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Mean-variance receding horizon control for discrete time linear stochastic systems

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Abstract: A control strategy based on a mean-variance objective and expected value constraints is proposed for systems with additive and multiplicative stochastic uncertainty. Subject to a mean square stabilizability condition, the receding horizon objective can be obtained by solving a system of Lyapunov equations. An algorithm is proposed for computing the unconstrained optimal control law, which is the solution of a pair of coupled algebraic Riccati equations, and conditions are given for its convergence. A receding horizon controller based on quasi-closed loop predictions is defined. The control law is shown to provide a form of stochastic convergence of the state, and to ensure that the time average of the state variance converges to known bounds.

Keywords: stochastic optimal control; receding horizon control; stability.

1. INTRODUCTION

The receding horizon control methodology provides computationally tractable optimal control laws by solving constrained control problems online. Most real-life control problems are not only subject to constraints but also involve multiplicative or additive stochastic uncertainty. However realistic formulations of such control problems do not usually admit analytical solutions, and this motivates the development of computational optimal control laws that take explicit account of stochastic uncertainty.

A well-established method of formulating an optimal control problem so as to incorporate information on the distribution of model uncertainty is to consider the expected value of a quadratic cost index. Earlier LQG formulations were generalized to incorporate system constraints by Lee and Cooley (1998) and Batina et al. (2002). More recent work has developed efficient non-conservative methods (van Hessem and Bosgra, 2002; Couchman et al., 2006a; Primbs, 2007) that have improved the applicability of the approach. To provide a means of incorporating probabilistic measures in the control problem, performance indices with mixed mean and variance terms have been proposed (Zhu et al., 2004; Freiling et al., 1999). Control objectives of this type enable conflicting requirements of nominal performance and minimum variance to be balanced. This paper is concerned with the use of a mean-variance objective in conjunction with a constrained receding horizon control framework for stochastic systems.

Earlier work (Couchman et al., 2006b) considered multiplicative and additive model uncertainty, and provided analytical expressions for a mean-variance cost and a basic framework for analyzing closed-loop stability. The current paper extends this work in two main respects. We present an algorithm for solving coupled algebraic Riccati equations (CARE) enabling the offline computation of the unconstrained optimal control law, and provide conditions for its convergence. In addition we extend the convergence analysis of the closed loop system under constrained receding horizon control. The paper is organized as follows. The plant model and control problem are defined in Sections 2 and 3, and the receding horizon performance objective is defined in Section 4. Section 5 considers the unconstrained LQ optimal control law, Section 6 analyzes closed loop stability and convergence of the receding horizon controller, and Section 7 concludes with a numerical example.

2. PLANT MODEL

Consider the stochastic discrete time plant model

$$x(k+1) = A_k x(k) + B_k u(k) + d_k, \quad y(k) = Cx(k) \qquad (1)$$

with $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, and

$$[A_k \ B_k \ d_k] = [\bar{A} \ \bar{B} \ 0] + \sum_{j=1}^{m} [A^{(j)} \ B^{(j)} \ d^{(j)}] q_k^{(j)}$$

where $q_k^{(j)}$ are random variables with known mean and variance. Let $q_k = [q_k^{(1)} \ \cdots \ q_k^{(m)}]^T$. It is assumed that $\{q_k, \ k = 0, 1, \ldots\}$ is stationary and that $\{q_k, q_j\}$ are statistically independent for all $k \neq j$. Without loss of generality we assume that q has mean $\mathbb{E}(q) = 0$ and covariance $\mathrm{Cov}(q) = I$. It is also assumed that (A, B) is mean square stabilizable, i.e. for any symmetric $\Sigma \succ 0$ there exist $K \in \mathbb{R}^{n_u \times n_x}$ and symmetric $\Pi \succ 0$ satisfying

$$\Pi - \mathbb{E}\big[(A + BK)^T \Pi (A + BK)\big] = \Sigma. \qquad (2)$$

Let r be a reference for the plant output y_k, and let $\bar{x}_{ss}$ satisfy the steady state conditions:

$$\bar{x}_{ss} = \bar{A}\bar{x}_{ss} + \bar{B}u_{ss}, \quad r = C\bar{x}_{ss}.$$

Let $w(k) = u(k) - u_{ss}$, $z(k) = y(k) - r$, and $v(k) = x(k) - \bar{x}_{ss}$ (in the following we also use v_k to denote $v(k)$), then

$$v(k+1) = A_k v(k) + B_k w(k) + \delta_k, \quad z(k) = Cv(k) \qquad (3)$$

where $\delta_k = (B_k - \bar{B})u_{ss} + d_k = \sum_j (\tilde{B}_j u_{ss} + \tilde{d}_j) q_k^j$. The aim of the controller is to regulate v_k about the origin.

3. CONTROL OBJECTIVE AND CONSTRAINTS

Denote the expectation of $v(k+i)$ conditional on v_k as $\bar{v}(k+i|k) = \mathbb{E}_k(v(k+i))$, and let $\tilde{v}(k+i|k) = v(k+i|k) - \bar{v}(k+i|k)$. Assuming that v_k is known at time k, the control objective is the minimization of a quadratic cost:

$$J(v_k, \mathbf{w}_k) = \sum_{i=0}^{\infty} \big[\|\bar{v}(k+i|k)\|_Q^2 + \|\bar{w}(k+i|k)\|_R^2\big] + \kappa^2 \sum_{i=0}^{\infty} \mathbb{E}_k \big[\|\tilde{v}(k+i|k)\|_Q^2 + \|\tilde{w}(k+i|k)\|_R^2\big] \quad (4)$$

for $Q, R \succ 0$. Here $\mathbf{w}_k = \{w(k+i|k),\ i = 0, 1, \ldots\}$ is a sequence of predicted future control inputs and $\kappa > 0$ defines a trade-off between mean and variance of state/input predictions. This form of objective was proposed for optimal portfolio selection (Zhu et al., 2004) and sustainable development problems (Couchman et al., 2006a,b). If $v(k+i|k), w(k+i|k)$ are normally distributed, then the stage cost in (4) has an interpretation in terms of bounds on state and input trajectories that hold with a probability specified by κ (Couchman et al., 2006b). The cost (4) is to be minimized subject to linear constraints:

$$F\bar{v}(k+i|k) + G\bar{w}(k+i|k) \le h, \quad i = 0, 1, \ldots, \quad (5)$$

on the expected values of predicted state/input variables.

4. COST FUNCTION

This section expresses the cost (4) in a form that can be optimized online. To obtain a finite parameterization of the infinite horizon cost, the input sequence $\mathbf{w}_k$ is specified as

$$\begin{aligned} w(k+i|k) &= Kv(k+i|k) + f(i|k), \quad i = 0, 1 \ldots \\ f(i|k) &= 0, \quad i = N, N+1 \ldots \end{aligned} \quad (6)$$

where $f(i|k)$, $i = 0, \ldots, N-1$ are optimization variables. The corresponding predicted state sequence satisfies

$$\begin{aligned} v(k+i+1|k) &= \Phi_i v(k+i|k) + B_i f(i|k) + \delta_i \\ \Phi &= A + BK. \end{aligned} \quad (7)$$

Assume that K is chosen so that Φ is mean square stable. Then $\bar{\Phi} = \bar{A} + \bar{B}K$ is stable and the first sum in (4) is therefore well defined. Furthermore it can be shown that the covariance of $v(k+i|k)$ converges if Φ is mean square stable: $\lim_{i\to\infty} \mathbb{E}[\tilde{v}(k+i|k)\tilde{v}(k+i|k)^T] = \Theta$, where

$$\Theta - \mathbb{E}[\Phi\Theta\Phi^T] = \mathbb{E}[\delta\delta^T]. \quad (8)$$

But this implies that the terms summed in (4) satisfy

$$\lim_{i\to\infty} \mathbb{E}\big[\|\tilde{v}(k+i|k)\|_Q^2 + \|\tilde{w}(k+i|k)\|_R^2\big] = \mathrm{tr}(\Theta S)$$
$$S = Q + K^T RK$$

and hence J is necessarily infinite. To obtain a finite cost we therefore redefine the objective function as

$$V(v_k, f_k) = \sum_{i=0}^{\infty} \big[\|\bar{v}(k+i|k)\|_Q^2 + \|\bar{w}(k+i|k)\|_R^2\big] + \kappa^2 \sum_{i=0}^{\infty} \mathbb{E}_k \big[\|\tilde{v}(k+i|k)\|_Q^2 + \|\tilde{w}(k+i|k)\|_R^2 - \mathrm{tr}(\Theta S)\big]. \quad (9)$$

Note that, although $V \not\succeq 0$, minimizing J over $\mathbf{w}_k$ is equivalent to minimizing V over $f_k = [f^T(0|k)\ \cdots\ f^T(N-1|k)]^T$. To evaluate $V(v_k, f_k)$ it is convenient to use an autonomous formulation of prediction dynamics (6)-(7):

$$z_{i+1} = \begin{bmatrix} \Phi_i & B_i E & \delta_i \\ 0 & M & 0 \\ 0 & 0 & 1 \end{bmatrix} z_i, \quad z_k = \begin{bmatrix} v_k \\ f_k \\ 1 \end{bmatrix}$$

with $v(k+i|k) = [I_{n_x}\ 0\ 0]z_{k+i}$, $w(k+i|k) = [K\ E\ 0]z_{k+i}$, and

$$M = \begin{bmatrix} 0 & I_{n_u} & 0 & \cdots & 0 \\ 0 & 0 & I_{n_u} & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad E = [I_{n_u}\ 0\ \cdots\ 0].$$

Lemma 1. Along trajectories of (7) the cost (9) is given by

$$V(v_k, f_k) = z_k^T P z_k, \quad P = \begin{bmatrix} P_v & P_{vf} & P_{v1} \\ P_{fv} & P_f & P_{f1} \\ P_{1v} & P_{1f} & P_1 \end{bmatrix} \quad (10)$$

where $P = (1-\kappa^2)X + \kappa^2 Y$ for matrices X, Y with blocks (conformal to the partitioning of z_k and P) defined by

$$\begin{bmatrix} X_v & X_{vf} \\ X_{fv} & X_f \end{bmatrix} - \begin{bmatrix} \bar{\Phi} & \bar{B}E \\ 0 & M \end{bmatrix}^T \begin{bmatrix} X_v & X_{vf} \\ X_{fv} & X_f \end{bmatrix} \begin{bmatrix} \bar{\Phi} & \bar{B}E \\ 0 & M \end{bmatrix} = \begin{bmatrix} S & K^T RE \\ E^T RK & E^T RE \end{bmatrix} \quad (11a)$$

$$[X_{1v}\ X_{1f}] = [0\ 0], \quad X_1 = 0 \quad (11b)$$

and

$$\begin{bmatrix} Y_v & Y_{vf} \\ Y_{fv} & Y_f \end{bmatrix} - \mathbb{E}\left(\begin{bmatrix} \Phi & BE \\ 0 & M \end{bmatrix}^T \begin{bmatrix} Y_v & Y_{vf} \\ Y_{fv} & Y_f \end{bmatrix} \begin{bmatrix} \Phi & BE \\ 0 & M \end{bmatrix}\right) = \begin{bmatrix} S & K^T RE \\ E^T RK & E^T RE \end{bmatrix} \quad (12a)$$

$$[Y_{1v}\ Y_{1f}] \left(I - \begin{bmatrix} \bar{\Phi} & \bar{B}E \\ 0 & M \end{bmatrix}\right) = \mathbb{E}\big(\delta^T Y_v\, [\Phi\ BE]\big) \quad (12b)$$

$$Y_1 = -\mathrm{tr}(\Theta Y_v). \quad (12c)$$

Proof: Using $\mathbb{E}(\|\tilde{v}\|^2) = \mathbb{E}(\|v\|^2) - \|\bar{v}\|^2$, V can be rewritten

$$V(v_k, f_k) = (1-\kappa^2) \sum_{i=0}^{\infty} \big[\|\bar{v}(k+i|k)\|_Q^2 + \|\bar{w}(k+i|k)\|_R^2\big] + \kappa^2 \sum_{i=0}^{\infty} \mathbb{E}_k \big[\|v(k+i|k)\|_Q^2 + \|w(k+i|k)\|_R^2 - \mathrm{tr}(\Theta S)\big] \quad (13)$$

Given that $\bar{\Phi}$ is stable and Φ is mean square stable by assumption, standard Lyapunov arguments can be used to show that the two sums in (13) are equal to $z_k^T X z_k$ and $z_k^T Y z_k$ respectively. □

5. UNCONSTRAINED LQ OPTIMAL CONTROL

This section discusses the offline computation of the gain K in (6). We show that, if constraints are inactive, then the minimizing control for (9) is affine state feedback with a linear gain matrix defined by a CARE. We propose, and analyze the convergence of, an iterative solution method similar to the Lyapunov iterations proposed for CAREs associated with related continuous time control problems (Gajic and Borno, 1995; Freiling et al., 1999).

Dynamic programming is not applicable since the cost defined in Section 4 is non-separable. In particular, (13) consists of two performance indices: one evaluated along trajectories of the stochastic plant model, the other along trajectories of the deterministic model given by the expected values of plant parameters. We therefore derive the optimal control law by considering the asymptotic properties of the problem of minimizing $V(v_k, f_k)$ as $N \to \infty$.

Lemma 2. The minimizer $f^*(v_k) = \arg\min_f V(v_k, f)$ is unique and satisfies

$$f^*(v_k) = -P_f^{-1} P_{fv} v_k - P_f^{-1} P_{f1}. \quad (14)$$

Proof: For $\Gamma = [K \;\; E]$, (11a) implies

$$\begin{bmatrix} X_v & X_{vf} \\ X_{fv} & X_f \end{bmatrix} - \bar{\Psi}^T \begin{bmatrix} X_v & X_{vf} \\ X_{fv} & X_f \end{bmatrix} \bar{\Psi} \succeq \Gamma\Gamma^T, \quad \Psi = \begin{bmatrix} \bar{\Phi} & \bar{B}E \\ 0 & M \end{bmatrix}$$

where, by construction, $\bar{\Psi}$ is stable and $(\bar{\Psi}, \Gamma)$ is observable. It follows that

$$\begin{bmatrix} X_v & X_{vf} \\ X_{fv} & X_f \end{bmatrix} \succ 0. \tag{15}$$

Similarly, subtracting (11a) from (12a) gives

$$\begin{bmatrix} Y_v - X_v & Y_{vf} - X_{vf} \\ Y_{fv} - X_{fv} & Y_f - X_f \end{bmatrix} - \bar{\Psi}^T \begin{bmatrix} Y_v - X_v & Y_{vf} - X_{vf} \\ Y_{fv} - X_{fv} & Y_f - X_f \end{bmatrix} \bar{\Psi} \succeq 0$$

which implies that

$$\begin{bmatrix} Y_v & Y_{vf} \\ Y_{fv} & Y_f \end{bmatrix} - \begin{bmatrix} X_v & X_{vf} \\ X_{fv} & X_f \end{bmatrix} \succeq 0. \tag{16}$$

From (15),(16) and $P = X + \kappa^2(Y - X)$, it follows that $P_f \succ 0$ for all $\kappa > 0$ and all $N \geq 1$. Hence $f^*(v_k)$ is defined uniquely by (14). □

The following theorem is based on the observation that if $f^*(v_k)$ is independent of v_k for all N, then the input sequence generated by $f^*(v_k)$ in (6) coincides with the closed loop optimal control law.

Theorem 3. The cost (13) is minimized over all input sequences $\mathbf{w}_k$ by the affine state feedback control law $w(k+i|k) = Kv(k+i|k) + b$ with

$$K = -D^{-1}\big[(1-\kappa^2)\bar{B}^T X_v \bar{A} + \kappa^2 \mathbb{E}(B^T Y_v A)\big] \tag{17a}$$

$$b = -D^{-1}\kappa^2\big[\bar{B}^T Y_{v1} + \mathbb{E}(B^T Y_v \delta)\big] \tag{17b}$$

and $D = R + (1-\kappa^2)\bar{B}^T X_v \bar{B} + \kappa^2 \mathbb{E}(B^T Y_v B)$.

Proof: From (14), f^* is independent of v_k iff $P_{fv} = 0$. But $P_{fv} - M^T P_{fv} \bar{\Phi} = E^T\big[RK + (1-\kappa^2)\bar{B}^T X_v \bar{\Phi} + \kappa^2 \mathbb{E}(B^T Y_v \Phi)\big]$ from (12a), and hence $P_{fv} = 0$ for all N if and only if K satisfies (17a). Under this condition (12b) implies that each element of f^* is equal to b defined in (17b). □

Corollary 4. The optimal feedback gain is defined by the solution of the coupled algebraic Riccati equations:

$$X_v = (\bar{A} + \bar{B}K)^T X_v (\bar{A} + \bar{B}K) + Q + K^T R K \tag{18a}$$

$$Y_v = \mathbb{E}\big[(A + BK)^T Y_v (A + BK)\big] + Q + K^T R K \tag{18b}$$

where $K = -D^{-1}[(1-\kappa^2)\bar{B}^T X_v \bar{A} + \kappa^2 \mathbb{E}(B^T Y_v A)]$.

Proof: (18a,b) follow directly from (11), (12) and (17a). □

The problem of solving (18) for (X_v, Y_v, K) is non-convex and, unlike the conventional algebraic Riccati equation, there is no transformation that leads to a LMI formulation. We propose the following iteration for computing K.

Algorithm 1. Set $X_v^{(0)} = Y_v^{(0)} = Q$. For $i = 0, 1, \ldots$ set:

$$K^{(i)} = \arg\min_{K^{(i)}} \operatorname{tr}(P_v^{(i+1)}) \tag{19a}$$

$$X_v^{(i+1)} = \bar{\Phi}^{(i)T} X_v^{(i)} \bar{\Phi}^{(i)} + Q + K^{(i)T} R K^{(i)} \tag{19b}$$

$$Y_v^{(i+1)} = \mathbb{E}(\Phi^{(i)T} Y_v^{(i)} \Phi^{(i)}) + Q + K^{(i)T} R K^{(i)} \tag{19c}$$

where $P_v^{(i)} = (1-\kappa^2)X_v^{(i)} + \kappa^2 Y_v^{(i)}$, $\bar{\Phi}^{(i)} = \bar{A} + \bar{B}K^{(i)}$, and $\Phi^{(i)} = A + BK^{(i)}$.

The proof that $\{X_v^{(i)}, Y_v^{(i)}\}$ converges to a solution of (18a,b) relies on the following result.

Lemma 5. $\operatorname{tr}(P_v^{(i)}) \leq \gamma$ for all i, for some $\gamma > 0$.

Proof: Given any mean square stabilizing linear feedback gain $\hat{K}$, there exist $\hat{X}_v, \hat{Y}_v \succ 0$ satisfying

$$\hat{X}_v = \hat{\bar{\Phi}}^T \hat{X}_v \hat{\bar{\Phi}} + Q + \hat{K}^T R \hat{K}, \qquad \hat{\bar{\Phi}} = \bar{A} + \bar{B}\hat{K} \tag{20a}$$

$$\hat{Y}_v = \mathbb{E}(\hat{\Phi}^T \hat{Y}_v \hat{\Phi}) + Q + \hat{K}^T R \hat{K}, \quad \hat{\Phi} = A + B\hat{K}. \tag{20b}$$

The optimality of $K^{(i)}$ in (19a) implies that $\operatorname{tr}(WP_v^{(i+1)})$ is minimized for any $W = W^T \succ 0$, so that any $\hat{K}$ yields

$$\operatorname{tr}(WP_v^{(i+1)}) \leq \kappa^2 \operatorname{tr}\big[\mathbb{E}(\hat{\Phi} W \hat{\Phi}^T) Y_v^{(i)}\big] + (1-\kappa^2)\operatorname{tr}\big[\hat{\bar{\Phi}} W \hat{\bar{\Phi}}^T X_v^{(i)}\big] + \operatorname{tr}\big[W(Q + \hat{K}^T R \hat{K})\big]. \tag{21}$$

For $\kappa \leq 1$, we therefore have

$$\operatorname{tr}(WP_v^{(i+1)}) \leq \operatorname{tr}\big[\mathbb{E}(\hat{\Phi} W \hat{\Phi}^T) P_v^{(i)}\big] + \operatorname{tr}\big[W(Q + \hat{K}^T R \hat{K})\big]$$

which, with (20a), gives

$$\operatorname{tr}[W(P_v^{(i+1)} - \hat{Y}_v)] \leq \operatorname{tr}\big[\mathbb{E}(\hat{\Phi} W \hat{\Phi}^T)(P_v^{(i)} - \hat{Y}_v)\big]$$

and since this holds for all $W = W^T \succ 0$, $P_v^{(i)} \preceq \hat{Y}_v$ therefore implies $P_v^{(i+1)} \preceq \hat{Y}_v$. Hence $P_v^{(i)} \preceq \hat{Y}_v$ for all i if $P_v^{(0)} \preceq \hat{Y}_v$. For $\kappa > 1$, it can be shown using (21) that there exist $\hat{K}$ and $W \succeq \mathbb{E}(\hat{\Phi} W \hat{\Phi}^T) \succ 0$ satisfying $\operatorname{tr}(WP_v^{(i+1)}) \leq \operatorname{tr}(WP_v^{(i)})$ whenever $\operatorname{tr}(WP_v^{(i)})$ is sufficiently large, and hence $\operatorname{tr}(P^{(i)})$ is bounded uniformly in i. □

For any sequence $\{K^{(i)}, i = 0, \ldots N-1\}$ generated by Algorithm 1, define $\{S_k,\ k = 0, \ldots N-1\}$ by

$$S_{k+1} = \mathbb{E}(\Phi^{(N-k-1)} S_k \Phi^{(N-k-1)T}) \tag{22}$$

with $S_0 = S_0^T \succeq 0$. We next show that S_N necessarily converges as $N \to \infty$ since $\{P^{(i)}\}$ is bounded, then use this result to demonstrate convergence of the iteration (19).

Theorem 6. For any $S_0 \succeq 0$, the sequence $\{S_k\}$ generated by (22) satisfies $\lim_{N\to\infty} S_N = 0$.

Proof: Pre-multiplying both sides of (19c) by S_k, where $k = N - i - 1$, extracting the trace and using (22) gives

$$\operatorname{tr}(S_k Y_v^{(i+1)}) = \operatorname{tr}(S_{k+1} Y_v^{(i)}) + \operatorname{tr}\big[S_k(Q + K^{(i)T} R K^{(i)})\big],$$

Since this equation holds for $i = 0, \ldots N-1$ we obtain

$$\operatorname{tr}(S_0 Y_v^{(N)}) = \sum_{i=0}^{N-1} \operatorname{tr}\big[S_{N-i-1}(Q + K^{(i)T} R K^{(i)})\big] + \operatorname{tr}(S_N Y_v^{(0)}).$$

However the definition of $P_v^{(i)}$ implies that $\kappa^2 Y_v^{(i)} \preceq P_v^{(i)}$ if $\kappa \leq 1$ and $Y_v^{(i)} \preceq P_v^{(i)}$ if $\kappa > 1$, and it follows from Lemma 5 that $\operatorname{tr}(S_0 Y_v^{(N)}) \leq \infty$ for all N. Therefore

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} \operatorname{tr}\big[S_{N-i-1}(Q + K^{(i)T} R K^{(i)})\big] + \operatorname{tr}(S_N Y_v^{(0)}) < \infty$$

and hence $S_N \to 0$ as $N \to \infty$ since $Q \succ 0$ and $S_k \succeq 0$. □

Corollary 7. $K^{(i)} \to K$ satisfying (17a) and $X_v^{(i)}$, $Y_v^{(i)}$ converge to solutions of (18a,b) as $i \to \infty$.

Proof: Theorem 6 implies that $\operatorname{tr}(S_0 Y_v^{(i)})$ converges to a finite limit for any $S_0 \succeq 0$, and therefore $\{X_v^{(i)}, Y_v^{(i)}, K^{(i)}\}$ converges to a fixed point of (19a-c) as $i \to \infty$. Moreover any fixed point of (19a-c) is a solution of (18a,b). □

6. RECEDING HORIZON CONTROL, STABILITY AND CONVERGENCE

The optimal control no longer has the form of a fixed affine feedback law when constraints (5) are active. We therefore propose a receding horizon control law which is to be computed online by minimizing (9) numerically subject to constraints. This section demonstrates that a stochastic form of stability holds, and derives bounds on the convergence of the time-average of the state covariance.

The problem formulation considered here does not assume that bounds are available on the stochastic plant parameters. Therefore it is not possible to guarantee the feasibility of constraints applied to predictions. Instead we denote the feasible set for the linear constraints (5) as $\mathcal{F}$, and define the following receding horizon optimization control law.

Algorithm 2. At times $k = 0, 1, \ldots$:

1. If $v_k \in \mathcal{F}$, minimize the cost (9) by computing
$$f^*(v_k) = \arg\min_f V(v_k, f) \quad \text{subject to (5)} \tag{23}$$
2. If $v_k \notin \mathcal{F}$, minimize the constraint violation via
$$f^*(v_k) = \arg\min_f \max_i F\bar{v}(k+i|k) + G\bar{w}(k+i|k) - h$$
$$\text{subject to } V(v_k, f) \leq V(v_k, Mf^*(v_{k-1})) \tag{24}$$
3. Implement $w_k = Kv_k + Ef^*(v_k)$.

Note that (23) is a linearly constrained quadratic program, whereas (24) is a quadratically constrained linear program, and the constraint in (24) is necessarily feasible. The purpose of the constraint in (24) is to enforce a Lyapunov-like inequality on the increment in predicted cost; this forms the basis of the convergence analysis below.

6.1 The case of $\kappa \geq 1$

For $\kappa \geq 1$ the following lemma shows that the predicted cost corresponding to the solution of (23) or (24) necessarily decreases whenever $\|(v_k, w_k)\|$ is sufficiently large.

Lemma 8. Let $V^*(v_k) = V(v_k, f^*(v_k))$, then the receding horizon application of Algorithm 2 ensures that
$$V^*(v_k) - \mathbb{E}_k V^*(v_{k+1}) \geq \|v_k\|_Q^2 + \|w_k\|_R^2 - \kappa^2 \mathrm{tr}(\Theta S). \tag{25}$$

Proof: Using (11) and (12) it can be shown that
$$V^*(v_k) - \mathbb{E}_k V(v_{k+1}, Mf^*(v_k))$$
$$= \begin{bmatrix} v_k \\ f^*(v_k) \end{bmatrix}^T \begin{bmatrix} S & K^T RE \\ E^T RK & E^T RE \end{bmatrix} \begin{bmatrix} v_k \\ f^*(v_k) \end{bmatrix} - \kappa^2 \mathrm{tr}(\Theta S)$$
$$+ (\kappa^2 - 1) \sum_{j=1}^m \|(A^{(j)} + B^{(j)}K)v_k + B^{(j)}Ef^*(v_k) + \delta^{(j)}\|_{X_v}^2 \tag{26}$$
However the objective in (23) and the constraint in (24) ensure that $V^*(v_k) \leq V(v_k, Mf^*(v_{k-1}))$, and therefore
$$\mathbb{E}_k V^*(v_{k+1}) \leq \mathbb{E}_k V(v_{k+1}, Mf^*(v_k)). \tag{27}$$
Combining (26),(27), and noting that the last term in (26) is non-negative for $\kappa \geq 1$ yields the inequality (25). □

From Lemma 8 we obtain the following bound on the mean of $\|(v_k, w_k)\|$ in closed loop operation.

Theorem 9. Under Algorithm 2 the closed loop state and input trajectories satisfy:
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_0(\|v_k\|_Q^2 + \|w_k\|_R^2) \leq \kappa^2 \mathrm{tr}(\Theta S). \tag{28}$$

Proof: Taking expectations and summing both sides of (25) over $k = 0, \ldots, n-1$ gives
$$\sum_{k=0}^{n-1} \mathbb{E}_0 \left[\|v_k\|_Q^2 + \|w_k\|_R^2 - \kappa^2 \mathrm{tr}(\Theta S) \right] \leq V^*(v_0) - \mathbb{E}_0 V^*(v_n),$$
which implies (28) since $V^*(v_0)$ is finite by assumption and, from (13), $V^*(v)$ has a finite minimum value. □

Remark 10. For $\kappa = 1$, Theorem 9 implies that the stage cost $\|v_k\|_Q^2 + \|w_k\|_R^2$ in closed loop operation converges in mean for to a value no greater than that obtained along the predicted trajectories of (6),(7). However for $\kappa > 1$ the receding horizon controller places greater emphasis on minimizing the variance $\mathbb{E}(\|v_k - \bar{v}_k\|_Q^2 + \|w_k - \bar{w}_k\|_R^2)$ at the expense of increased expected values $\|\bar{v}_k\|_Q$ and $\|\bar{w}_k\|_R$.

If the additive disturbance δ_k in (3) were non-persistent, with $\lim_{k\to\infty} \mathbb{E}(\delta_k \delta_k^T) = 0$, so that $\Theta = 0$ in (8), then (25) would imply mean square stability of the closed loop system and hence ensure convergence: $v_k \to 0$ with probability 1 (w.p.1) (Kushner, 1971). For the more general problem considered here mean square stability does not apply, however it is possible to obtain an alternative characterization of stability based on convergence of v_k to a set Ω defined by $\Omega = \{v : v^T Q v \leq \kappa^2 \mathrm{tr}(\Theta S)\}$.

Theorem 11. If $v_0 \notin \Omega$, then the closed loop system under Algorithm 2 satisfies $v_k \in \Omega$ for some $i > k$ w.p.1.

Proof: Define a sequence $\{\hat{v}_k\}$ by
$$\hat{v}_k = \begin{cases} v_k & \text{if } \|v_i\|_Q^2 > \kappa^2 \mathrm{tr}(\Theta S) \ \forall i = 0, \ldots, k-1 \\ \hat{v}_{k-1} & \text{if } \|v_i\|_Q^2 \leq \kappa^2 \mathrm{tr}(\Theta S) \text{ for any } i = 0, \ldots, k-1 \end{cases}$$
then (25) implies
$$V^*(\hat{v}_k) - \mathbb{E}_k V^*(\hat{v}_{k+1}) \geq \|\hat{v}_k\|_Q^2 - \kappa^2 \mathrm{tr}(\Theta S) \geq 0$$
for all k, implying that $V^*(\hat{v}_k)$ is a supermartingale. Since $V^*(\hat{v}_k)$ is lower-bounded, it follows that $\|\hat{v}_k\|_Q^2 \to \kappa^2 \mathrm{tr}(\Theta S)$ w.p.1. (Kushner, 1971). □

Remark 12. Theorem 11 demonstrates that every state trajectory of the closed loop system (3) converges to the set Ω, although subsequently it may not remain in Ω. The same result shows that the state continually returns to Ω.

6.2 The case of $\kappa < 1$

With $\kappa < 1$ the expected value of $V^*(v_k)$ in (25) may not decrease even when the state and input are large. In order to quantify the convergence of the closed loop system in this case, we therefore use an input-state stability argument that does not rely on monotonicity of the optimal cost $V^*(v_k)$. Throughout this section it is assumed that K is defined as the unconstrained LQ optimal feedback gain.

The approach uses bounds on the first element $f^*(0|k)$ of the optimal solution $f^*(v_k)$ for (23) or (24) in Algorithm 2. These bounds are derived below by considering only the terms in the cost (10) that depend on f_k. If K is LQ optimal, then for given v_k, minimizing $V(v_k, f_k)$ is equivalent to minimizing
$$V_f(f_k) = f_k^T P_f f_k + 2 f_k^T P_{f1}$$
where
$$P_f = \begin{bmatrix} D & & \\ & \ddots & \\ & & D \end{bmatrix}, \quad P_{f1} = \begin{bmatrix} g \\ \vdots \\ g \end{bmatrix},$$
D is defined in Theorem 3 and $g = \kappa^2[\bar{B}^T Y_{v1} + \mathbb{E}(B^T Y_v \delta)]$.

Lemma 13. f^* computed by Algorithm 2 satisfies

$$\sum_{k=0}^{n-1} \mathbb{E}_0\big[\|f^*(0|k)\|_W^2 + 2g^T f^*(0|k)\big] \leq V_{f,0}^* - \mathbb{E}_0 V_{f,n-1}^* \quad (29)$$

where $V_{f,k}^*$ denotes $V_f\big(f^*(v_k)\big)$.

Proof: The objective in (23) and the constraint in (24) ensure that $V(f^*(v_k)) \leq V(Mf^*(v_{k-1}))$ for all k. Given the structure of P_f and P_{f1} it follows that

$$V_{f,k}^* - \mathbb{E}_k V_{f,k+1}^* \geq \|f^*(0|k)\|_W^2 + 2g^T f^*(0|k), \quad (30)$$

and (29) is obtained by taking expectations and summing both sides of this inequality for $k = 0, \dots, n-1$. □

We next show that the dynamics of the closed loop system mapping $f^*(0|k)$ to v_k have finite l_2-gain.

Theorem 14. There exist $\beta > 0$ and $\hat{Y} \succ 0$ satisfying

$$\sum_{k=0}^{n-1}\big(\mathbb{E}_0[v_k^T S v_k] - \mathrm{tr}(\Theta S)\big) \leq \sum_{k=0}^{n-1}\mathbb{E}_0\big[\beta\|f^*(0|k)\|^2 + 2\hat{g}^T f^*(0|k)\big] + v_0^T \hat{Y}_v v_0 - \mathbb{E}_0(v_n^T \hat{Y}_v v_n) \quad (31)$$

for any given $\hat{g} \in \mathbb{R}^{n_u}$.

Proof: Since $\Phi = A + BK$ is mean square stable by construction, there necessarily exists $\hat{Y}_v \succ 0$ satisfying

$$\hat{Y}_v - \mathbb{E}\big[\Phi^T \hat{Y}_v \Phi\big] - S \succ 0.$$

It follows that the condition:

$$\begin{bmatrix} \hat{Y}_v - S & 0 & 0 \\ 0 & \beta I & \hat{g} \\ 0 & \hat{g}^T & \gamma \end{bmatrix} - \mathbb{E}\left\{ \begin{bmatrix} \Phi^T \\ B^T \\ \delta^T \end{bmatrix} \hat{Y}_v \, [\Phi \;\; B \;\; \delta] \right\} \succeq 0 \quad (32)$$

is satisfied for $\gamma \geq \mathbb{E}(\delta^T \hat{Y}_v \delta) \geq \mathrm{tr}(\Theta S)$ and sufficiently large (but finite) β. Pre- and post-multiplying (32) by $(v_k, f^*(0|k), 1)$ gives

$$v_k^T \hat{Y}_v v_k - \mathbb{E}_k(v_{k+1}^T \hat{Y}_v v_{k+1}) + \beta\|f^*(0|k)\|^2 + 2\hat{g}^T f^*(0|k) \leq v_k^T S v_k - \mathrm{tr}(\Theta S).$$

Taking expectations and summing both sides of this inequality for $k = 0, \dots, n-1$ yields (31). □

Bounds on the convergence of the state in closed loop operation can be obtained by combining Lemma 13 and Theorem 14.

Theorem 15. The closed loop state trajectories under Algorithm 2 satisfy

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_0(v_k^T S v_k) \leq \mathrm{tr}(\Theta S) \quad (33)$$

Proof: If $\hat{g}$ in (31) is chosen as $\beta g/\underline{\sigma}(W)$ (where $\underline{\sigma}(W)$ denotes the minimum singular value of W), then

$$\mathbb{E}_0\big[\beta\|f^*(0|k)\|^2 + 2\hat{g}^T f^*(0|k)\big] \leq \frac{\beta}{\underline{\sigma}(W)} \mathbb{E}_0\big[\|f^*(0|k)\|_W^2 + 2g^T f^*(0|k)\big].$$

From (29) and (31) we therefore have

$$\sum_{k=0}^{n-1}\big(\mathbb{E}_0[v_k^T S v_k] - \mathrm{tr}(\Theta S)\big) \leq \frac{\beta}{\underline{\sigma}(W)}(V_{f,0}^* - \mathbb{E}_0 V_{f,n-1}^*) + v_0^T \hat{Y}_v v_0 - \mathbb{E}_0(v_n^T \hat{Y}_v v_n). \quad (34)$$

Each term on the RHS is finite since $V_{f,0}^*$, $v_0^T \hat{Y} v_0$ are finite by assumption and $V_{f,n-1}^*$ has lower bound $-P_{1f} P_f^{-1} P_{f1}$. Hence (34) implies (33). □

Remark 16. The bound (33) implies that the time-average of $\mathbb{E}(v_k^T S v_k)$ for the closed loop system is asymptotically no greater than the state covariance under the fixed linear feedback law $w_k = K v_k$. This bound is achieved even though Algorithm 2 allows for the handling of constraints that are not accounted for explicitly by the linear feedback law.

7. EXAMPLE

A randomly selected, open loop unstable plant is used to demonstrate the action of the proposed control law:

$$\bar{A} = \begin{bmatrix} 1.00 & 0.16 & -0.50 \\ -0.06 & 0.35 & -0.18 \\ -0.52 & 0.10 & 0.74 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 \\ -0.28 \\ -0.12 \end{bmatrix} \quad C^T = \begin{bmatrix} 0.11 \\ -1.86 \\ 0.71 \end{bmatrix}$$

$$A^{(1)} = \begin{bmatrix} -0.09 & -0.05 & 0.07 \\ -0.09 & 0 & 0.02 \\ 0.06 & 0 & 0.07 \end{bmatrix} \quad B^{(1)} = \begin{bmatrix} 0.01 \\ -0.02 \\ 0.02 \end{bmatrix} \quad d^{(1)} = \begin{bmatrix} 0.03 \\ 0.04 \\ 0.09 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} -0.06 & 0.07 & 0.06 \\ -0.03 & 0.03 & 0.09 \\ 0.10 & 0.07 & 0.05 \end{bmatrix} \quad B^{(2)} = \begin{bmatrix} 0.02 \\ -0.09 \\ 0.01 \end{bmatrix} \quad d^{(2)} = \begin{bmatrix} 0.07 \\ 0.10 \\ 0.09 \end{bmatrix}$$

The reference is set as $r = 10$, which gives $x^{ss} = [-4.1466 \;\; -6.4038 \;\; -2.0492]^T$ and $u^{ss} = 17.1$. The linear feedback gain K is chosen to be LQ optimal, computed using Algorithm 1. Figure 1 shows the evolution of the eigenvalues of $P_v{}^{(k)}$ at iteration k of Algorithm 1 for $\kappa = 1.9$ (for which $K = [-11.7 \;\; -0.913 \;\; 9.02]$) and for $\kappa = 0.5$ (for which $K = [-10.1 \;\; -0.750 \;\; 7.87]$).

For initial condition $x(0) = [15.9 \;\; -6.40 \;\; -2.05]^T$, Fig. 2 shows the ensemble average of the system output response $y(k)$ under receding horizon control with $\kappa = 1.9$. Note from this plot that the output mean converges to a lower value than r, this is due to the trade off in the cost between mean and the variance. For comparison the output response under the linear feedback law is also shown, and the dashed lines show the bounds of mean ± 1 standard deviation. Clearly linear feedback has much greater variance but achieves zero mean error in steady state. The horizontal black lines indicate $r \pm 1$ standard deviation as given by $(C\Theta C^T)^{0.5}$.

If constraints are inactive, the difference between output responses for receding horizon and linear feedback laws becomes less significant as κ is reduced (Fig. 3). However the variance of the receding horizon controller is again reduced when constraints are active. This is shown in Figure 4, where the constraint that the expected value of $y(k)$ should be less than 15 is imposed.

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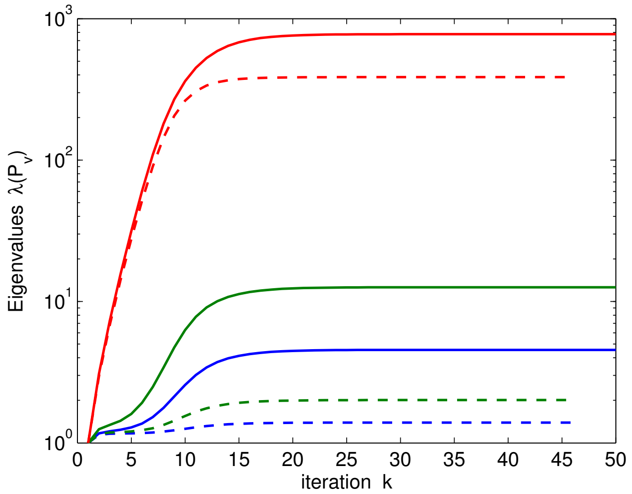

Fig. 1. Variation of eigenvalues of P_v with iteration of Algorithm 1, for $\kappa = 1.9$ (solid lines) and $\kappa = 0.5$ (dashed lines)

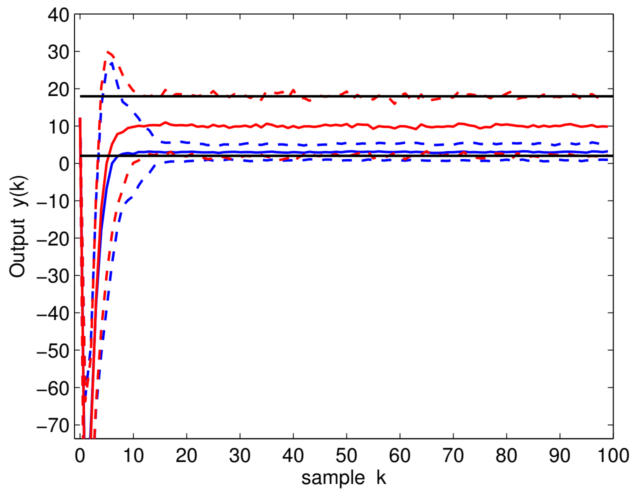

Fig. 2. For $\kappa = 1.9$: ensemble averages (solid lines) and mean ± 1 standard deviation (dashed) for output responses $y(k)$ under Algorithm 2 (blue) and linear feedback (red); bounds $r \pm \sqrt{C\Theta_{ss}C^T}$ shown in black.

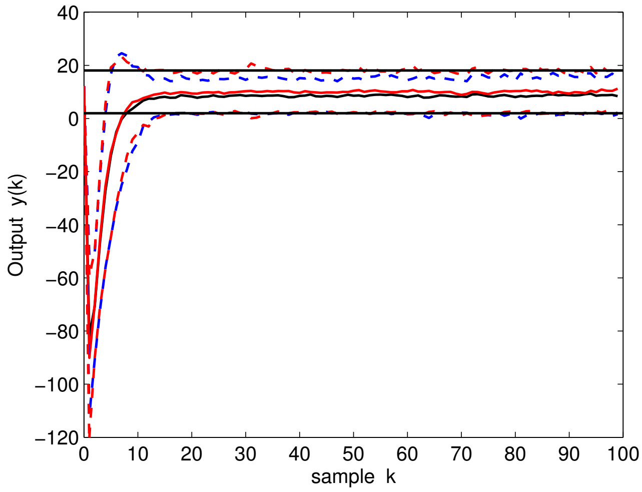

Fig. 3. For $\kappa = 0.5$: ensemble averages (solid lines) and mean ± 1 standard deviation (dashed) for $y(k)$ under receding horizon control (blue) and linear feedback (red).

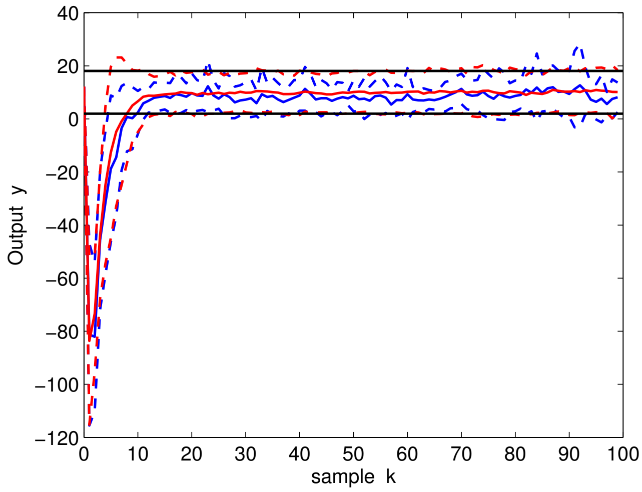

Fig. 4. For $\kappa = 0.5$ and constraint $\mathbb{E}(y) \leq 15$: ensemble averages (solid) and bounds showing mean $\pm$ one variance (dashed) for $y(k)$ under Algorithm 2 (blue) and linear feedback (red).

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