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Numerical Solution of the Isaacs Equation for Differential Games with State Constraints ⋆

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Abstract: We present a numerical approximation for differential games with state constraints. The scheme is based on dynamic programming and on the discretization of the Isaacs equation which describes the value function of the game. Once the approximate value function has been computed we can construct a numerical synthesis of feedback controls in order to reconstruct the corresponding optimal trajectories. Some numerical tests are presented and discussed.

1. INTRODUCTION

In this paper we present a numerical approximation scheme for general differential games with state constraints. In fact, we want to extend our approach for 2-player pursuit-evasion games with state constraints presented in [13] to more general situations where the dynamics is coupled and the effect of the strategies chosen by every player affects all the components (a more precise description of the dynamics will be presented in Section 2). The scheme is based on the dynamic programming approach and derives from a natural generalization of the unconstrained approximation scheme (see the survey papers [5, 14] for a general introduction). Unfortunately, we are not able to give a proof of convergence for our algorithm due to the fact that a precise definition of viscosity solution for the general constrained case is still missing. Our contribution here is mainly at the experimental and numerical level. However, we will show (in Section 3) some interesting examples where our algorithm is able to build a solution of the Isaacs equations and the corresponding optimal trajectories for the two players. The qualitative behaviour of the optimal strategies looks rather accurate and this motivates an additional effort to analyze the problem and prove a convergence theorem. In order to set our paper into perspective, note that in [5] the convergence of the fully-discrete solution to the solution of the continuous problem was proved in the free (i.e. unconstrained) case, but this result can not be directly extended to the constrained case. In [8] a convergence result is proved for constrained control problems, but it strictly relies on the fact that the time-discrete value function is continuous so we can not apply the same ideas here.

To deal with generalized differential games, we adapt to the discrete problem the definitions of admissible controls presented in [17] since they are not restricted to pursuit-evasion games, *i.e.* games where each player controls only his own dynamics. It should be noted that very few results on constrained differential games are available although several interesting problems with state constraints have been studied in the literature by Isaacs [16] and Breakwell in [7]. The aim of those contributions is mainly to compute the optimal trajectories without solving the Isaacs equation. The main theoretical contributions to the characterization of the value function for state constrained problems are, at our knowledge, the papers by Alziary de Roquefort [1], Bardi *et alia* [6] and by Cardaliaguet, Quincampoix and Saint-Pierre [9]. From the numerical point of view the list of contributions is even shorter. The first examples of computed optimal trajectories for pursuit-evasion games have appeared in the work by Alziary de Roquefort [2]. In Bardi et al. [5] there are some interesting tests in $\Omega \subset \mathbb{R}^2$ with state constraints and discontinuous value function. In [3] the effect of the boundary conditions for the free problem in $\mathbb{R}^4$ is studied. In the paper Cardaliaguet, Quincampoix and Saint-Pierre [10] a convergence result for an approximation scheme is presented for a modified viability kernel algorithm (see [11] for more details on this approach). Finally, in [13] we have shown convergence of our algorithm for pursuit-evasion games.

2. THEORETICAL BACKGROUND AND NOTATIONS

Let us start introducing the problem and our notations. A target set $\mathcal{T} \subset \mathbb{R}^n$ is given and it is assumed to be closed. The system describing the dynamics is

$$\begin{cases} \dot{y}(t) = f(y(t), a(t), b(t))\,,\ t > 0 \\ y(0) = x \end{cases} \tag{1}$$

where $y(t)$ is the state of the system, $a(\cdot) \in \mathcal{A}$ and $b(\cdot) \in \mathcal{B}$ are respectively the controls of the first and the second player, $\mathcal{A}$ and $\mathcal{B}$ being the sets of admissible controls defined as

$$\mathcal{A} = \{a(\cdot) : [0, +\infty) \to A,\ \text{measurable}\}\,,$$
$$\mathcal{B} = \{b(\cdot) : [0, +\infty) \to B,\ \text{measurable}\}\,,$$

and A and B are given compact sets of $\mathbb{R}^m$.

We will always assume that

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$$\begin{cases} f:\mathbb{R}^n\times A\times B\to\mathbb{R}^n \text{ is continuous in the three} \\ \text{variables and there exists } L>0 \text{ such that} \\ |f(y_1,a,b)-f(y_2,a,b)|\le L|y_1-y_2| \\ \text{for all } y_1,y_2\in\mathbb{R}^n,\ a\in A,\ b\in B. \end{cases} \tag{2}$$

We will denote the solution of (1) by $y_x(t;a(\cdot),b(\cdot))$. In our generalized Pursuit-Evasion game the first player, called the *Pursuer* and denoted by P, wants to drive the system to $\mathcal{T}$. The second player, called the *Evader* and denoted by E, wants to drive the system away.
Note that in [13] we have proposed an algorithm for the special case where $y=(y_P,y_E)$ and $f(y,a,b))=(f_P(y_P,a),f_E(y_E,b))$. We deal with the natural extension of the minimum time problem, so we define the *payoff* of the game as the first time of arrival $T(x)$ (if any) on the target $\mathcal{T}$ for the solution trajectory of (1) starting at x. Note that, as usual, we set $T(x)=+\infty$ if the trajectory will not reach the target.
As we said in the introduction we want to construct a numerical approximation for *differential games with state constraints*. This means that both the players have to keep the system in a given bounded domain Q and satisfy additional state constraints (if any) described by a set C. We will denote by $\overline{\Omega}\subset\mathbb{R}^n$ the set describing *all* the constraints (i.e. $\overline{\Omega}\equiv Q\cap C$). The analysis of the continuous model with state constraints via dynamic programming techniques which is the basis for our approximation can be found in [17, 6]. In order to simplify the presentation, we will assume in the sequel that $\mathcal{T}\subset Q$ (this is not restrictive since we can always redefine the target to be $\mathcal{T}\cap Q$).
Let us start giving the time-discrete and the corresponding fully-discrete version of the differential game with state constraints.

3. THE FULLY-DISCRETE APPROXIMATION SCHEME

We will consider a discrete version of the dynamics based on the Euler scheme, namely

$$\begin{cases} y_{n+1}=y_n+hf(y_n,a_n,b_n) \\ y_0=x \end{cases}$$

We denote by $y(n;x,\{a_n\},\{b_n\})$ its solution at time nh. The state constraints require that $y(n;x,\{a_n\},\{b_n\})\in\overline{\Omega}$ for all $n\in\mathbb{N}$.
Let us define

$$\mathcal{A}^h:=\{\{a_n\}:a_n\in A\,,\ \text{for all } n\}$$
$$\mathcal{B}^h:=\{\{b_n\}:b_n\in B\,,\ \text{for all } n\}.$$

Adapting to the discrete case definitions in [17], we define the set of *admissible pairs* of controls at $x\in\overline{\Omega}$

$$AP(x)\equiv\{(\{a_n\},\{b_n\})\in\mathcal{A}^h\times\mathcal{B}^h: \\ y(n;x,\{a_n\},\{b_n\})\in\overline{\Omega} \text{ for all } n\}$$

and then the sets of *admissible controls* for each player

$$\mathcal{A}^h_x\equiv\{\{a_n\}\in\mathcal{A}^h:\exists\{b_n\}\in\mathcal{B}^h|(\{a_n\},\{b_n\})\in AP(x)\}$$
$$\mathcal{B}^h_x\equiv\{\{b_n\}\in\mathcal{B}^h:\exists\{a_n\}\in\mathcal{A}^h|(\{a_n\},\{b_n\})\in AP(x)\}.$$

We will always assume that $\mathcal{A}^h_x\neq\emptyset$ (or equivalently $\mathcal{B}^h_x\neq\emptyset$) for all $x\in\overline{\Omega}$.
Let us also define the following subsets of A and B:

$$A_h(x,b):=\{a\in A:x+hf(x,a,b)\in\overline{\Omega}\}\,,\qquad x\in\overline{\Omega}$$

and

$$B_h(x,a):=\{b\in B:x+hf(x,a,b)\in\overline{\Omega}\}\,,\qquad x\in\overline{\Omega}.$$

We will also assume that

$$\begin{cases} \exists h_0>0:A_h(x,b)\neq\emptyset \text{ and } B_h(x,a)\neq\emptyset \\ \forall(h,x)\in(0,h_0]\times\overline{\Omega}\,,\ a\in A\,,\ b\in B \end{cases} \tag{3}$$

Definition 1.
A *strategy* for the first player is a map $\alpha_x:\mathcal{B}^h_x\to\mathcal{A}^h_x$. It is *nonanticipating* if $\alpha_x\in\Gamma^h_x$, where

$$\Gamma^h_x:=\{\alpha_x:\mathcal{B}^h_x\to\mathcal{A}^h_x:b_n=\tilde{b}_n \text{ for all } n\le n' \\ \text{implies } \alpha_x[\{b_k\}]_n=\alpha_x[\{\tilde{b}_k\}]_n \text{ for all } n\le n'\}. \tag{4}$$

Let us define the reachable set as the set of starting points from which the system can be driven to the target

$$\mathcal{R}^h:=\Big\{x\in\mathbb{R}^n:\forall\{b_n\}\in\mathcal{B}^h\exists\alpha\in\Gamma^h \text{ and } \bar{n}\in\mathbb{N} \text{ s.t.} \\ y(\bar{n};x,\alpha[\{b_n\}],\{b_n\})\in\mathcal{T}\Big\}. \tag{5}$$

Then, we define for $x\in\mathcal{R}^h$

$$n_{min}(x,\{a_n\},\{b_n\})\equiv\min\{n:\ y(n;x,\{a_n\},\{b_n\})\in\mathcal{T}\}$$

and

$$n_h(x,\{a_n\},\{b_n\})\equiv\begin{cases} n_{min}(x,\{a_n\},\{b_n\}) & \text{for } x\in\mathcal{R}^h \\ +\infty & x\notin\mathcal{R}^h \end{cases}$$

We will consider for our approximation the discrete lower value of the game, which is

$$T_h(x):=\inf_{\alpha_x\in\Gamma^h_x}\sup_{\{b_n\}\in\mathcal{B}^h_x} h\,n_h(x,\alpha_x[\{b_n\}],\{b_n\})$$

and its Kružkov transform

$$v_h(x):=1-e^{-T_h(x)}\,,\quad x\in\overline{\Omega}. \tag{6}$$

Note that a similar construction can be done for the upper value of the game. The Dynamic Programming Principle (DPP) for differential games with state constraints (under rather restrictive assumptions) is proved in [17] which also gives a characterization of the lower and upper value of the game in terms of the Isaacs equation. The discrete version of the DPP should lead to the following characterization of the time-discrete value function v_h, For every $x\in\overline{\Omega}\backslash\mathcal{T}$

$$v_h(x)=\max_{b\in B}\min_{a\in A_h(x,b)}\{\beta v_h(x+hf(x,a,b))\}+1-\beta \tag{7}$$

whereas

$$v_h(x)=0 \text{ for } x\in\mathcal{T} \tag{8}$$

where $\beta\equiv e^{-h}$. Unfortunately, the resulting Hamilton-Jacobi-Isaacs equation (7)-(8) is not very general and does not include simple games like pursuit-evasion games we studied for example in [13]. In fact in [17] it is assumed that the second player can choose his control in B without any restriction due to the state constraints and then only the first player has the responsibility to maintain the state of the system in $\overline{\Omega}$. Although we can not prove at this stage a more general DPP we try to solve numerically the Hamilton-Jacobi-Isaacs equation in a more general framework in which every player must consider the choice of the other player in order to choose an admissible pairs of controls (a,b). So we substitute (7) by the following equation

$$v_h(x)=\max_{b\in B_h(x)}\min_{a\in A_h(x,b)}\{\beta v_h(x+hf(x,a,b))\}+1-\beta \tag{9}$$

where

$$B_h(x):=\{b\in B:\exists a\in A_h(x,b)|x+hf(x,a,b)\in\overline{\Omega}\}$$

for all $x\in\overline{\Omega}$. This choice seems to be reasonable and it seems to be the right choice to solve differential games with coupled dynamics as we will see in the next section.

In order to achieve the fully-discrete equation we build a regular triangulation of $\overline{\Omega}$ denoting by X the set of its nodes x_i, $i = 1, \ldots, N$ and by S the set of simplices S_j, $j \in J \equiv 1, \ldots, L$. $V(S)$ will denote the set of the vertices of a simplex S and the space discretization step will be denoted by k where $k := \max_j\{diam(S_j)\}$.
The *fully-discrete approximation scheme* is, for $x_i \in (\overline{\Omega}\backslash\mathcal{T}) \cap X$,

$$v_h^k(x_i) = \max_{b\in B_h(x)} \min_{a\in A_h(x_i,b)} \left\{\beta v_h^k(x_i + hf(x_i,a,b))\right\} + 1 - \beta \tag{10}$$

whereas the homogeneous Dirichlet boundary condition (8) becomes

$$v_h^k(x_i) = 0\,, \quad x_i \in \mathcal{T} \cap X\ . \tag{11}$$

The local reconstruction of the term $v_h^k(x_i + hf(x_i,a,b))$ is obtained by linear interpolation, i.e.

$$\begin{aligned} &v_h^k(x) = \sum_j \lambda_j(x) v_h^k(x_j)\,, \text{ where} \\ &0 \le \lambda_j(x) \le 1\,, \qquad \sum_j \lambda_j(x) = 1 \quad x \in \overline{\Omega}. \end{aligned} \tag{12}$$

As in the unconstrained problem, the choice of linear interpolation is not an obligation and it was made here just to simplify the presentation.
Let us denote by W^k the set

$$W^k := \left\{w \in C(\overline{\Omega}) : \nabla w(x) = \text{ constant for } x \in S_j\,, j \in J\right\}.$$

The proof of the following theorem can be obtained with simple adaptations of the standard proof for the free fully-discrete scheme (see e.g. [5]).

Theorem 1. The problem (10), (11) has a unique solution $v_h^k \in W^k$ such that $v_h^k : \overline{\Omega} \to [0,1]$.

Finally we note that the theorem of convergence of v_h^k to v_h for k tends to 0 in [13] can be easily adapted to equation (10) although it was first stated in the particular case of pursuit-evasion games.

4. NUMERICAL EXPERIMENTS

In this section we present some numerical experiments for pursuit-evasion games as well as for general differential games. The code is written in C++ using OpenMP directives. The algorithm ran on an IBM system p5 575 equipped with 8 processors Power5 at 1.9 GHz and 32 GB RAM located at CASPUR (`www.caspur.it`).
We denote by N the number of nodes in each dimension. In every case the controls a and b are chosen in the boundary of the two-dimensional unit ball $B(0,1)$ plus the central point $(0,0)$. We denote by N_c the number of admissible directions/controls for each player.
We always solve the problem on a structured grid with four-dimensional cells of volume $\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4$ and we choose the (fictitious) time step h such that

$$\|hf(x,a,b)\| \le \min\{\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4\}$$

for all x, a, b (so that the interpolation is made in the neighboring cells of the considered point). We adopt

$$\|V^{(p+1)} - V^{(p)}\|_\infty \le \varepsilon\,,\ \varepsilon > 0$$

as stopping criterion for the fixed point iteration $V^{p+1} = F(V^p)$ (where $V_i = v_h^k(x_i)$). We denote by $v(x)$ the approximate value function and by $T(x) = -\ln(1 - v(x))$ the time needed to reach the target. In the following we name "CPU time" the sum of the times taken by the CPUs and by "wallclock time" the elapsed time.

Test 1 (Tag–Chase game)
In this test we consider two boys P and E running one after the other in the same two-dimensional domain. The real game is played in a square $[-2,2]^2$ so the problem is set in $Q = [-2,2]^4$. The coordinates (x_1,x_2) represent the position of the Pursuer and (x_3,x_4) represent the position of the Evader. The Pursuer's dynamics is

$$\begin{cases} f_1(x,a,b) = 2a_1 \\ f_2(x,a,b) = 2a_2 \end{cases} \tag{13}$$

and the Evader's dynamics is

$$\begin{cases} f_3(x,a,b) = b_1 \\ f_4(x,a,b) = b_2 \end{cases} \text{if } b_2 \ge 0 \tag{14}$$

$$\begin{cases} f_3(x,a,b) = 3b_1 \\ f_4(x,a,b) = 3b_2 \end{cases} \text{otherwise} \tag{15}$$

so the Evader can run faster than the Pursuer when he goes down. We consider the state constraints due to boundary of the square (players can not exit the admissible domain Q) and, in addition, another constraint $C \equiv \{x \in \mathbb{R}^4 : x_4 > x_2\}$ so that the Evader must remains above the Pursuer. Although the dynamics is split in the sense that the choice of a player does not affect the position of the other, the state constraints are coupled and depend on the global state of the system.
The numerical target is $\mathcal{T} = \{(i,j,k,l) \in \{1,\ldots,N\}^4 : |i-k| \le 1 \text{ and } |j-l| \le 1\}$ so the target is reached when the capture occurs. We plot some flags on the approximate optimal trajectories every some time steps. This allows to follow the position of one player with respect to the other during the game.
We choose $\varepsilon = 10^{-3}$, $N = 50$ and $N_c = 32+1$. Convergence was reached in 137 iterations. The CPU time was 1d 01h 26m, the wallclock time was 3h 44m. Fig. 1 shows the optimal trajectory corresponding to the starting point $P = (-1.8, -1.9)$, $E = (-1.5, 1.5)$. We compare this solution with the solution of the problem in which we removed the constraints $x_4 > x_2$ (see Fig. 2). It is immediately seen that the behavior is completely different.
If the Evader is constrained above the Pursuer, he goes a little bit down just to increase his velocity but after a while he is pushed to the north boundary by the Pursuer.

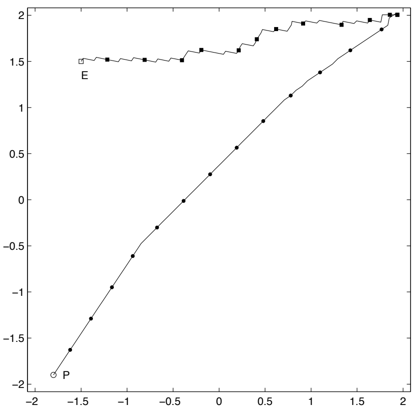

Fig. 1. Optimal trajectories for Test 1. Constrained case with $x_4 > x_2$.

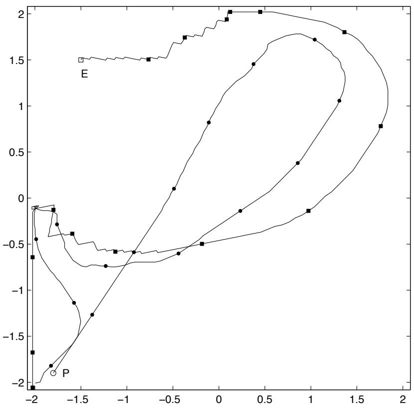

Fig. 2. Optimal trajectories for Test 1. Constrained case without the requirement $x_4 > x_2$

In absence of that constraints the Evader waits until the Pursuer approaches the north boundary and then he goes down faster than the Pursuer so he is captured only when he touches the south boundary.

Test 2

In this test we consider a completely coupled dynamics. A ball is free to move on the plane $[-5,5]^2$. We indicate the position of the ball by (x_1,x_2) and its velocity by (x_3,x_4). The two players can move the ball applying a force which depends on (x_1,x_2). The first player wants to steer the ball to the target $\mathcal{T} = \{(x_1,x_2) \mid x_1 \geq 4, x_2 \geq 0\}$ while the second player wants to steer the ball away. The dynamics is

$$\begin{cases} f_1(x,a,b) = x_3 \\ f_2(x,a,b) = x_4 \end{cases} \tag{16}$$

$$f_3(x,a,b) = \begin{cases} 4a_1 + b_1 & x_1 \leq 0 \\ 2a_1 + 3b_1 & x_2 > 0 \end{cases} \tag{17}$$

$$f_4(x,a,b) = \begin{cases} 4a_2 + b_2 & x_1 \leq 0 \\ 2a_2 + 3b_2 & x_2 > 0 \end{cases} \tag{18}$$

This means that the first player can completely control the ball in the left side of the domain but not in the right side. We choose $\varepsilon = 10^{-3}$, $N = 40$ and $N_c = 24+1$. Convergence was reached in 132 iterations. The CPU time was 6h 14m, the wallclock time was 52m. Fig. 3 shows

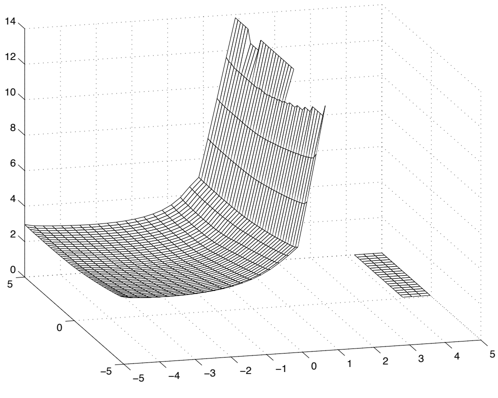

Fig. 3. Test 2. Value function $T(x_1,x_2,0,0)$

the value function $T(x_1,x_2,0,0)$ (we fix the initial velocity equal to $(0,0)$). It is immediately seen that if the ball starts from the right-hand side the first player can not steer it to the target so the optimal time T is $+\infty$. Fig. 4 shows an optimal trajectory corresponding to the starting point $(0,0,0,0)$. We can see that at the beginning the first player moves the ball toward the left side of the domain in such a way he can control and accelerate the ball. After that, he pushes the ball toward the target. When the ball enters the right side of the domain the second player tries to slow down the ball and to move it away but at this point the velocity of the ball is too high so it can reach the target despite the second player.

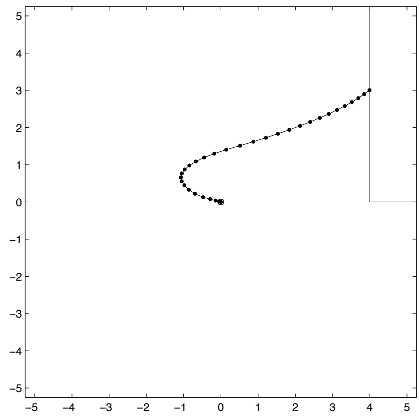

Fig. 4. Test 2. An optimal trajectory

Test 3

In this test we consider again a completely coupled dynamics. The aim is to stabilize a dynamical system. The dynamics is

$$\begin{cases} f_1(x,a,b) = (-3+a_1)x_1 \\ f_2(x,a,b) = (-3+a_2)x_2 \\ f_3(x,a,b) = (-3+a_1-2b_1)x_3 \\ f_4(x,a,b) = (-3+a_2-2b_2)x_4 \end{cases} \tag{19}$$

We choose $\varepsilon = 10^{-3}$, $N = 38$ and $N_c = 2$ (a_i,$b_i = \pm 1$). Convergence was reached in 36 iterations. Fig. 5 shows

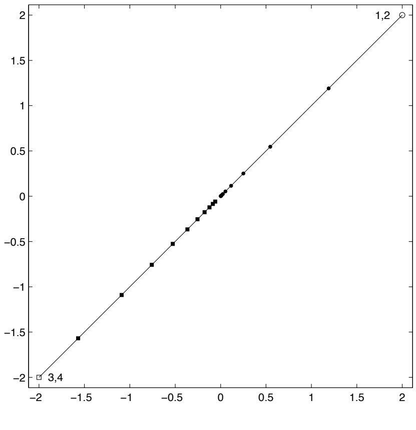

Fig. 5. Test 3. An optimal trajectory

an optimal trajectory corresponding to the starting point $(2,2,-2,-2)$. We plotted the coordinates (x_1,x_2) and (x_3,x_4) separately for the reader's convenience (the first is plotted by circles, the second by squares). We can see that the two curves approach the origin in different time due to the action of the second player which slows down the evolution of the system.

5. CONCLUSION

We have proposed an approximation scheme for general differential games with state constraints. According to the numerical tests the numerical approximation gives an appropriate qualitative description of the value function and of the corresponding optimal trajectories. These results push toward a further analysis in order to prove that the approximate solution computed by the algorithm converges, for h and k tending to 0, to the viscosity solution of the Isaacs equation.

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