Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

A Lyapunov Approach for Discrete-time Sliding Hyperplane Design and Robust Sliding Mode Control using MROF

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Abstract: In this paper, we propose a generalized method to design sliding hyperplane for variable structure control in the discrete time domain. The well-known Lyapunov inequality of full order will be used for generating the *stable* sliding hyperplanes without loss of generality, which shows the necessary and sufficient condition for the existence of stable sliding hyperplane for multi-input-multi-output systems. Also, we derive a desirable reaching law that guarantees the attractiveness of the boundary layer. Then, it will be shown that the results (obtained with full state feedback information) can be generalized for the discrete time output feedback case by adopting multi-rate output feedback (MROF).

Keywords: variable structure control; sliding hyperplane, Lyapunov matrix, LMIs, discrete-time, multi-rate output feedback.

1. INTRODUCTION

Variable structure control has been one of the major concerns in control theory because of the robustness to external disturbances satisfying the *input-matching* condition. The conventional approaches in variable structure control have been defined in the continuous time domain, which enables the assumption for infinite switching at the sliding surface. Even though there would exist the chattering phenomenon around the sliding surface, the robustness obtained is remarkable. However, this benefit cannot be expected in the discrete time domain approaches. In order to recover the robustness in the discrete time approach, several methods such as the sliding sector by Furuta and Pan [2000] and the attractive boundary layer by Tang and Misawa [2000] have been proposed. In particular, to achieve the "quasi-sliding" mode, the saturation function instead of the signum function has been used for designing the reaching law in literature. See Tang and Misawa [1998], Janardhanan and Bandyopadhyay [2006], Furuta and Pan [2000], and references therein.

Nevertheless of the remarkable progress in the discrete time sliding mode control (mainly focused on the reaching law design), there have been few approaches for the sliding hyperplane design methods. As to the sliding mode (or, hyperplane) design for discrete time systems, the eigenvalue constraint methods for the equivalent dynamics matrix (e.g., see Tang and Misawa [1998] and Spurgeon [1992]) have been used. In particular, Tang and Misawa [1998, 2000] adopts the single sliding hyperplane even for the systems having multi-inputs.

In this paper, we aim at proposing, in the discrete time domain, a novel approach to design sliding hyperplanes systematically for variable structure control with full state information, and expanding the result with multi-rate output feedback. To this end, first, the necessary and sufficient condition for the existence of discrete time sliding hyperplane is newly formulated by utilizing the discrete time Lyapunov inequality. The main idea starts from the extension of the Lyapunov approach in continuous time by Kim and Park [2004]. Then, the suitable reaching law that confines the sliding function within certain bounds is derived. It will be shown that the proposed control can be generalized for output feedback thanks to the multi-rate output feedback (MROF) approach by Janardhanan and Bandyopadhyay [2006].

The preliminaries will be introduced in Section 2. Then, the discrete time variable structure control utilizing full state measurement is derived and illustrated in Section 3. In Section 4, the full state feedback results in Section 3 will be extended for the output feedback based on MROF. The parametric uncertainties will be further discussed in Section 5. Then, the conclusion follows in Section 6.

2. PRELIMINARIES

2.1 Matrix manipulations

Let us introduce useful matrix manipulations related to the sign definiteness of the augmented matrices.

⋆ This work was supported in part by the Ministry of Commerce, Industry & Energy (MOCIE) under *Mid-term Tech. Development* project and by the Ministry of Education & Human Resources Development (MOE) under the BK21 program, KOREA.

Lemma 1. The matrix $M := \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$ is negative definite if and only if $M_{11} < 0$ and $M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0$.

The proof of Lemma 1 is straight forward by using the relationship

$$\begin{bmatrix} -M_{11}^{-1}M_{12} & I \\ I & 0 \end{bmatrix}^T \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \begin{bmatrix} -M_{11}^{-1}M_{12} & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} M_{22} - M_{12}^T M_{11}^{-1} M_{12} & 0 \\ 0 & M_{11} \end{bmatrix}$$

Lemma 2. The matrix $H := \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}$ is positive definite if and only if $H_{22} > 0$ and $H_{11} - H_{12}^T H_{22}^{-1} H_{12} > 0$.

The proof of Lemma 2 is omitted due to the similarity to that of Lemma 1.

2.2 Lyapunov inequalities - stabilizability

Consider the discrete time linear system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $x(k) \in R^n$ is the state variable and $u(k) \in R^m$ is the control input. The stabilizability of the pair (A, B) is assumed. Then, it is straight forward to have the following theorem from literature.

Theorem 3. The system (1) is stabilizable by the full state feedback $u_k = -Kx_k$ *if and only if* there exist matrices $P > 0$ and K, for a $Q \geq 0$, satisfying

$$(A - BK)^T P(A - BK) - P + Q < 0 \tag{2}$$

Note that the positive (semi-) definite matrix Q does not impose any restriction on the existence of *stabilizing* full state feedback gain matrix. Only does the *stabilizability* of the pair (A, B). Therefore, the matrix Q may be used for a design parameter for obtaining the specific performances.

3. VARIABLE STRUCTURE CONTROL DESIGN

3.1 Discrete-time sliding hyperplane

Without loss of generality, consider the system of regular form

$$x(k+1) = Ax(k) + Bu(k) + Fw(k) \tag{3}$$

where $x(k) \in \mathrm{R}^n$, $u(k) \in \mathrm{R}^m$, $w(k) \in \mathrm{R}^l$, and,

$$B = \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix}, \; F = \begin{bmatrix} 0_{(n-m)\times l} \\ F_2 \end{bmatrix}$$

for invertible $B_2 \in \mathrm{R}^{m\times m}$, and, $F_2 \in \mathrm{R}^{m\times l}$. Let us further assume that the disturbance is bounded as follows:

$$|w_j(k)| < \overline{w}_j, \quad \forall\, j = 1, \cdots, l.$$

In this paper, we use the system description in the regular form in order for the simplicity of description. With no loss of generality, it is always possible to transform the system in general coordinate into the regular form system by properly choosing the coordinate transformation. For example, one may have the singular value decomposition of B as $B = U_1 \Sigma V^T$ for the unitary matrix $[U_1,\ U_2] \in R^{n\times n}$. Then, the state transformation such as $x' = Tx$ leads to the regular form system, where $T = [U_2,\ U_1]^T$. Note that the disturbance resides with the control inputs, which implies the so-called *matching condition.*

Define the sliding function as

$$s(k) = x_2(k) + Cx_1(k) \tag{4}$$

Then, when the control input satisfying the *reaching law* is applied, we have the sliding behavior as follows:

$$x_1(k+1) = (A_{11} - A_{12}C)\, x_1(k) \tag{5}$$

$$x_2(k) = -Cx_1(k) \tag{6}$$

This shows that the coefficient matrix C for the sliding function has the role of the full state feedback gain matrix for the reduced order system defined by (A_{11}, A_{12}).

Theorem 4. Given a matrix $Q \geq 0$, let us define a set of matrices as follows:

$$\Omega(Q) \triangleq \{P > 0 | (A - BK)^T P(A - BK) - P + Q < 0\} \tag{7}$$

Then, the matrix defined as

$$C \triangleq P_{22}^{-1} P_{12}^T \tag{8}$$

for any matrix $P \in \Omega(Q)$, is the *stable* sliding function coefficient, where

$$P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \in \begin{bmatrix} \mathrm{R}^{(n-m)\times(n-m)} & \mathrm{R}^{(n-m)\times m} \\ \mathrm{R}^{m\times(n-m)} & \mathrm{R}^{m\times m} \end{bmatrix}.$$

Moreover, for any *stable* sliding function coefficient matrix C, there exists a matrix $P \in \Omega(Q)$ satisfying (8) and (7).

Proof: (1st argument) For a $P \in \Omega(Q)$, one may show that

$$\begin{aligned} & A^T PA - A^T PBR^{-1}B^T PA - P + Q \\ &= (A - BK)^T P(A - BK) - P + Q + K^T B^T PA \\ &\quad + A^T PBK - K^T B^T PBK - A^T PBR^{-1}B^T PA \\ &= (A - BK)^T P(A - BK) - P + Q \\ &\quad - (RK - B^T PA)^T R^{-1}(RK - B^T PA) \\ &\leq (A - BK)^T P(A - BK) - P + Q < 0 \end{aligned} \tag{9}$$

where $R = B^T PB$.

Now, define a transformation matrix such that $T_r = \left[I_{(n-m)} \ \ -P_{12}P_{22}^{-1} \right]$, and, pre- and post multiply (9) by T_r and T_r^T, respectively. Then, we have

$$\begin{aligned} 0 &> T_r \left(A^T PA - A^T PBR^{-1}B^T PA - P + Q \right) T_r \\ &= (A_{11} - A_{12}P_{22}^{-1}P_{12}^T)^T P_r (A_{11} - A_{12}P_{22}^{-1}P_{12}^T) \\ &\quad - P_r + Q_r \end{aligned} \tag{10}$$

where $P_r = P_{11} - P_{12}P_{22}^{-1}P_{12}^T$ and $Q_r = T_r Q T_r^T$. This explicitly shows that the matrix $P_{22}^{-1}P_{12}^T$ is the stabilizing state feedback matrix for the reduced order system by *Theorem* 1 since the matrix P_r is positive-definite for the matrix $Q_r \geq 0$.

(2nd argument): The stable sliding coefficient matrix $C \in \mathrm{R}^{m\times(n-m)}$ guarantees the existence of a matrix P_r satisfying

$$(A_{11} - A_{12}C)^T P_r(A_{11} - A_{12}C) - P_r + \overline{Q}_{11} + \delta I_{n-m} < 0 \quad (11)$$

for a scalar $\delta > 0$ and the matrix $\overline{Q}_{11} \geq 0$, where $\overline{Q} \triangleq TQT^T$, and,

$$\overline{Q} \triangleq \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} \\ \overline{Q}_{12}^T & \overline{Q}_{22} \end{bmatrix} \in \begin{bmatrix} \mathrm{R}^{(n-m)\times(n-m)} & \mathrm{R}^{(n-m)\times m} \\ \mathrm{R}^{m\times(n-m)} & \mathrm{R}^{m\times m} \end{bmatrix}, \quad (12)$$

$$T \triangleq \begin{bmatrix} I_{n-m} & -C^T \\ 0_{m\times(n-m)} & I_m \end{bmatrix} \quad (13)$$

Let us choose a matrix $0 < P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ such that

$$P_{22} = \epsilon I_m + \overline{Q}_{22} + A_{12}^T P_r A_{12} + \frac{1}{\delta}(A_r^T P_r A_{12} + \overline{Q}_{12})^T (A_r^T P_r A_{12} + \overline{Q}_{12}) \quad (14)$$

$$P_{11} = P_r + C^T P_{22} C \quad (15)$$

$$P_{12} = C^T P_{22} \quad (16)$$

for an $\epsilon > 0$, where $A_r \triangleq A_{11} - A_{12}C$ for simplicity. Note that it is straight forward to show the positiveness of P since $P_{22} > 0$ and $P_r > 0$, using *Lemma* 2. Now, we define the matrix $K \triangleq [K_1 \ K_2]$ with

$$K_1 = B_2^{-1}(A_{21} + CA_{11}) \quad (17)$$

$$K_2 = B_2^{-1}(A_{22} + CA_{12}) \quad (18)$$

Using the equations from (11) to (18), one may show the following:

$$\begin{aligned} &T\left[(A - BK)^T P(A - BK) - P + Q\right] T^T \\ &= \begin{bmatrix} A_r^T P_r A_r - P_r + \overline{Q}_{11} & A_r^T P_r A_{12} + \overline{Q}_{12} \\ A_{12}^T P_r A_r + \overline{Q}_{12}^T & A_{12}^T P_r A_{12} + \overline{Q}_{22} - P_{22} \end{bmatrix} \\ &\leq \begin{bmatrix} -\delta I_{n-m} & A_r^T P_r A_{12} + \overline{Q}_{12} \\ A_{12}^T P_r A_r + \overline{Q}_{12}^T & A_{12}^T P_r A_{12} + \overline{Q}_{22} - P_{22} \end{bmatrix} \\ &\triangleq \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \end{aligned} \quad (19)$$

Using *Lemma 1*, the matrix M is negative definite since $M_{11} < 0$ and

$$M_{22} - M_{12}^T M_{11}^{-1} M_{12} = -\epsilon I_m < 0 \quad (20)$$

Hence, given a matrix $Q \geq 0$, we have, with the matrices $P > 0$ and K,

$$(A - BK)^T P(A - BK) - P + Q < 0 \quad (21)$$

which proves the 2nd argument. □

The result of *Theorem* 4 provides the generalized parameterization approach to the sliding mode design in the discrete time domain. As long as the stabilizability (i.e., controllability in the strong sense) is guaranteed, there exist discrete sliding modes and any of stable sliding function coefficients can be found by solving the Lyapunov inequality in the full order system state.

3.2 Reaching law design

In the previous section, we have assumed the sliding mode (i.e., $s(k+1) = s(k) = 0$) in order to design the sliding hyperplane. However, in the discrete time system, the complete sliding mode can not be obtained. Instead, the concept of quasi-sliding mode has been adopted in general. Given the sliding function (4), one may have

$$x_1(k+1) = (A_{11} - A_{12}C)x_1(k) + A_{12}s(k) \quad (22)$$

Also, it can be easily shown that

$$\begin{aligned} \|x_2(k)\| &= \|s(k) - Cx_1(k)\| \\ &\leq \|s(k)\| + \|Cx_1(k)\| \end{aligned} \quad (23)$$

Those equations implies that the system state vector $x(k)$ should be confined within a certain bound as long as so is the sliding function $s(k)$. To derive the control law that drives the sliding function within a bound, we propose the control law as follows:

$$u(k) = -(GB)^{-1}\{GAx(k) - \widetilde{\beta} \cdot s(k) + \widetilde{z} \cdot f(s(k), \phi)\} \quad (24)$$

where $G \triangleq [C, \ I_m] \in \mathrm{R}^{m\times n}$, $\widetilde{\beta} = diag[\beta_1, \cdots, \beta_m]$, $\widetilde{z} = diag[z_1, \cdots, z_m]$, $\phi = [\phi_1, \cdots, \phi_m]$, and

$$f(s(k), \phi) \triangleq \left[sat\left(\frac{s_1(k)}{\phi_1}\right), \cdots, sat\left(\frac{s_m(k)}{\phi_m}\right)\right]^T \in \mathrm{R}^m$$

$$z_i \triangleq \sum_{j=1}^{l} |(GF)_{ij}|\overline{w}_j \quad (25)$$

$$\frac{2z_i}{\phi_i} - 1 < \beta_i < 1 \quad (26)$$

for a scalar $\phi_i > 0$ which necessarily implies that $\phi_i > z_i$.

With the control law (24), we have

$$s_i(k+1) = \beta_i s_i(k) - z_i sat\left(\frac{s_i(k)}{\phi_i}\right) + d_i(k) \quad (27)$$

where

$$d_i(k) = \sum_{j=1}^{l} (GF)_{ij} w_j(k).$$

Then, to show the quasi-sliding mode behavior, we rely on the quadratic function as

$$\begin{aligned} \Delta V_i(k) &= s_i^2(k+1) - s_i^2(k) \\ &= \Delta s_i(k)\left(\Delta s_i(k) + 2s_i(k)\right) \end{aligned} \quad (28)$$

where $\Delta s_i(k) = s_i(k+1) - s_i(k)$.

First, consider the case when $s_i(k) > \phi_i$. From (27), it holds that

$$\begin{aligned} \Delta s_i(k) &= (\beta_i - 1)s_i(k) - z_i + d_i(k) \\ &< (\beta_i - 1)\phi_i - z_i + \sum_{j=1}^{l} |(GF)_{ij}|\overline{w}_j \\ &= (\beta_i - 1)\phi_i < 0 \end{aligned} \quad (29)$$

since $\beta_i - 1 < 0$. Moreover,

$$\begin{aligned} (\Delta s_i(k) + 2s_i(k)) &= (\beta_i + 1)s_i(k) - z_i + d_i(k) \\ &\geq (\beta_i + 1)s_i(k) - 2z_i \\ &> \frac{2z_i}{\phi_i} \times \phi_i - 2z_i = 0 \end{aligned} \quad (30)$$

Table 1. Continuous-time sliding mode design

System	$\dot{x} = Ax + Bu + Fw$
Condition	*Given a* $Q \geq 0$, *find* $P > 0$ *satisfying* $(A - BK)^T P + P(A - BK) + Q < 0$ *for* $K \in \mathrm{R}^{m \times n}$
Sliding fn.	$s = Cx_1 + x_2$, $C = P_{22}^{-1} P_{12}^{-1}$

which shows that $\Delta V_i(k) < 0$. In the similar way, one may show that $\Delta V_i(k) < 0$ in case of $s_i(k) < -\phi_i$

Secondly, let's investigate the sliding function behavior in case of $|s_i(k)| < \phi_i$. From (27), we have

$$s_i(k+1) = \left(\beta_i - \frac{z_i}{\phi_i}\right) s_i(k) + d_i(k) \tag{31}$$

Using the fact that, from the condition (26),

$$\left|\beta_i - \frac{z_i}{\phi_i}\right| < \frac{\phi_i - z_i}{\phi_i} \tag{32}$$

it holds

$$\begin{aligned} |s_i(k+1)| &< \left|\beta_i - \frac{z_i}{\phi_i}\right| |s_i(k)| + |d_i(k)| \\ &< \frac{\phi_i - z_i}{\phi_i} \times \phi_i + z_i = \phi_i \end{aligned} \tag{33}$$

This implies that the sliding function should remain inside the bound once it enters the bound.

3.3 Revisit to the continuous-time sliding mode design

In order to show the similarity between the designs in the continuous-time domain and the discrete-time domain, let us revisit the results by Kim et al. [2000] and Kim and Park [2004] in the following. Consider the system

$$\dot{x} = Ax + Bu + Fw \tag{34}$$

where the matrices A, B and F are defined in the same manner as in (3). Then, based on Kim and Park [2004], we have the following result.

Theorem 5. (Kim and Park [2004]) Given a matrix $Q \geq 0$, define the set

$$\Omega(Q) \triangleq \left\{ P \mid \begin{matrix} (A-BK)^T P + P(A-BK) + Q < 0, \\ P > 0,\ K \in \mathrm{R}^{m \times n} \end{matrix} \right\} \tag{35}$$

Then, any stabilizing sliding function coefficient exists in the form of (8) (i.e., $C = P_{22}^{-1} P_{12}^T$), and, moreover, for any $P \in \Omega(Q)$, the composite matrix $C = P_{22}^{-1} P_{12}^T$ is a stabilizing sliding function coefficient.

For the reaching law, let us apply *Theorem* 1 of Kim et al. [2000] to the continuous system (34) to derive the control law.

$$u(t) = \begin{cases} 0 & if\ s(t) = 0 \\ -(GB)^{-1} \left\{ GAx + \widetilde{\beta} s + \widetilde{z} \cdot sgn(s) \right\} & if\ s(t) \neq 0 \end{cases} \tag{36}$$

where $\widetilde{\beta} = diag[\beta_1, \cdots, \beta_m]$ for $\beta_i > 0$, and, $\widetilde{z} = diag[z_1, \cdots, z_m]$ for $z_i = \sum_{j=1}^{l} |(GF)_{ij}| \overline{w}_j$.

Overall, the sliding hyperplane design methods in the continuous time and the discrete time are summarized in Tables 1 and 2.

Table 2. Discrete-time sliding mode design

System	$x(k+1) = Ax(k) + Bu(k) + Fw(k)$
Condition	*Given a* $Q \geq 0$, *find* $P > 0$ *satisfying* $(A - BK)^T P(A - BK) - P + Q < 0$ *for* $K \in \mathrm{R}^{m \times n}$
Sliding fn.	$s(k+1) = Cx_1(k) + x_2(k)$, $C = P_{22}^{-1} P_{12}^{-1}$

4. MULTI-RATE OUTPUT FEEDBACK CASE

4.1 state estimate via multi-rate output vector

Consider the output feedback system in which only partial states are measured:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + Fw(k) \\ y(k) = Cx(k) \end{cases} \tag{37}$$

Note that (37) is obtained by sampling the continuous-time system at every T sec. In order to apply the MROF approach introduced in Janardhanan and Bandyopadhyay [2006], let also assume that high rate sampling with the time interval $\tau = T/N$ for an integer N is available. The high rate sampling occurs N times in one T sampling period. Then, the state space description for the τ-system is shown as

$$x(q+1) = A_\tau x(q) + B_\tau u(q) + F_\tau w(q) \tag{38}$$

where

$$A = A_\tau^N, B = \sum_{i=0}^{N-1} A_\tau^i B_\tau, F = \sum_{i=0}^{N-1} A_\tau^i F_\tau.$$

Then, we have the multi-rate output vector as

$$y_k = C_0 x(k) + D_0 u(k) + C_d w(k) \tag{39}$$

where

$$C_0 = \begin{bmatrix} C \\ CA_\tau \\ CA_\tau^2 \\ \vdots \\ CA_\tau^{N-1} \end{bmatrix}, D_0 = \begin{bmatrix} 0 \\ CB_\tau \\ CA_\tau B_\tau + CB_\tau \\ \vdots \\ C\sum_{i=0}^{N-2} A_\tau^i B_\tau \end{bmatrix}, \quad C_d = \begin{bmatrix} 0 \\ CF_\tau \\ CA_\tau F_\tau + CF_\tau \\ \vdots \\ C\sum_{i=0}^{N-2} A_\tau^i F_\tau \end{bmatrix} \tag{40}$$

and

$$y_k = \left[\, y((k-1)T)\ \ y((k-1)T+\tau)\ \cdots\ y(kT-\tau) \,\right]^T.$$

From (37) and (39), $x(k)$ can be obtained from the output samples as

$$x(k) = L_y y_k + L_u u(k-1) + L_d w(k-1), \tag{41}$$

where

$$\begin{aligned} L_y &= A\left(C_0^T C_0\right)^{-1} C_0^T, L_u = B - L_y D_0, \\ L_d &= F - L_y C_d. \end{aligned} \tag{42}$$

(41) provides an important idea to extract the static state estimate from the multi-rate output samples y_k as follows:

$$\overline{x}(k) = L_y y_k + L_u u(k-1) \tag{43}$$

Remark 6. As a matter of fact, more accurate state estimate can be obtained by incorporating the mean value of the disturbance as follows.

$$\overline{x}(k) = L_y y_k + L_u u(k-1) + l_0 \tag{44}$$

where

$$l_{l,i} < (L_d w(k-1))_i < l_{u,i},\ l_0 = \frac{l_l + l_u}{2},$$

when the upper- and the lower bounds are available. However, for simplicity of descriptions, we assume that l_0 is equal to (or very near to) zero since the disturbance bias known can be easily canceled out by the static compensation.

The sliding function is defined in terms of output as

$$\overline{s}(k) = G\overline{x}(k) = GL_y y_k + GL_u u(k-1). \tag{45}$$

4.2 MROF reaching law design

To achieve the quasi-sliding mode, we propose the reaching law obtained from (24) by simple replacements of $x(k)$ and $s(k)$ by $\overline{x}(k)$ and $\overline{s}(k)$, respectively, as

$$u(k) = -(GB)^{-1}\{GA\overline{x}(k) - \widetilde{\beta}\cdot\overline{s}(k) + \widetilde{z}\cdot f(\overline{s}(k),\phi)\} \tag{46}$$

where (26) holds with

$$z_i = \sum_{j=1}^{l}\{|(GAL_d)_{ij}| + |(GL_yC_d)_{ij}|\}\,\overline{w}_j \tag{47}$$

As shown in Section 3.2, for the Lyapunov function $V_i(k) = \overline{s}_i(k)^2$, we are aiming at proving the negative definiteness of the change in the Lyapunov function

$$V_i(k+1) - V_i(k) = [2\overline{s}_i(k) + \Delta\overline{s}_i(k)]\,\Delta\overline{s}_i(k). \tag{48}$$

Using the relationship, from (43) and (41),

$$\overline{x}(k) = x(k) - L_d w(k-1),$$

the incremental change in sliding function can be obtained as

$$\begin{aligned}\overline{s}(k+1) - \overline{s}(k) &= G(x(k+1) - L_d w(k)) - \overline{s}(k)\\ &= G[Ax(k) + Bu(k) + Fw(k)\\ &\quad -(F - L_yC_d)w(k)] - \overline{s}(k)\\ &= (\widetilde{\beta} - I)\overline{s}(k) + d(k) - \widetilde{z}\cdot f(\overline{s}(k),\phi)\end{aligned} \tag{49}$$

where

$$d(k) = GAL_d w(k-1) + GL_yC_d w(k).$$

Hence, one may obtain

$$\Delta\overline{s}_i(k) = (\beta_i - 1)\overline{s}_i(k) + d_i(k) - z_i \mathrm{sat}\left(\frac{\overline{s}_i(k)}{\phi_i}\right) \tag{50}$$

where

$$d_i(k) = \sum_{j=1}^{l}\{(GAL_d)_{ij} w_j(k-1) + (GL_yC_d)_{ij} w_j(k)\}$$

.

Let us consider the case of $\overline{s}_i(k) > \phi_i$. Using the fact that $d_i(k) \le z_i$, we have

$$\begin{aligned}\Delta\overline{s}_i(k) &\le (\beta_i - 1)\overline{s}_i(k) + z_i - z_i\\ &= (\beta_i - 1))\overline{s}_i(k) < 0\end{aligned} \tag{51}$$

since $\beta_i - 1 < 0$. Then, now,

$$\begin{aligned}[2\overline{s}_i(k) + \Delta\overline{s}_i(k)] &= (\beta_i + 1)\overline{s}_i(k) + d_i(k) - z_i\\ &\ge (\beta_i + 1)\overline{s}_i(k) - 2z_i\\ &> (\beta_i + 1)\phi_i - 2z_i \ge 0\end{aligned} \tag{52}$$

thanks to the constraint (26). So, $\overline{s}_i(k)$ decreases as k increases.

The proof procedures for the case of $\overline{s}_i(k) < -\phi$ are omitted due to the similarity.

Let us consider the case of $|\overline{s}_i(k)| < \phi_i$. From (50), we have

$$\overline{s}_i(k+1) = \left(\beta_i - \frac{z_i}{\phi_i}\right)\overline{s}_i(k) + d_i(k) \tag{53}$$

Using (26), it can be shown that

$$\begin{aligned}|\overline{s}_i(k+1)| &\le \left|\beta_i - \frac{z_i}{\phi_i}\right|\cdot|\overline{s}_i(k)| + z_i\\ &< \frac{\phi_i - z_i}{\phi_i}\cdot\phi_i + z_i = \phi_i\end{aligned} \tag{54}$$

which proves that $\overline{s}(k)$ remains within the band ϕ once the trajectory enters the band.

Now, to investigate the *true* sliding function behavior, one may show that

$$\overline{s}(k) - s(k) = G(\overline{x}(k) - x(k)) = -GL_d w(k-1) \tag{55}$$

which results in

$$|s_i(k)| < \phi_i + \psi_i \tag{56}$$

where

$$\psi_i = \sum_{j=1}^{l}|(GL_d)_{ij}|\overline{w}_j$$

This implies that the sliding function $s(k)$ should remain within the extended band only defined by the external disturbance magnitude.

5. FURTHER TOPICS: EXTENSION TO UNCERTAIN LINEAR SYSTEMS

Consider the uncertain system

$$x(k+1) = (A + \Delta A)x(k) + (B + \Delta B)u(k) + Fw(k) \tag{57}$$

where the uncertainties of ΔA and ΔB shall be defined later on. Then, based on the results in the previous section, there exist stable sliding surfaces if there exist matrices $P > 0$ and K satisfying

$$A_{cl,\Delta}^T P A_{cl,\Delta} - P + Q < 0 \tag{58}$$

for any $\Delta A \in \Delta A_{all}$ and $\Delta B \in \Delta B_{all}$, where $A_{cl,\Delta} = A - BK + \Delta A - \Delta BK$. Then, with the Lyapunov matrix P, we can naturally generate a sliding hyperplane by the composition rule as $C = P_{22}^{-1}P_{12}^T$. One of the important results of the paper is that the parameter uncertainties may not be *matched* to the range space of input matrix B for the existence of stable sliding surface as far as the Lyapunov inequality (i.e., *quadratic stabilizability*) is met.

For last several decades, the parameter uncertainties have been widely studied and the most of results are available from the references. Nevertheless, in the paper, we revisit an LMIs approach with over-bounding technique for the completeness of the paper, in sense of the sliding surface design.

Let us define the uncertainties of interest as follows:

$$[\Delta A, \Delta B] = \sum_{i=1}^{N} \delta_i(t) E_i \tag{59}$$

where nontrivial matrices $E_i \in \mathrm{R}^{n\times(n+m)}$ and the scalar uncertain parameters possibly time-varying but the magnitude limited as $|\delta_i(t)| \leq 1 \ \forall t$. The advantages of the definition are the capabilities to express (i) the structural information of uncertainties (i.e., specific location of uncertainties) and (ii) uncertain parameters shared in the system matrices A and B. Then, using the spectral decomposition of E_i, it is straight forward to have the following form:

$$[\Delta A, \Delta B] = M\Delta\,[N_A,\ N_B] \tag{60}$$

where $\Delta = blkdiag\,[\delta_1(t)I_{r_1}, \cdots, \delta_N(t)I_{r_N}]$ and $r_i = rank(E_i)$, $\forall i \in [1, \cdots, N]$. In order to derive the sufficient condition for (58), let us introduce the change of variables such that $Y := P^{-1}$ and $F := KP^{-1}$. Then, we have, from (58), equivalently,

$$A_{Y,F}^T Y^{-1} A_{Y,F} - Y + YU_Q U_Q^T Y < 0 \tag{61}$$

where $Q = U_Q Q_Q^T,\ rank(Q) = rank(U_Q) = r_Q,\ A_{Y,F} = AY - BF + M\Delta(N_A Y - N_B F)$, which leads to the inequality

$$\begin{bmatrix} -Y & YU_Q & A_{Y,F}^T \\ U_Q^T Y & -I_{r_Q} & 0 \\ A_{Y,F} & & \end{bmatrix} < 0 \tag{62}$$

Then, to eliminate the uncertainty terms, consider the over-bounding technique:

$$z^T M\Delta(N_A Y - N_B F)x + x^T(N_A Y - N_B F)^T \Delta M^T z \leq$$
$$z^T MXM^T z + x^T(N_A Y - N_B F)^T X^{-1}(N_A Y - N_B F)x$$

for any x and z, where $X = blkdiag\,[X_1, \cdots, X_N]$ for positive-definite $X_i \in \mathrm{R}^{r_i\times r_i}$. Using the bounding inequality and the matrix augmentation technique, it is straight forward to derive a sufficient condition as follows:

$$\begin{bmatrix} -Y & YN_A^T - F^T N_B^T & YU_Q & AY - BF \\ (1,2)^T & -X & 0 & 0 \\ (1,3)^T & 0 & -I_{r_Q} & 0 \\ (1,4)^T & 0 & 0 & -Y + MXM^T \end{bmatrix} < 0 \tag{63}$$

where (i, j) denotes the component at i-th raw and j-th column position. Based on the result above, we propose the following theorem.

Theorem 7. Given the uncertain system (57) with (59) (or, (60)), there exist stable sliding surfaces if there exist matrices $Y > 0$, F and $X \in S_\Delta$ satisfying the linear matrix inequality (63). Also, the stable sliding function coefficients are given by $C = P_{22}^{-1} P_{12}^T$, where $P := Y^{-1}$.

Regardless of successful extension of the proposed sliding hyperplane design approach to the uncertain parametric systems, the robust reaching law design remains open. In case of approaches in the continuous time, Kim et al. [2000] has provided the reaching law that guarantees the asymptotic stability of the sliding function.

6. CONCLUSION

In this paper, we have shown that the Lyapunov inequality for the linear full state feedback is generically the necessary and sufficient condition for the existence of discrete-time sling hyperplane. Considering that the same argument has been made for the continuous-time case, the results can be viewed as an extension of the continuous-time approach to the discrete-time one. Also, the reaching law was derived to satisfy the attractiveness of the boundary layer. Then, the proposed control law (based on the full state information) has been extended to the output feedback case by adopting the multi-rate output feedback.

The verifications of the proposed approach by simulations (or experiments) and the robust reaching law design remain as further studies.

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