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Max–Min Optimal Control of Constrained Discrete-time Systems

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Abstract: This paper considers the optimal control problem for constrained discrete–time systems affected by measured and bounded disturbances and uncertainties in the underlying system equations. This problem setting leads to the *sup–inf* robust optimal control problems. Three classes of discrete–time systems permitting the characterization of the *sup–inf* value functions and robust optimal control policies are examined. The corresponding *max–min* optimal control problems are solved by using the dynamic programming. *Copyright© 2008 IFAC.*

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1. INTRODUCTION

Optimal control of constrained discrete–time systems affected by bounded and unknown disturbances and/or uncertainties in the underlying system equations is a well-studied topic. Pioneering contributions in the optimal control setup (Witsenhausen, 1968; Glover and Schweppe, 1971; Bertsekas and Rhodes, 1971) considered the *inf–sup* robust optimal control problem and obtained its solution via dynamic programming. The contemporary research has resulted in the development of several control techniques offering meaningful solutions to problems of robust control synthesis. Fairly reasonable robust control synthesis methods utilize set–theoretic techniques (Aubin, 1991; Blanchini and Miani, 2008) or an adequate, somehow complementary, game–theoretic framework (Başar and Olsder, 1995). The robust and optimal control problems for some specific classes of constrained discrete–time systems have been recently reconsidered by utilizing parametric programming techniques (Bank *et al.*, 1983). These recent advances in the synthesis of robust optimal controllers for constrained discrete time systems include (Bemporad *et al.*, 2003; Mayne *et al.*, 2006*a*). An alternative, notable, robust control synthesis technique is the design of tube-based model predictive controllers (Raković and Mayne, 2005; Mayne *et al.*, 2006*b*).

We consider the optimal control problem for constrained discrete–time systems subject to the uncertainty in the case when future realizations of disturbances and uncertain sequences are bounded, while their *current realization is known.* This setup covers, for instance, control of linear parameter-varying systems (Lu and Arkun, 2000; Besselmann *et al.*, 2008), linear time-varying systems, control of supply chains and multi-inventory systems (Laumanns and Lefeber, 2006). We aim to promote the point that when information of the current disturbance/uncertainty is available to the controller, it is more natural to define such problems as *max–min optimal control problems.*

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Outline of the paper: Section 2 introduces preliminaries. Sections 3, 4 and 5 discuss the max–min optimal control problems for constrained linear: (i) time invariant, (ii) time-varying, and (iii) parameter-varying systems. Section 6 presents concluding remarks.

Notation and basic definitions: The set of non–negative and positive integers are denoted, respectively, by $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ := \{1, 2, \ldots\}$. Let $\mathbb{N}_{[q_1,q_2]} := \{q_1, q_1 + 1, \ldots, q_2 - 1, q_2\}$ for given $q_1 \in \mathbb{N}$ and $q_2 \in \mathbb{N}$ such that $q_1 < q_2$; $\mathbb{N}_q$ denotes $\mathbb{N}_{[0,q]}$ for $q \in \mathbb{N}$. A set of non–negative real numbers is denoted by $\mathbb{R}_+$. The positive orthant in the d−dimensional Euclidean space is denoted by $\mathbb{R}^d_+$. For vectors we use the following notation: x_i denotes the i^{th} vector, while $x_{[i]}$ denotes the i^{th} component of the vector x. In general, we write $f(\cdot)$ or f for a function and $f(x)$ for its value at the point x. The symbol Δ^n denotes the standard n–simplex: $\Delta^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_+ : \sum_{i=1}^{n+1} x_{[i]} = 1\}$. A *polyhedron* is a set described by the intersection of finitely many half-spaces. A *polytope* is a closed and bounded polyhedron. A function $f : \mathcal{D} \to \mathbb{R}$ is a *polyhedral function* if its epigraph $\mathcal{E}_f := \{(x, \gamma) : \gamma \geq f(x),\ x \in \mathcal{D}\}$ is a closed polyhedron. The set of vertices of a polytope $\mathcal{P}$ is denoted as $\text{vert}(\mathcal{P})$ and the convex hull of a set of points $\mathcal{V}$ as $\text{convh}(\mathcal{V})$. A *polytopal complex* $\mathcal{C}$ is a finite collection of polytopes such that: (i) the empty polytope is in $\mathcal{C}$, (ii) $\mathcal{P} \in \mathcal{C}$ implies that the faces of $\mathcal{P}$ are in $\mathcal{C}$ and (iii) for $\mathcal{P}, \mathcal{Q} \in \mathcal{C}$ the intersection $\mathcal{P} \cap \mathcal{Q}$ is a face of both $\mathcal{P}$ and $\mathcal{Q}$. A polytopal subdivision of a polytope $\mathcal{Q}$ is a polytopal complex $\mathcal{C} = \{\mathcal{P}_0, \ldots, \mathcal{P}_n\}$ such that $\mathcal{Q} = \bigcup_{i=0}^{q} \mathcal{P}_i$. Given a polytope $\mathcal{P}$, a function f is called *continuous piecewise-affine over* $\mathcal{P}$ (CPWA over $\mathcal{P}$) if f is continuous and there exists a polytopal subdivision $\mathcal{C} = \{\mathcal{P}_k : k \in \mathbb{N}_q\}$ of the set $\mathcal{P}$ such that f is affine in each $\mathcal{P}_k$. The Minkowski set addition and the Minkowski (Pontryagin) difference of two (non–empty) sets X and Y, such that $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$, are denoted by $X \oplus Y := \{x + y : x \in X,\ y \in Y\}$ and $X \ominus Y := \{z \in \mathbb{R}^n : z \oplus Y \subseteq X\}$.

2. PRELIMINARIES

We consider the discrete–time system:

$$x^+ = f(x, u, w), \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $x^+ \in \mathbb{R}^n$ are, respectively, the current and the successor state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^p$ is the disturbance and the current state transition mapping is $f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$. The system variables x, u and w are subject to constraints:

$$x \in \mathcal{X} \subset \mathbb{R}^n,\ u \in \mathcal{U} \subset \mathbb{R}^m, \text{ and } w \in \mathcal{W} \subset \mathbb{R}^p, \tag{2.2}$$

where state, control and disturbance constraint sets $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{W}$ are compact sets. The state transition function $f(\cdot,\cdot,\cdot)$ is constrained to belong to a set of functions $\mathcal{F}$ given either as a discrete set of a finite number of maps or as its (closed) convex hull:

$$f \in \mathcal{F} \text{ with } \mathcal{F} = \widetilde{\mathcal{F}} \text{ or } \mathcal{F} = \text{convh}(\widetilde{\mathcal{F}}), \text{ where} \tag{2.3a}$$

$$\widetilde{\mathcal{F}} = \{f_i(\cdot,\cdot,\cdot) : i \in \mathbb{N}_q\}, \tag{2.3b}$$

and, for each $i \in \mathbb{N}_q$, $f_{s_i}(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$. Hence, $x_{k+1} = f_{s_k}(x_k, u_k, w_k)$ is the state at time $k+1$, if at time k the state is x_k, the applied input is u_k, the state transition mapping is $f_{s_k}(\cdot,\cdot,\cdot)$ and the disturbance is w_k (hereafter s_k is an indicator at time k associated with the mapping $f_{s_k}(\cdot,\cdot,\cdot)$). Robust control problems of our interest are characterized by the following essential interpretation:

Interpretation 1. When the decision concerning the control input u_k is taken (at time k) the state x_k, the disturbance w_k and the state transition mapping $f_{s_k}(\cdot,\cdot,\cdot)$ are known, while the only available information of the future disturbances w_{k+i}, $i \in \mathbb{N}_+$, and the future state transition maps $f_{s_{k+i}}(\cdot,\cdot,\cdot)$, $i \in \mathbb{N}_+$, is that they can take any arbitrary values in their respective constraint sets $\mathcal{W}$ and $\mathcal{F}$. Furthermore, realizations of the future disturbances w_{k+i}, $i \in \mathbb{N}_+$, and the future state transition maps $f_{s_{k+i}}(\cdot,\cdot,\cdot)$, $i \in \mathbb{N}_+$, *will be known at future times* $k+i$, $i \in \mathbb{N}_+$ *but are unknown at time* k.

We formalize the notion of *knowledge* available for the control synthesis from the previous interpretation by introducing the *information vector* $z(x, w, f)$ which aggregates relevant knowledge of the current values of the state x, the disturbance w and the state transition mapping f with $(x, w, f) \in \mathcal{X} \times \mathcal{W} \times \mathcal{F}$. Consequently, we introduce the set $\mathcal{Z}$ specified by:

$$\mathcal{Z} := \{z(x, w, f) : (x, w, f) \in \mathcal{X} \times \mathcal{W} \times \mathcal{F}\}$$

and refer to the set $\mathcal{Z}$ as the *information set.* A control policy, i.e. a sequence of control laws $\pi_i : \mathcal{Z} \to \mathcal{U}$, over the horizon of length $N \in \mathbb{N}_+$ is denoted by $\Pi_N := \{\pi_i(\cdot) : i \in \mathbb{N}_{N-1}\}$. The set of all control policies over the horizon of length N is denoted by $\mathbf{\Pi}_N$. An admissible disturbance sequence over the horizon of length N is denoted by $\mathbf{w}_N := \{w_0, w_1, \ldots, w_{N-1}\}$ where $w_i \in \mathcal{W}$ for all $i \in \mathbb{N}_{N-1}$. The set of all admissible disturbance sequences over the horizon of length N is denoted by $\mathbf{W}_N$. An admissible state transition mapping sequence over the horizon of length N is denoted by $\mathbf{f}_N := \{f_{s_0}, f_{s_1}, \ldots, f_{s_{N-1}}\}$ where, as above, $f_{s_i} \in \mathcal{F}$ for all $i \in \mathbb{N}_{N-1}$. The set of all admissible state transition mapping sequences over the horizon of length N is denoted by $\mathbf{F}_N$. Also, let $\phi(i; x, \Pi_N, \mathbf{w}_N, \mathbf{f}_N)$ denote the solution to (2.1) at time instant i, $i \in \mathbb{N}_{N-1}$, given the initial state x (at time 0), a control policy Π_N and admissible disturbance and state transition mapping sequence $\mathbf{w}_N \in \mathbf{W}_N$ and $\mathbf{f}_N \in \mathbf{F}_N$ (by convention $\phi(0; x, \Pi_N, \mathbf{w}_N, \mathbf{f}_N) = x$).

The cost $V_N(x, \Pi_N, \mathbf{w}_N, \mathbf{f}_N)$, for the initial state x, the control policy Π_N, the disturbance sequence $\mathbf{w}_N$ and the state transition mapping sequence $\mathbf{f}_N$, is:

$$V_N(x, \Pi_N, \mathbf{w}_N, \mathbf{f}_N) := V_f(x_N) + \sum_{i=0}^{N-1} \ell(x_i, u_i), \tag{2.4}$$

where, for each $i \in \mathbb{N}$, $x_i := \phi(i; x, \Pi_N, \mathbf{w}_N, \mathbf{f}_N)$ and $u_i := \pi_i(z(\phi(i; x, \Pi_N, \mathbf{w}_N, \mathbf{f}_N), w_i, f_{s_i}))$ and functions $V_f(\cdot)$ and $\ell(\cdot,\cdot)$, representing, respectively, the terminal and the path cost, are continuous and non–negative (finite) valued.

Problem 1. (The N–horizon robust control problem). Given an integer $N \in \mathbb{N}_+$, characterize the set of states $x \in \mathcal{X}$ and the corresponding control policy (possibly a set of control policies) $\Pi_N \in \mathbf{\Pi}_N$ such that for all $\mathbf{w}_N \in \mathbf{W}_N$, all $\mathbf{f}_N \in \mathbf{F}_N$ and all $i \in \mathbb{N}_{N-1}$:

$$\phi(i; x, \Pi_N, \mathbf{w}_N, \mathbf{f}_N) \in \mathcal{X} \text{ and } \phi(N; x, \Pi_N, \mathbf{w}_N, \mathbf{f}_N) \in \mathcal{X}_f,$$

where $\mathcal{X}_f \subseteq \mathcal{X}$ is the terminal set (assumed to be compact), and, in addition, the control policy Π_N results in the guaranteed cost specified by:

$$V_N^0(x) = \sup_{(\mathbf{w}_N, \mathbf{f}_N) \in \mathbf{W}_N \times \mathbf{F}_N} \ \inf_{\Pi_N \in \mathbf{\Pi}_N} V_N(x, \Pi_N, \mathbf{w}_N, \mathbf{f}_N).$$

We consider Problem 1 as a dynamic game (Başar and Olsder, 1995). At each time j the first player (the *controller*) can choose a control $u_j \in \mathcal{U}$ within rules of the form $u_j = u_j(z(x_j, w_j, f_{s_j}))$ and the second player (the *adversary*) can choose a disturbance $w_j \in \mathcal{W}$ and a state transition mapping $f_{s_j} \in \mathcal{F}$. At time j, the adversary declares his choice prior to the *controller* and hence the controller can, in view of Interpretation 1, utilize the information vector $z(x_j, w_j, f_{s_j})$ when declaring his control action $u_j = u_j(z(x_j, w_j, f_{s_j}))$. The controller synthesizes the control policy in accordance with Interpretation 1 and aims, in addition, to ensure the solvability of Problem 1 no matter what triplet (x_k, w_k, f_k) occurs at time k when the control policy Π_N is adopted. As a result of our setup, the controller is concerned with the *sup–inf robust optimal control problem.* The following example illustrates the value of the information available to the controller.

Example 1. Consider the scalar system:

$$x^+ = x + u + w,$$

where $\mathcal{F} = \{x + u + w\}$, with the constraint sets:

$$\mathcal{X} = [-10, 10],\ \mathcal{U} = [-3, 3],\ \mathcal{W} = [-2, 2] \text{ and } \mathcal{X}_f = [-1, 1].$$

In view of Interpretation 1, the controller employs the information vector $z(x, w) = (x, w)$ *for the control synthesis.* The control law $u(x, w) = 3$ when $x \leq -3$, $u(x, w) = -(x + w)$ when $-3 \leq x + w \leq 3$ and $u(x, w) = -3$ when $x \geq 3$ ensures that any state $x \in \mathcal{X}$ reaches the terminal constraint set $\mathcal{X}_f$ in at most 9 time steps (for any admissible disturbance sequence). Hence, the *sup–inf* robust optimal control problem is feasible for any $x \in \mathcal{X}$ (for any horizon length $N \geq 9$). *The* inf–sup *robust optimal control problem corresponds to the case when the information vector* $z(x, w)$ *is merely the state* x. Since the terminal constraint set $\mathcal{X}_f$ is a proper subset of the disturbance set $\mathcal{W}$ the *inf–sup* robust optimal control problem is, clearly, not solvable.

Essentially, the controller utilizes *sup–inf* dynamic programing (DP) in order to obtain the solution to Problem 1. More precisely, given a horizon length $N \in \mathbb{N}_+$, the controller is concerned with the computation of the sequence of partial return functions $\{V_j^0(\cdot)\}_{j=1}^N$, the sequence of control laws $\{\kappa_j(\cdot)\}_{j=1}^N$ and the sequence of the controllability sets $\{\mathcal{X}_j\}_{j=1}^N$ specified by, for all $j \in \mathbb{N}_{[1,N]}$,:

$$V_j^0(x) = \sup_{(w,f)\in\mathcal{W}\times\mathcal{F}} \inf_{u\in\mathcal{U}} \{\ell(x,u) + V_{j-1}^0(f(x,u,w)) \ : \ f(x,u,w) \in \mathcal{X}_{j-1}\}, \ x \in \mathcal{X}_j \quad (2.5a)$$

$$\kappa_j^0(x,w,f) = \arg\inf_{u\in\mathcal{U}} \{\ell(x,u) + V_{j-1}^0(f(x,u,w)) \ : \ f(x,u,w) \in \mathcal{X}_{j-1}\}, \ (x,w,f) \in \mathcal{X}_j \times \mathcal{W} \times \mathcal{F} \quad (2.5b)$$

$$\mathcal{X}_j = \{x \in \mathcal{X} \ : \ \forall (w,f) \in \mathcal{W}\times\mathcal{F}, \ \exists u \in \mathcal{U} \text{ such that } f(x,u,w) \in \mathcal{X}_{j-1}\} \quad (2.5c)$$

with the boundary conditions $\mathcal{X}_0 := \mathcal{X}_f$ and $V_0^0(x) := V_f(x), \ x \in \mathcal{X}_f$. The control laws $\kappa_j^0(\cdot)$ are employed to construct the corresponding *sup–inf* optimal control policy $\{\pi_i^0(\cdot) \ : \ i \in \mathbb{N}_{N-1}\}$ via relations $\pi_{N-j}^0(x,w,f) = \kappa_j^0(x,w,f)$ (or $\pi_{N-j}^0(x,w,f) \in \kappa_j^0(x,w,f)$ when $\kappa_j^0(\cdot)$ is set–valued) for all $(x,w,f) \in \mathcal{X}_j\times\mathcal{W}\times\mathcal{F}$ and all $j \in \mathbb{N}_{[1,N]}$.

3. LINEAR TIME INVARIANT SYSTEMS

Consider the discrete-time, linear time-invariant, system:

$$x^+ = Ax + Bu + Dw \quad (3.1)$$

where $x \in \mathbb{R}^n$ and $x^+ \in \mathbb{R}^n$ are, respectively, the current and the successor state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^p$ is the disturbance and matrices A, B and D are of appropriate dimensions. Constraint sets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{X}_f$ satisfy the following assumption invoked merely for computational reasons:

Assumption 1. The state constraint set $\mathcal{X}$ is a polytope in $\mathbb{R}^n$. The control constraint set $\mathcal{U}$ is a polytope in $\mathbb{R}^m$. The disturbance constraint set $\mathcal{W}$ is a polytope in $\mathbb{R}^p$. The terminal constraint set $\mathcal{X}_f \subseteq \mathcal{X}$ is a polytope in $\mathbb{R}^n$. The sets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{X}_f$ all contain the origin.

The cost function is given by (2.4) and, in addition, the terminal and path functions are specified by:

$$V_f(x) = \|Px\| \ \text{ and } \ \ell(x,u) = \|Qx\| + \|Ru\|, \quad (3.2)$$

where $\|\cdot\|$ denotes a polyhedral norm and matrices P, Q and R are of appropriate dimensions. *Clearly, in this setting the information vector $z(x,w)$ is the pair (x,w).*

3.1 Exact DP Recursion for Linear–Polytopic Case

The DP recursion (2.5) reduces to, for $j \in \mathbb{N}_{[1,N]}$,:

$$V_j^0(x) = \max_{w\in\mathcal{W}} \min_{u\in\mathcal{U}} \{\ell(x,u) + V_{j-1}^0(Ax+Bu+Dw) \ : \ Ax+Bu+Dw \in \mathcal{X}_{j-1}\}, \ x \in \mathcal{X}_j \quad (3.3a)$$

$$\kappa_j^0(x,w) = \arg\min_{u\in\mathcal{U}} \{\ell(x,u) + V_{j-1}^0(Ax+Bu+Dw) \ : \ Ax+Bu+Dw \in \mathcal{X}_{j-1}\}, \ (x,w) \in \mathcal{X}_j \times \mathcal{W} \quad (3.3b)$$

$$\mathcal{X}_j = \{x \in \mathcal{X} \ : \ \forall w \in \mathcal{W}, \ \exists u \in \mathcal{U} \text{ such that } Ax+Bu+Dw \in \mathcal{X}_{j-1}\} \quad (3.3c)$$

where the endpoint conditions are $\mathcal{X}_0 := \mathcal{X}_f$ and $V_0^0(x) := V_f(x), \ x \in \mathcal{X}_f$ and the corresponding *max–min* optimal control policy $\{\pi_i^0(\cdot) \ : \ i \in \mathbb{N}_{N-1}\}$ is obtained via relations $\pi_{N-j}^0(x,w) = \kappa_j^0(x,w)$ (or $\pi_{N-j}^0(x,w) \in \kappa_j^0(x,w)$ when $\kappa_j^0(\cdot)$ is set–valued) for all $(x,w) \in \mathcal{X}_j \times \mathcal{W}$ and $j \in \mathbb{N}_{[1,N]}$. The controllability sets $\mathcal{X}_j$ are directly computable by utilizing the set–theoretic calculus:

$$\mathcal{X}_j = \{x \in \mathcal{X} \ : \ Ax \in [(\mathcal{X}_{j-1} \oplus (-B\mathcal{U})) \ominus D\mathcal{W}]\}, \quad (3.4)$$

where $\mathcal{X}_0 := \mathcal{X}_f$.

Remark 1. When a control law $\kappa_f(\cdot,\cdot) \ : \ \mathcal{X}_f \times \mathcal{W} \to \mathcal{U}$ associated with the terminal set $\mathcal{X}_f$ is such that the following invariance condition holds:

$$\forall (x,w) \in \mathcal{X}_f \times \mathcal{W}, \ Ax + B\kappa_f(x,w) + w \in \mathcal{X}_f,$$

then the controllability sets $\mathcal{X}_j$ are all non–empty, nested ($\mathcal{X}_j \subseteq \mathcal{X}_{j+1}$) and also invariant (in the sense that for all $(x,w) \in \mathcal{X}_j \times \mathcal{W}$ there exists a $u_j = u_j(x,w) \in \mathcal{U}$ such that $Ax + Bu_j(x,w) + w \in \mathcal{X}_{j-1} \subseteq \mathcal{X}_j$).

The optimization problem $\mathbb{P}_{\min}(x,w)$ associated with the DP recursion (3.3):

$$J_j^0(x,w) = \min_{u\in\mathcal{U}} \{\ell(x,u) + V_{j-1}^0(Ax+Bu+Dw) \ : \ Ax+Bu+Dw \in \mathcal{X}_{j-1}\}, \ (x,w) \in \mathcal{X}_j \times \mathcal{W} \quad (3.5a)$$

$$\kappa_j^0(x,w) = \arg\min_{u\in\mathcal{U}} \{\ell(x,u) + V_{j-1}^0(Ax+Bu+Dw) \ : \ Ax+Bu+Dw \in \mathcal{X}_{j-1}\}, \ (x,w) \in \mathcal{X}_j \times \mathcal{W} \quad (3.5b)$$

can be written transparently as a parametric linear programming problem (Bank *et al.*, 1983) whenever Assumption 1 holds, $\ell(\cdot,\cdot)$ is as specified in (3.2) and $V_{j-1}^0(\cdot)$ is a polyhedral function over $\mathcal{X}_{j-1}$. Furthermore, it is also known (assuming that $\mathcal{X}_j$ is a non–empty set) that function $J_j^0(\cdot,\cdot)$ is a polyhedral function over $\mathcal{X}_j \times \mathcal{W}$ and that there exists a polytopal subdivision (polytopal complex) $\mathcal{C}_j := \{\mathcal{P}_{(1,j)}, \ldots, \mathcal{P}_{(q_j,j)}\}$ of the polytope $\mathcal{X}_j \times \mathcal{W}$ and a continuous selection $\tilde{\kappa}_j^0(x,w) \in \kappa_j^0(x,w)$ affine in each polytope $\mathcal{P}_{(i,j)} \in \mathcal{C}$ (Bank *et al.*, 1983). Functions $V_j^0 \ : \ \mathcal{X}_j \to \mathbb{R}_+$ can be obtained, as the maximum of a finite number of polyhedral functions as specified by the optimization problem $\mathbb{P}_{\max}(x)$:

$$V_j^0(x) = \max_{w\in\text{vert}(\mathcal{W})} J_j^0(x,w), \ x \in \mathcal{X}_j. \quad (3.6)$$

Consequently, it follows that $V_j^0(\cdot)$ is a polyhedral function over $\mathcal{X}_j$ (Rockafellar, 1970). Since $V_f(\cdot)$ is a polyhedral function over $\mathcal{X}_f = \mathcal{X}_0$ a direct argument based on the principle of mathematical induction allows us to summarize properties of functions $V_j^0(\cdot), \ j \in \mathbb{N}_{[1,N]}$ and control laws $\kappa_j^0(\cdot,\cdot), \ j \in \mathbb{N}_{[1,N]}$.

Proposition 1. Suppose Assumption 1 holds, fix an integer $N \in \mathbb{N}_+$ and assume that the controllability set $\mathcal{X}_N$, given by set recursion (3.4), is non–empty. Consider Problem 1 for the system (3.1) with the terminal and path cost given by (3.2). Then (i) Problem 1 is solvable, (ii) the partial return functions $V_j^0(\cdot)$ are polyhedral functions over $\mathcal{X}_j$ for all $j \in \mathbb{N}_{[1,N]}$, and (iii) there exist control laws $\tilde{\kappa}_j^0(\cdot,\cdot)$, which are CPWA functions over $\mathcal{X}_j \times \mathcal{W}$, such that $\tilde{\kappa}_j^0(x,w) \in \kappa_j^0(x,w)$ for all $(x,w) \in \mathcal{X}_j \times \mathcal{W}$ and $j \in \mathbb{N}_{[1,N]}$.

Remark 2. Utilizing linearity and convexity and the fact that functions $J_{j-1}^0(\cdot,\cdot)$ are polyhedral functions over $\mathcal{X}_{j-1} \times \mathcal{W}$, the following parametric optimization problem yields functions $J_j^0(\cdot,\cdot)$ (and also control laws $\kappa_j(\cdot,\cdot)$) and it does not require the explicit computation of functions $V_{j-1}^0(\cdot)$ for $j \in \mathbb{N}_{[2,N]}$:

$$J_j^0(x,w) := \min_{(u,\gamma)}\{\ell(x,u)+\gamma \ : (u,\gamma)\in\mathcal{D}(x,w)\} \text{ where}$$
$$\mathcal{D}(x,w) := \{(u,\gamma)\in\mathcal{U}\times\mathbb{R}_+ \ : \ \forall\tilde{w}\in\text{vert}(\mathcal{W}),\ J_{j-1}^0(Ax+Bu+Dw,\tilde{w})\le\gamma,\ Ax+Bu+Dw\in\mathcal{X}_{j-1}\},\ (x,w)\in\mathcal{X}_j\times\mathcal{W}.$$

and, in addition, can be solved by an adequate parametric linear program. The function $J_1^0(\cdot,\cdot)$ (and the control law $\kappa_1^0(\cdot,\cdot)$) are directly computable from (3.3) or (3.5). In addition, the epigraph of a function $V_j^0(\cdot)$ corresponds to the irredundant representation of the polyhedron:

$$\{(x,\gamma)\in\mathcal{X}_j\times\mathbb{R}_+ \ : \ \forall\tilde{w}\in\text{vert}(\mathcal{W}),\ J_j^0(x,\tilde{w})\le\gamma\}.$$

Our next example, borrowed from (Laumanns and Lefeber, 2006), illustrates benefits of the adequate utilization of the information available for control synthesis.

Example 2. Consider the model of a demand-driven supply network (a variant of the Beer Distribution Game) which can be described by:

$$x^+ = \begin{bmatrix}1&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}x + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}u + \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}w$$

subject to the state, control and disturbance constraints:

$$x\ge 0,\quad u\in[0,8],\quad w\in[0,8].$$

In addition, the states are restricted to set $\{x\in\mathbb{R}^4 \ : \ \|x\|_\infty \le 100\}$ for computational reasons. The state x denotes the amount of goods at different stages of the supply chain, the additive disturbance w models customers demands and the control input u is the order rate at the chain input. The control objective is to devise a strategy for ordering new goods ensuring that the amount of goods in the chain is minimized while the customers demands are satisfied. The corresponding control objective is reflected via the cost function:

$$V(x_0,\Pi_\infty,\mathbf{w}_\infty) = \sum_{i=0}^{\infty} 0.5\|x_i\|_1,$$

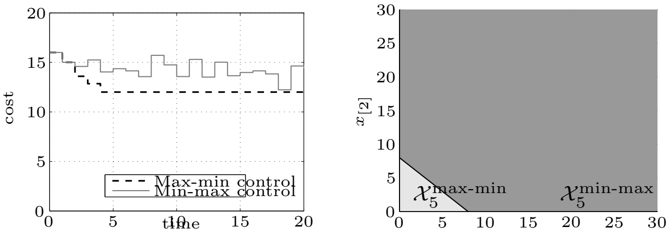
(here x_i is, as before, the solution of the underlying state update equation given the initial state x, the control policy Π_∞ and the disturbance sequence $\mathbf{w}_\infty$). In (Laumanns and Lefeber, 2006) the authors utilize a dynamic programming approach to compute the *min–max* robust optimal control policy. *The information vector $z(x,w)$ employed in(Laumanns and Lefeber, 2006) corresponds to the state x and yields the time invariant control law*:

$$\pi_\infty^{\min-\max}(x) = \max\{32-\|x\|_1\ ,\ 0\}.$$

Utilizing the pair (x,w) as the information vector $z(x,w)$ leads to the max–min *robust optimal control problem and yields the* max–min *robust optimal control law:*

$$\pi_\infty^{\max-\min}(x,w) = \max\{24+w-\|x\|_1\ ,\ 0\}.$$

The min–max and the max–min control laws were tested in a simulation with uniformly distributed random customers demands. Figure 1 (a) shows the evolution of the actual cost $\ell(x_i)=0.5\|x_i\|_1$ over time. It can be seen that utilizing the information about the current customers demands when synthesizing the control policy yields an average cost reduction of about 15%. The set of feasible states for *max–min* ordering strategy is "larger" as illustrated in Figure 1 (b), where projections of the *min–max* and *max–min* feasible sets onto $x_{[1]}-x_{[2]}$ subspace are shown in darker and lighter gray (the min–max feasibility set overlaps partially the max–min feasibility set since it is its subset).



(a) Cost (b) Feasibility Sets

Fig. 1. Comparison of the min–max and max–min controls.

3.2 Interpolation Based DP Recursion

The exact DP recursion (3.3) requires a solution to a parametric linear programming problem in (x,w) space. It is desirable, from the computational point of view, to obtain simplified DP procedure (if possible) that operates in lower dimensional space. When the disturbance constraint set $\mathcal{W}$ is given as the (closed) convex hull of the set:

$$\tilde{\mathcal{W}} := \{\tilde{w}_i\in\mathbb{R}^p \ : \ i\in\mathbb{N}_q\},$$

where q is a finite integer we consider the modification of the DP equations (3.3):

$$V_{(j,i)}^0(x) = \min_{u\in\mathcal{U}}\{\ell(x,u)+V_{j-1}^0(Ax+Bu+D\tilde{w}_i) \ : \ Ax+Bu+D\tilde{w}_i\in\mathcal{X}_{j-1}\},\ x\in\mathcal{X}_j \quad (3.7a)$$
$$\nu_{(j,i)}^0(x) = \arg\min_{u\in\mathcal{U}}\{\ell(x,u)+V_{j-1}^0(Ax+Bu+D\tilde{w}_i) \ : \ Ax+Bu+D\tilde{w}_i\in\mathcal{X}_{j-1}\},\ x\in\mathcal{X}_j \quad (3.7b)$$
$$V_j^0(x) = \max_{i\in\mathbb{N}_q} V_{(j,i)}^0(x),\ x\in\mathcal{X}_j. \quad (3.7c)$$

Underlying linearity, convexity and polyhedral nature of involved functions, yield the fact that functions $V_j^0(\cdot)$ obtained by (3.3a) and (3.7c) coincide (their values are equal for any $x\in\mathcal{X}_j$). Control laws $\kappa_j^0(\cdot,\cdot)$ and $\nu_{(j,i)}(\cdot)$ are, however, defined over different spaces ($\mathcal{X}_j\times\mathcal{W}$ and $\mathcal{X}_j$ respectively), but the *max–min* interpolated control laws, say $\nu_j(\cdot,\cdot) \ : \ \mathcal{X}_j\times\mathcal{W}\to\mathcal{U}$ can be obtained by:

$$\nu_j(x,w) := \sum_{i=0}^{q}\lambda_{[i]}^0(w)\tilde{\nu}_{(j,i)}(x),\ (x,w)\in\mathcal{X}_j\times\mathcal{W} \text{ with}$$
$$\lambda^0(w) := \arg\min_{\lambda}\{\lambda'\lambda \ : \ w=\sum_{i=0}^{q}\lambda_{[i]}\tilde{w}_i,\ \lambda\in\Delta^q\}, \quad (3.8)$$

and where $\tilde{\nu}_{(j,i)}(\cdot)$ are CPWA functions over $\mathcal{X}_j$ satisfying $\tilde{\nu}_{(j,i)}(x)\in\nu_{(j,i)}^0(x)$ for all $x\in\mathcal{X}_j$.

Remark 3. Polyhedral nature of involved functions, the linearity of the state update equation, polytopic structure of state, control, disturbance and terminal constraint sets (sets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{X}_f$) and sets $\mathcal{X}_j$ ensures that the *max–min* interpolated control laws $\nu_j(\cdot,\cdot)$, specified by (3.8), satisfy all the constrains and yield the guaranteed *max–min* cost specified by functions $V_j^0(\cdot)$. However, the *max–min* interpolated control laws $\nu_j(\cdot,\cdot)$ are essentially different from the *max–min* exact control laws or their adequate selections, say $\tilde{\kappa}_j(\cdot,\cdot)$, (obtained by (3.3b) or (3.5b)) in the sense that, for all $(x,w) \in \mathcal{X}_j \times \mathcal{W}$,:

$$J_j^0(x,w) = \ell(x, \tilde{\kappa}_j(x,w)) + V_{j-1}^0(Ax + B\tilde{\kappa}_j(x,w) + w)$$

and

$$J_j^0(x,w) \leq \ell(x, \nu_j(x,w)) + V_{j-1}^0(Ax + B\nu_j(x,w) + w),$$

where function $J_j^0(\cdot,\cdot)$ is specified by (3.5a).

4. LINEAR TIME-VARYING SYSTEMS

Consider a linear time-varying system when the state transition equations belong to a discrete set of finite cardinality:

$$x^+ = f(x,u),\ f \in \mathcal{F} = \{A_i x + B_i u\ :\ i \in \mathbb{N}_q\}, \qquad (4.1)$$

where q is a (finite) integer. The system (4.1) is subject to constraints satisfying Assumption 1 where, for simplicity, we consider the case in which $\mathcal{W} = \{0\}$. The cost is specified by (2.4) with the terminal and path cost given by (3.2).

The integer $s_k \in \mathbb{N}_q$ denotes the indicator, at time k, associated with the set of matrix pairs $\{(A_0,B_0),\ldots,(A_q,B_q)\}$, i.e. at time k:

$$x_{k+1} = A_{s_k} x_k + B_{s_k} u_k.$$

In this case, the information vector $z(x,f)$ is the state–indicator pair, i.e. $z(x,f) = (x,s)$ with $(x,s) \in \mathcal{X} \times \mathbb{N}_q$.

A more detailed form of DP equations (2.5), in this case, is given by, for $j \in \mathbb{N}_{[1,N]}$,:

$$J_j^0(x,i) := \min_{u\in\mathcal{U}}\{\ell(x,u) + V_{j-1}^0(A_i x + B_i u)\ :\ A_i x + B_i u \in \mathcal{X}_{j-1}\},\ x \in \mathcal{X}_j(i),\ i \in \mathbb{N}_q, \qquad (4.2a)$$

$$\nu_j^0(x,i) := \arg\min_{u\in\mathcal{U}}\{\ell(x,u) + V_{j-1}^0(A_i x + B_i u)\ :\ A_i x + B_i u \in \mathcal{X}_{j-1}\},\ x \in \mathcal{X}_j(i),\ i \in \mathbb{N}_q, \qquad (4.2b)$$

$$\mathcal{X}_j(i) := \{x \in \mathcal{X}\ :\ A_i x \in [\mathcal{X}_{j-1} \oplus (-B_i\mathcal{U})]\},\ i \in \mathbb{N}_q, \qquad (4.2c)$$

$$\mathcal{X}_j := \bigcap_{i\in\mathbb{N}_q} \mathcal{X}_j(i) \qquad (4.2d)$$

$$V_j^0(x) := \max_{i\in\mathbb{N}_q} J_j^0(x,i),\ x \in \mathcal{X}_j, \qquad (4.2e)$$

$$\kappa_j^0(x,s) := \nu_j^0(x,s),\ (x,s) \in \mathcal{X}_j \times \mathbb{N}_q, \qquad (4.2f)$$

with boundary conditions: $\mathcal{X}_0 := \mathcal{X}_f$ and $V_0(x) := V_f(x),\ x \in \mathcal{X}_f$. As before the corresponding *max–min* optimal control policy $\{\pi_i^0(\cdot)\ :\ i \in \mathbb{N}_{N-1}\}$ is obtained via relations $\pi_{N-j}^0(x,s) = \kappa_j^0(x,s)$ (or $\pi_{N-j}^0(x,s) \in \kappa_j^0(x,s)$ when $\kappa_j^0(\cdot)$ is set–valued) for all $(x,s) \in \mathcal{X}_j \times \mathbb{N}_q$ and $j \in \mathbb{N}_{[1,N]}$.

Similarly to Proposition 1, we have:

Proposition 2. Suppose Assumption 1 holds, fix an integer $N \in \mathbb{N}_+$ and assume that the controllability set $\mathcal{X}_N$, given by set recursion (4.2d), is non–empty. Consider Problem 1 for the system (4.1) with the terminal and path cost given by (3.2). Then (i) Problem 1 is solvable, (ii) the partial return functions $V_j^0(\cdot)$ are polyhedral functions over $\mathcal{X}_j$ for all $j \in \mathbb{N}_{[1,N]}$, and (iii) for any $s \in \mathbb{N}_q$ there exist control laws $\tilde{\kappa}_j^0(\cdot,s)$, which are CPWA functions over $\mathcal{X}_j(s)$, such that $\tilde{\kappa}_j^0(x,s) \in \nu_j^0(x,s)$ for all $x \in \mathcal{X}_j(s)$ and $j \in \mathbb{N}_{[1,N]}$.

Remark 4. Similarly to Remarks 1 and 2, under Assumption 1 and when the control law $\kappa_f(\cdot,\cdot)\ :\ \mathcal{X}_f \times \mathbb{N}_q \to \mathcal{U}$ associated with the terminal constraint set $\mathcal{X}_f$ is such that:

$$\forall (x,s) \in \mathcal{X}_f \times \mathbb{N}_q,\ A_s x + B_s \kappa_f(x,s) \in \mathcal{X}_f$$

the controllability sets $\mathcal{X}_j$ are non–empty polytopes in $\mathbb{R}^n$, nested ($\mathcal{X}_j \subseteq \mathcal{X}_{j+1}$) and invariant (for all $(x,s) \in \mathcal{X}_j \times \mathbb{N}_q$ there exists a $u_j = u_j(x,s) \in \mathcal{U}$ such that $A_s x + B_s u_j(x,s) \in \mathcal{X}_{j-1} \subseteq \mathcal{X}_j$). In this case, functions $J_j^0(\cdot,\cdot)$ and control laws $\nu_j^0(\cdot,\cdot)$ can be obtained for $j \in \mathbb{N}_{[2,N]}$ as follows. Let, for each j, $\mathbf{u_j} := \{u_0, u_1, \ldots, u_q\}$ and $\mathbf{\Gamma_j} := \{\gamma_0, \gamma_1, \ldots, \gamma_q\}$ and consider the following parametric optimization problem:

$$L_j^0(x) = \min_{(\mathbf{u_j},\Gamma_\mathbf{j})}\{\sum_{i=0}^{q} \gamma_i\ :\ (u_i,\gamma_i) \in \mathcal{D}_i(x),\ i \in \mathbb{N}_q\} \text{ where}$$

$$\mathcal{D}_i(x) := \{(u,\gamma) \in \mathcal{U} \times \mathbb{R}_+\ :\ \forall k \in \mathbb{N}_q,\ \ell(x,u) + J_{j-1}^0(A_i x + B_i u, k) \leq \gamma,\ A_i x + B_i u \in \mathcal{X}_{j-1}\},$$

whose solution can be obtained from an adequate parametric linear programming problem. It is not difficult to see that, given functions $J_{j-1}^0(\cdot,\cdot)$, functions $J_j^0(\cdot,\cdot)$ and control laws $\nu_j^0(\cdot,\cdot)$ can be obtained directly from the optimizer (or its selection) $(\mathbf{u_j^0}(\cdot), \Gamma_\mathbf{j}^0(\cdot))$ of the function $L_j^0(\cdot)$. Functions $J_1^0(\cdot,\cdot)$ and $\nu_1^0(\cdot,\cdot)$ are computable directly by using (4.2). As in Remark 2 it is direct to obtain, if necessary, the epigraph of functions $V_j^0(\cdot)$ (and consequently functions $V_j^0(\cdot)$ themselves) given functions $J_j^0(\cdot,\cdot)$. Control laws $\kappa_j^0(\cdot,\cdot)$ can be constructed transparently by utilizing functions $\mathbf{u_j^0}(\cdot)$ and (4.2f).

5. LINEAR PARAMETER-VARYING SYSTEMS

Consider linear parameter-varying system with the uncertain state transition matrix:

$$x^+ = A(\lambda)x + Bu,\ A(\lambda) := \sum_{j=0}^{q} \lambda_{[j]} A_j,\ \lambda \in \Delta^q. \qquad (5.1)$$

In this case, according to the Interpretation 1, at time k values of the scheduling parameters $\lambda_k \in \Delta^q$ and the state x_k are available to the controller. The system (5.1) is, as in the previous subsection, subject to constraints satisfying Assumption 1 with $\mathcal{W} = \{0\}$ and the cost function is given by (2.4) and (3.2). Note that, in this setting, the state transition equation (5.1) remains linear in $y := A(\lambda)x$:

$$x^+ = y + Bu \text{ where } y := A(\lambda)x, \qquad (5.2)$$

and $A(\lambda)$ is such that $A(\lambda) = \sum_{j=0}^{q} \lambda_{[j]} A_j,\ \lambda \in \Delta^q$. Furthermore, the path cost function $\ell(\cdot,\cdot)$ is, clearly, a separable function in x and u:

$$\ell(x,u) = \ell_x(x) + \ell_u(u) \text{ where}$$
$$\ell_x(x) := \|Qx\| \text{ and } \ell_u(u) := \|Ru\| \qquad (5.3)$$

The linearity of equation (5.1) in $A(\lambda)x$ and separability of the path cost $\ell(\cdot,\cdot)$ expressed in, respectively, (5.2) and (5.3) suggest that it is convenient and natural to consider the information vector $z(x,\lambda)$ specified by:

$$z(x,\lambda) = y \text{ with}$$

$$y = A(\lambda)x \text{ and } A(\lambda) = \sum_{j=0}^{q} \lambda_{[j]} A_j,\ \lambda \in \Delta^q.$$

Due to the underlying "y–linearity" and convexity (i.e. polytopic nature of involved constraint sets), DP equations (2.5) in this case take the following form, for $j \in \mathbb{N}_{[1,N]}$,:

$$\tilde{J}_j^0(y) := \min_{u \in \mathcal{U}} \{\ell_u(u) + V_{j-1}^0(y + Bu) \ : \ y + Bu \in \mathcal{X}_{j-1}\},\ y \in \tilde{\mathcal{Y}}_j \tag{5.4a}$$

$$\kappa_j^0(y) := \arg\min_{u \in \mathcal{U}} \{\ell_u(u) + V_{j-1}^0(y + Bu) \ : \ y + Bu \in \mathcal{X}_{j-1}\},\ y \in \tilde{\mathcal{Y}}_j \tag{5.4b}$$

$$\tilde{\mathcal{Y}}_j := \{y \in \mathbb{R}^n \ : \ \exists u \in \mathcal{U} \text{ such that } y + Bu \in \mathcal{X}_{j-1}\} \tag{5.4c}$$

$$\mathcal{X}_j := \{x \in \mathcal{X} \ : \ \forall \lambda \in \Delta^q,\ \exists u \in \mathcal{U} \text{ such that } A(\lambda)x + Bu \in \mathcal{X}_{j-1}\}, \tag{5.4d}$$

$$V_j^0(x) := \ell_x(x) + \max_{i \in \mathbb{N}_q} \tilde{J}_j^0(A_i x),\ x \in \mathcal{X}_j \tag{5.4e}$$

with boundary conditions, as before, $\mathcal{X}_0 = \mathcal{X}_f$ and $V_0^0(x) = V_f(x),\ x \in \mathcal{X}_f$. In this case, the corresponding *max–min* optimal control policy $\{\pi_i^0(\cdot) \ : \ i \in \mathbb{N}_{N-1}\}$ is obtained via relations $\pi_{N-j}^0(y) = \kappa_j^0(y)$ (or $\pi_{N-j}^0(y) \in \kappa_j^0(y)$ when $\kappa_j^0(\cdot)$ is set–valued) for all $y \in \tilde{\mathcal{Y}}_j$ and all $j \in \mathbb{N}_{[1,N]}$. In this case, it is important to observe that sets $\tilde{\mathcal{Y}}_j$ are convex (polyhedral under Assumption 1) and that not all points $y \in \tilde{\mathcal{Y}}_j$ are of interest to the controller. In fact, given an $x \in \mathcal{X}_{j-1}$, the controller is merely interested in points y such that $y \in \mathcal{Y}(x) := \text{convh}(\{A_i x \ : \ i \in \mathbb{N}_q\})$. It is hopefully clear that $\bigcup_{x \in \mathcal{X}_j} \mathcal{Y}(x) \subseteq \tilde{\mathcal{Y}}_j$ and that the controller is concerned with functions $\tilde{J}_j^0(\cdot)$ and control laws $\kappa_j^0(\cdot,\cdot)$ only for points y such that $y \in \bigcup_{x \in \mathcal{X}_j} \mathcal{Y}(x) \subseteq \tilde{\mathcal{Y}}_j$ as well as functions $V_j^0(\cdot)$ for $x \in \mathcal{X}_j$. As in Propositions 1 and 2 we have:

Proposition 3. Suppose Assumption 1 holds, fix an integer $N \in \mathbb{N}_+$ and assume that the controllability set $\mathcal{X}_N$, given by set recursion (5.4c), is non–empty. Consider Problem 1 for the system (5.1) with the terminal and path cost given by (3.2). Then (i) Problem 1 is solvable, (ii) the partial return functions $V_j^0(\cdot)$ are polyhedral functions over $\mathcal{X}_j$ for all $j \in \mathbb{N}_{[1,N]}$, and (iii) there exists control laws $\tilde{\kappa}_j^0(\cdot)$, which are CPWA functions over $\tilde{\mathcal{Y}}_j$, such that $\tilde{\kappa}_j^0(y) \in \kappa_j^0(y)$ for all $y \in \tilde{\mathcal{Y}}_j$ and $j \in \mathbb{N}_{[1,N]}$.

Remark 5. As before and similarly to Remarks 1, 2 and 4, under Assumption 1 and when the control law $\kappa_f(\cdot,\cdot) \ : \ \mathcal{X}_f \times \Delta^q \to \mathcal{U}$ associated with the terminal constraint set $\mathcal{X}_f$ is such that $\forall (x,\lambda) \in \mathcal{X}_f \times \Delta^q,\ A(\lambda)x + B\kappa_f(x,\lambda) \in \mathcal{X}_f$, the controllability sets $\mathcal{X}_j$ are non–empty, nested, polytopes in $\mathbb{R}^n$ and invariant. In this case, functions $\tilde{J}_j^0(\cdot)$ and control laws $\kappa_j^0(\cdot)$ can be obtained for $j \in \mathbb{N}_{[2,N]}$ by the following parametric optimization problem:

$$\tilde{J}_j^0(y) = \min_{(u,\gamma)} \{\gamma \ : \ (u,\gamma) \in \mathcal{D}(y)\} \text{ where}$$

$$\mathcal{D}(y) := \{(u,\gamma) \in \mathcal{U} \times \mathbb{R}_+ \ : \ y + Bu \in \mathcal{X}_{j-1},$$

$$\forall k \in \mathbb{N}_q,\ \ell_u(u) + \ell_x(y + Bu) + \tilde{J}_{j-1}^0(A_k(y + Bu)) \leq \gamma\},$$

which can be casted as an parametric linear programming problem. Functions $\tilde{J}_1^0(\cdot)$ and $\kappa_1^0(\cdot)$ are computable directly by using (5.4). As in Remark 2 it is direct to obtain, if necessary, the epigraph of functions $V_j^0(\cdot)$ (and consequently functions $V_j^0(\cdot)$ themselves) given functions $\tilde{J}_j^0(\cdot)$.

6. CONCLUSION

We considered the optimal control problem for constrained discrete–time systems affected by the measured and bounded uncertainty and obtained its solution by employing *max–min* dynamic programming. We examined the characterization and discussed the computation of the *max–min* value function and robust optimal control policies for several particular classes of discrete time–systems.

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