rho+\mathbf{i}\,\omega=\rho+\mathbf{i}\tan\frac{x}{2}$, it follows that

$$J_{I\rho} = \int_{-\pi}^{\pi} \frac{F(\rho+\mathbf{i}\tan\frac{x}{2})}{\left[h_1+\rho+\mathbf{i}\left(h_2+\tan\frac{x}{2}\right)\right]^k F(\rho+\mathbf{i}\,h_2)} \frac{\mathbf{i}\,dx}{(2\cos^2\frac{x}{2})}. \tag{10}$$

By defining $G_\rho(x)$ as the integrand in (10), we arrive at the following initial value problem

$$\begin{cases} \dfrac{dy_{r\rho}(x)}{dx} = \mathrm{Re}\left\{G_\rho(x)\right\}, \quad y_{r\rho}(-\pi)=0, \\ \dfrac{dy_{i\rho}(x)}{dx} = \mathrm{Im}\left\{G_\rho(x)\right\}, \quad y_{i\rho}(-\pi)=0, \end{cases} \tag{11}$$

with $J_{I\rho} = y_{r\rho}(\pi) + \mathbf{i}\,y_{i\rho}(\pi)$.

The remaining task is to compute $J_{1\rho}$ and $J_{2\rho}$. As $r\to 0$, we obtain

$$J_{1\rho} = \int_{\rho}^{0} \frac{F(-x\,e^{-\mathbf{i}\pi})\,dx}{(-x\,e^{-\mathbf{i}\pi}+h_1+\mathbf{i}h_2)^k\,F(\rho+\mathbf{i}\,h_2)}, \tag{12}$$

$$J_{2\rho} = \int_{\rho}^{0} \frac{-F(-x\,e^{\mathbf{i}\pi})\,dx}{(-x\,e^{\mathbf{i}\pi}+h_1+\mathbf{i}h_2)^k\,F(\rho+\mathbf{i}\,h_2)}. \tag{13}$$

Following the approach used in computing $J_{I\rho}$, we easily deduce that $J_{1\rho}$ and $J_{2\rho}$ can be obtained by solving the following initial value problems.

$$\begin{cases} \dfrac{dy_{rj}(x)}{dx} = \mathrm{Re}\left\{G_j(x)\right\}, \quad y_{rj}(\rho)=0, \\ \dfrac{dy_{ij}(x)}{dx} = \mathrm{Im}\left\{G_j(x)\right\}, \quad y_{ij}(\rho)=0, \end{cases} \tag{14}$$

and

$$J_{j\rho} = y_{rj}(0) + \mathbf{i}\,y_{ij}(0), \quad \text{for } j=1,2, \tag{15}$$

where $G_j(x)$ denote the integrands in (12) and (13).

2.2 Case II: $\rho > 0$

For $\rho > 0$, we define the contour as follows.

$$\begin{aligned} \Gamma_\rho &\triangleq \Gamma_{I\rho}\cup\Gamma_{R\rho}, \\ \Gamma_{I\rho} &= \left\{s=\rho+\mathbf{i}\,\omega : \omega\in[-R,R]\right\}, \\ \Gamma_{R\rho} &= \left\{s=\rho+R\,e^{\mathbf{i}\theta} : -\pi/2\le\theta\le\pi/2\right\}, \end{aligned} \tag{16}$$

where $R\to\infty$. See Fig. 2. Define

$$J(\rho) \triangleq \int_{\Gamma_\rho} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\,F(\rho+\mathbf{i}h_2)}ds. \tag{17}$$

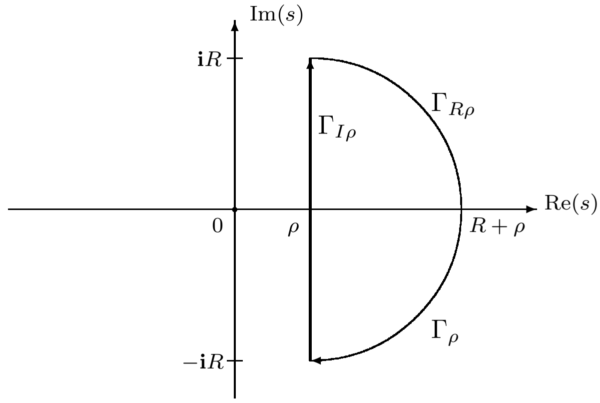

Fig. 2. The contour Γ_ρ for $\rho > 0$.

Then, we can easily deduce that $f(s)$ is $H(\rho)$-stable if and only if $J(\rho)=0$ for any $(-h_1-\mathbf{i}h_2)$ outside Γ.

In the following, how to compute $J(\rho)$ is described. It is easy to see that as $R\to\infty$, the integral along the contour $\Gamma_{R\rho}$ vanishes. Hence,

$$\begin{aligned} J(\rho) &= \int_{\Gamma_{I_\rho}} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\,F(\rho+\mathbf{i}h_2)}ds \\ &= \int_{\rho-\mathbf{i}\infty}^{\rho+\mathbf{i}\infty} \frac{F(s)}{(s+h_1+\mathbf{i}h_2)^k\,F(\rho+\mathbf{i}h_2)}ds. \end{aligned} \tag{18}$$

Again, by the change of variable $s=\rho+\mathbf{i}\,\omega=\rho+\mathbf{i}\tan\frac{x}{2}$, we have

$$J(\rho) = \int_{-\pi}^{\pi} \frac{F(\rho+\mathbf{i}\tan\frac{x}{2})}{\left[h_1+\rho+\mathbf{i}\left(h_2+\tan\frac{x}{2}\right)\right]^k F(\rho+\mathbf{i}\,h_2)} \frac{\mathbf{i}\,dx}{(2\,\cos^2\frac{x}{2})}. \tag{19}$$

By defining $G_\rho(x)$ as the integrand in (19), we obtain the following initial value problem.

$$\begin{cases} \dfrac{dy_{r\rho}(x)}{dx} = \mathrm{Re}\left\{G_\rho(x)\right\}, \quad y_{r\rho}(-\pi)=0, \\ \dfrac{dy_{i\rho}(x)}{dx} = \mathrm{Im}\left\{G_\rho(x)\right\}, \quad y_{i\rho}(-\pi)=0, \end{cases} \tag{20}$$

and $J(\rho) = y_{r\rho}(\pi) + \mathbf{i}\,y_{i\rho}(\pi)$.

It is found that sometimes the initial value problems (11) and (20) can be stiff, especially when the path $\Gamma_{I\rho}$ gets very close to the right-most pole of $F(s)$, whose real part is the abscissa of stability. This frequently happens when the stability test is used repeatedly in computing the abscissa of stability (see Section 3). Therefore, it is advisable to solve the initial value problems (11) and (20) using stiff ODE solvers, for example, Radau5 code [Hairer and Wanner, 1996]. See more discussion on computational issues in Section 4.

3. COMPUTING THE ABSCISSA OF STABILITY

In this section, the method for computing the abscissa of stability of $f(s)$ in (1) is described. It is based on Zakian [1979b]'s iteration, which makes repeated use of a stability test. Here, the Routh test of the shifted polynomial in Zakian [1979b] is replaced by the $H(\rho)$-stability test of $f(s)$ developed in Section 2.

The essentials of the iteration are as follows. First, determine an interval (a_0, b_0) that contains α; i.e., $f(s)$ is

$H(b_0)$-stable but $H(a_0)$-unstable. Then, let $x_1 = (a_0 + b_0)/2$ and determine whether $f(s)$ is $H(x_1)$-stable. If so, then $\alpha \in (a_0, x_1)$; otherwise, $\alpha \in (x_1, b_0)$. The procedure is repeated until α is located within a sufficiently small interval.

More explicitly, the iteration is expressed as follows. Let $n = 0, 1, 2, \ldots$ and let $\{x_n\}$ be a sequence of real numbers generated by

$$x_{n+1} = x_n + h_{n+1},$$
$$h_{n+1} = \begin{cases} |h_n|/2 & \text{if } f(s) \text{ is } H(x_n)\text{-unstable} \\ -|h_n|/2 & \text{if } f(s) \text{ is } H(x_n)\text{-stable} \end{cases} \qquad (21)$$

where

$$h_0 = b_0 - a_0 \quad \text{and} \quad x_0 = a_0.$$

It can be shown [Zakian, 1979b] that the infinite sequence $\{x_n\}$ in (21) converges to the abscissa of stability, α, with the property

$$|\alpha - x_{n+1}| \le \frac{|h_0|}{2^{n+1}}. \qquad (22)$$

The iteration (21) has the following noteworthy feature [Zakian, 1979b]. It always converges with the rate of convergence given by (22). The convergence is disrupted, but not catastrophically, if failure to obtain the correct result of the $H(x_n)$-stability test of $f(s)$ occurs in the iteration. The error $|\alpha - x_n|$ achieves its minimal value just prior to the first failure to obtain the correct result in the $H(x_n)$-stability test, and thereafter the error always remains less than twice the minimal value, irrespective of further miscalculation. See Section 4.3 for examples.

It now remains to explain how to determine a_0 and b_0. Given $a < 0$ and $b > 0$, set $a_0 = a$ and $b_0 = b$. There are only three possibilities. (I) If $f(s)$ is $H(a_0)$-unstable and is $H(b_0)$-stable, then $a_0 = a$ and $b_0 = b$. (II) If $f(s)$ is $H(a_0)$-stable, we keep setting $a_0 = 2a_0$ until $f(s)$ is $H(a_0)$-unstable and then set $b_0 = a_0/2$. (III) If $f(s)$ is $H(b_0)$-unstable, we keep setting $b_0 = 2b_0$ until $f(s)$ is $H(b_0)$-stable and then set $a_0 = b_0/2$.

Following is the pseudo-code of the algorithm.

```
input:  permissible error ε > 0, a < 0, b > 0,
output: the abscissa of stability α

begin
  n = 0; a_n = a; b_n = b;
  % First, find an initial interval containing α
  if f(s) is H(a_n)-stable,
    while f(s) is H(a_n)-stable,
      a_n = 2 * a_n;
    end
    b_n = a_n/2;
  elseif f(s) is H(b_n)-unstable,
    while f(s) is H(b_n)-unstable,
      b_n = 2 * b_n;
    end
    a_n = b_n/2;
  end
  c = (a_n + b_n)/2;
  % Now, start doing bisection
  while |a_n − b_n| > ε,
    n = n + 1;
    if f(s) is H(c)-stable,
      a_n = a_{n−1};    b_n = c;
    else
      a_n = c;    b_n = b_{n−1};
    end
    c = (a_n + b_n)/2;
  end
  α = c;
end
```

4. NUMERICAL EXAMPLES

In this section, numerical examples are given to show the results of the application of the methods developed in this paper, and to demonstrate how to stabilize an RFDDS by numerical methods. In solving the initial value problems, we use the FORTRAN code Radau5 [Hairer and Wanner, 1996].

4.1 Example 1

Consider the following characteristic function [Ozturk and Uraz, 1985, Hwang and Cheng, 2006].

$$f(s) = (\sqrt{s})^3 - 1.5(\sqrt{s})^2 - 1.5(\sqrt{s})^2 e^{-\tau s} + 4\sqrt{s} + 8. \quad (23)$$

It is shown [Ozturk and Uraz, 1985] that for $\tau \in (0.99830, 1.57079)$, the system is BIBO stable. Therefore, $f(s)$ is $H(0)$-unstable for $\tau = 0.99$ and $H(0)$-stable for $\tau = 1$.

k	h_1	h_2	J	N_c
$\tau =$	0.99			
1	3.9204572	−2.1184570	$0.1425918 + \mathbf{i}\,0.7859342$	21374
2	2.2235993	1.0918359	$-0.0405419 + \mathbf{i}\,0.1515234$	7472
3	5.4785590	−8.8245291	$-0.0531885 + \mathbf{i}\,0.0381619$	4107
k	h_1	h_2	$J \times 10^8$	N_c
$\tau =$	1.00			
1	3.0051836	−2.4528789	$-2.1067 - \mathbf{i}\,3.6143$	23891
2	0.4782589	1.5748952	$-0.3593 - \mathbf{i}\,0.2598$	9039
3	1.2774739	4.7541852	$-0.4067 + \mathbf{i}\,3.0108$	6186

Table 1. Stability test for Example 1, where N_c is the number of function calls.

We perform the $H(0)$-stability test of $f(s)$ in (23) the result of which is given in Table 1. Furthermore, we verify the stability result by computing the abscissa of stability of $f(s)$ for the various values of τ with the permissible error $\epsilon = 10^{-8}$. The computed abscissae of stability (α_{comp}), which are close to zero, are shown in Table 2. This demonstrates the effectiveness of the proposed method.

τ	0.99830	0.99840	1.57078	1.57080
α_{comp}	0.74×10^{-5}	-0.14×10^{-4}	-0.17×10^{-5}	0.38×10^{-6}

Table 2. The computed α for Example 1.

4.2 Example 2

In this example, we consider the characteristic function of a temperature control system, where the process is described by a heat equation (see the Appendix for the detail.) The characteristic function $f_1(s)$ is

$$f_1(s) = \sigma A\sqrt{\lambda s}(1 - e^{-2L\sqrt{\lambda s}}) + 2p\, e^{-L\sqrt{\lambda s}}, \qquad (24)$$

where p is the adjustable controller gain and

$$A = 2, \quad L = 1, \quad \sigma = 0.5, \quad \lambda = 1.$$

By using the graphical stability test proposed by Callier and Desoer [1972], Lertsatienchai [2003] shows that the system is stable for $p < 17.7985$ and unstable for $p >$

p	h_1	h_2	$J(0)$
15	6.0818	-0.9552	$(-0.04 - \mathbf{i}\,0.54) \times 10^{-8}$
	4.2558	-2.3985	$(0.04 + \mathbf{i}\,3.41) \times 10^{-9}$
17.798	10.9483	7.3350	$(-0.16 - \mathbf{i}\,0.27) \times 10^{-8}$
	5.8581	8.0661	$(0.20 - \mathbf{i}\,0.37) \times 10^{-8}$
17.799	1.2031	2.7357	$1.7967 + \mathbf{i}\,0.5787$
	10.4406	-9.1844	$0.1107 + \mathbf{i}\,0.0043$
18.5	2.7785	3.8160	$1.3455 - \mathbf{i}\,0.3424$
	3.6394	-0.8452	$-1.3771 + \mathbf{i}\,1.3702$

Table 3. Stability test results with $k = 2$.

17.7985. Hence, $p = 17.7985$ is the critical gain of the closed loop system.

We perform the $H(0)$-stability test of $f_1(s)$ in (24) for various values of the gain p using the method in Section 2. The values of $J(0)$ are shown in Table 3. The results show that for all stable cases $J(0) \approx 0$, and that the critical gain must be in the interval $(17.798, 17.799)$. This agrees well with Lertsatienchai [2003]'s result.

4.3 Example 3

Consider the functions $f_2(s)$ and $f_3(s)$ given by

$$\begin{aligned} f_2(s) &\triangleq (s+1)^n f_1(s), \\ f_3(s) &\triangleq (s^2 - 2s + 5)^n f_1(s), \end{aligned} \tag{25}$$

where $f_1(s)$ is given by (24) with $p = 10$ and $n \geq 1$ is an integer. It readily follows from (25) that

$$\begin{aligned} f_2(s) &= \sqrt{s}(s+1)^n + 20(s+1)^n e^{-\sqrt{s}} \\ &\quad - \sqrt{s}(s+1)^n e^{-2\sqrt{s}}, \\ f_3(s) &= \sqrt{s}(s^2-2s+5)^n + 20(s^2-2s+5)^n e^{-\sqrt{s}} \\ &\quad - \sqrt{s}(s^2-2s+5)^n e^{-2\sqrt{s}}. \end{aligned} \tag{26}$$

It should be noted that the abscissa of stability of $f_1(s)$ with $p = 10$ is equal to -1.61. Hence, from the definition of $f_2(s)$ and $f_3(s)$ in (25), it is clear that the abscissae of stability of $f_2(s)$ and $f_3(s)$ are respectively equal to -1 and 1 for all $n \geq 1$. Notice that when $n > 1$, the right-most roots are repeated and thus we expect severe effect of roundoff errors during computation.

The method developed in Section 3 is applied to the computation of the abscissae of stability of $f_2(s)$ and $f_3(s)$ expressed in (26) for $n = 1, 2, 3, 4$. Computational results are shown in Tables 4 and 5, where

$$\text{actual error} = \begin{cases} |\alpha_{\text{comp}} + 1| & \text{for } f_2(s) \\ |\alpha_{\text{comp}} - 1| & \text{for } f_3(s) \end{cases}.$$

From Tables 4 and 5, we can see that the actual errors are greater than the permissible errors in most of the cases where $n > 1$. This indicates that, for those difficult cases, the iteration (21) fails to converge to the correct value of α owing to the wrong determination of $H(\rho)$-stability effected by the roundoff errors. However, it should be noted that such cases hardly happen in practice. As shown in this example, even if they happen, the iteration (21) is still able to produce results with reasonable accuracy of 2–3 decimal digits. Note that in practice, this level of accuracy is adequate for many applications.

n	permissible error	actual error	α_{comp}
1	1×10^{-4}	0.42×10^{-4}	-0.9999581333
	1×10^{-5}	0.25×10^{-5}	-0.9999975049
	1×10^{-6}	0.34×10^{-6}	-1.0000003377
2	1×10^{-4}	0.21×10^{-3}	-0.9997943714
	1×10^{-5}	0.13×10^{-3}	-0.9998723965
	1×10^{-6}	0.35×10^{-3}	-0.9996510458
3	1×10^{-4}	0.11×10^{-1}	-0.9891804825
	1×10^{-5}	0.16×10^{-1}	-0.9844916687
	1×10^{-6}	0.15×10^{-1}	-0.9847371004
4	1×10^{-4}	0.67×10^{-1}	-0.9328956603
	1×10^{-5}	0.42×10^{-1}	-0.9584787572
	1×10^{-6}	0.69×10^{-1}	-0.9308129399

Table 4. The abscissae of stability of $f_2(s)$.

n	permissible error	actual error	α_{comp}
1	1×10^{-4}	0.24×10^{-4}	1.0000242478
	1×10^{-5}	0.29×10^{-5}	1.0000028711
	1×10^{-6}	0.31×10^{-6}	1.0000003056
2	1×10^{-4}	0.16×10^{-4}	1.0000159501
	1×10^{-5}	0.45×10^{-4}	1.0000449319
	1×10^{-6}	0.37×10^{-4}	1.0000372677
3	1×10^{-4}	0.12×10^{-2}	1.0011814091
	1×10^{-5}	0.16×10^{-2}	1.0015600809
	1×10^{-6}	0.14×10^{-2}	1.0014076181
4	1×10^{-4}	0.30×10^{-2}	1.0029858523
	1×10^{-5}	0.62×10^{-2}	1.0061657232
	1×10^{-6}	0.47×10^{-2}	1.0046522891

Table 5. The abscissae of stability of $f_3(s)$.

4.4 Example 4

Consider a unity feedback control system shown in Fig. 3, where the plant transfer function $G(s)$ is given by

$$G(s) = \frac{e^{-\sqrt{s}}}{s(s-1)}. \tag{27}$$

Obviously, the plant is unstable.

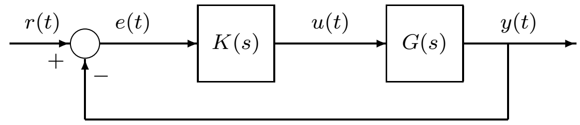

Fig. 3. A unity feedback control system.

The problem is to stabilize the control system using a PD controller

$$K(s) = p_1 + p_2 s \quad \text{where} \quad p_1, p_2 > 0.$$

The characteristic function of the closed loop system is

$$f(s) = s(s-1) + (p_1 + p_2 s)e^{-\sqrt{s}}. \tag{28}$$

In conjunction with the method of inequalities, a stabilizing controller is obtained by solving the inequality

$$\alpha(p) \leq \varepsilon, \quad \varepsilon = -0.001. \tag{29}$$

p_0	(3, 2)	(1, 4)	(1.5, 20)
α_{comp}	0.5657	0.0709	0.3602
p	$(0.7162, 4.3345)$	$(0.6850, 4.3220)$	$(0.8760, 7.0325)$
α_{comp}	−0.0119	−0.0172	−0.0612

Table 6. Stabilization of an unstable RFDDS.

For the details on this, see Arunsawatwong [1996]. By using the moving boundaries process algorithm [see Zakian and Al-Naib, 1973 and also Zakian, 2005], the solutions of (29) are located. The stabilizing controllers are obtained from three different starting points. The results are shown in Table 6.

This example demonstrates that the stabilization problem for RFDDSs can be solved easily by iterative numerical methods once a practical method of computing the abscissa of stability is available.

5. CONCLUSIONS

This paper has developed computational tools for solving stability problems associated with the design of linear RFDDS by using the method of inequalities and other numerical optimization approaches. The new stability test is extended from Hwang and Cheng [2006]'s test. By modifying Zakian [1979b]'s technique, a practical method for computing the abscissa of stability is established. The numerical results show that the method developed here is effective, even when the iteration is disrupted by serious round-off error during computation. Also, the numerical example shows that the stabilization problem of RFDDS can be easily solved by numerical methods once a method of computing the abscissa of stability is available.

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Appendix A. MODEL OF TEMPERATURE CONTROL SYSTEM

The heat conduction plant is shown in Fig. A.1. The rod has length L, cross-sectional area A and is made of a material with density ρ, heat capacity C and thermal conductivity σ.

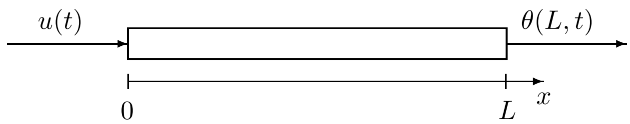

Fig. A.1. The heating metallic rod.

The control signal $u(t)$ is the heat flow injected at $x = 0$. The output $y(t) = \theta(L, t)$ is the temperature measured at $x = L$.

Schwarz and Friedland [1965] show that with appropriate boundary conditions, the transfer function

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{1}{\sigma A\sqrt{\lambda s}.\sinh(\sqrt{\lambda s}\,L)}, \tag{A.1}$$

where the parameter $\lambda \triangleq C\rho/\sigma$.

The heat conduction process is controlled by a heat flow at $x = 0$ so that the temperature at $x = L$ is kept as close as possible to a reference signal. With the control structure in Fig. 3 and $K(s) = p$, the characteristic function of the closed-loop system is

$$f_1(s) = \sigma A\sqrt{\lambda s}(1 - e^{-2L\sqrt{\lambda s}}) + 2pe^{-L\sqrt{\lambda s}}. \tag{A.2}$$