Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Binary control of Volterra integral equations

S. A. Belbas*

** Mathematics Department, University of Alabama, Tuscaloosa, AL 35487-0350 USA (e-mail: sbelbas@bama.ua.edu).*

Abstract: We analyze the problem of optimally controlling a system governed by a Volterra integral equation, when the controls take the values 0 or 1 and induce what we term here "amplified memory effect". Under certain conditions, we derive a set of Hamiltonian equations and a set of necessary conditions for optimality.

1. INTRODUCTION

The classical theory of optimal control deals with systems governed by ordinary differential equations or, in the case of discrete-time systems, finite difference equations. That basic theory has been extended in many directions: for example, we may have ssystems governed by partial differential equations, systems governed by sets of inequalities that involve partial derivatives, continuous-time systems with discrete-time controls (switching control or hybrid control), systems governed by impulsive differential equations, or delay-differential equations, or various types of integral equations, etc.
In this paper, we focus on systems governed by Volterra integral equations. Volterra integral equations are used to model a variety of systems with memory effects, i.e. hereditary systems, including applications to population dynamics, epidemiology, economics, continuum mechanics of visoelastic bodies, etc.
The particular type of control systems we consider here are characterized by three important effects: (i) switching nature of the control, (ii) what we have termed "amplified memory effects", and (iii) multiplicity of Hamiltonians. The concept of amplified memory effects means, roughly, that the Hamiltonians that become relevant for these problems have memory with respect to both state and co-state; this will be explained later in this paper. (By contrast, for other types of controlled Voltera integral equations, the Hamiltonian has memory only with respect to the co-state, but not with respect to the state.) The multiplicity of Hamiltonians is also a new phenomenon, and it refers to the fact that one set of Hamiltonians is used for the equations that determine the co-state, and a different set of Hamiltonians is used for an extremum principle akin to Pontryagin's maximum principle. The combination of these phenomena is peculiar to hereditary systems, and it has no counterpart in the theory of controlled ordinary differential equations.

2. STATEMENT OF THE PROBLEM

We consider a system governed by a Volterra integral equation:

$$x(t) = x_0(t) + \int_0^t f(t,s,x(s),u(t,s))\,ds \qquad (1)$$

The state x(t) takes values in a finite-dimensional Euclidean space, and the control $u(t,s)$ is a piecewise-continuous function of two variable, defined on the domain $D_T := \{(t,s): 0 \le s \le t \le T\}$, where T is the finite time-horizon of the control problem.
In addition, we require that u takes values in a finite set $\mathbf{A} := \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$. For simplicity, in this paper we shall restrict our attention to the case in which **A** has exactly two members, which we designate as 0 and 1.
This type of controlled Volterra integral equation has an interpretation in the context of systems theory.
A Volterra operator $(Vx)(t) = \int_0^t g(t,s,x(s))\,ds$ can be considered as a parallel connection of an infinite number of nonlinear gains, one gain for each value of s. If Vx is approximated by a finite sum, say

$$(V_h x)(t) = \sum_{k=0}^{N-1} h\, g(t,s_k,x(s_k)) \equiv \sum_{k=0}^{N-1} G_k(t,x(\cdot)) \ \ (\text{where} \ \ h = \tfrac{t}{N}) \qquad (2)$$

then this discretized operator is the connection in parallel of the gains G_k.
Let us set $f_0(t,s,x) := f(t,s,x,0)$, $f_1(t,s,x) := f(t,s,x,1)$; at each time-instant t, we choose to utilize f_0 for the integration over s (in the Volterra integral equation) in a certain part of the interval [$0, t$], and f_1 in the remaining part of [0, t]. In terms of the interpretation of a connection of gains in parallel, this amounts to having two sets of gains, those based on f_0 and those based on f_1, say $G_k^{(0)}(t,x(\cdot))$ and $G_k^{(1)}(t,x(\cdot))$. Now, for each t and for each k, we choose one of $G_k^{(0)}(t,x(\cdot))$, $G_k^{(1)}(t,x(\cdot))$, with different choices for different values of t and k, and we connect our chosen gains in parallel, disregarding the other gains. This also amounts to selective use of past information about the system to be utilized in the system dynamics. Actual applications to real-world problems may arise in a variety of contexts; one example is an advertising campaign, which is dynamically

evolving, and different aspects of the past history (those events that could be relevant to the campaign) are utilized to affect the current presentations at different times.

The objective of an optimal control problem is to extremize (here, for definiteness, to minimize) a cost functional of the following form:

$$J := \int_0^T \int_0^t F_2(t,s,x(t),x(s),u(t,s))\,ds\,dt + \\ + \int_0^T F_1(t,x(t))\,dt + F_0(x(T)) \quad . \tag{3}$$

3. BACKGROUND FROM CONTROLLED ODES AND ORDINARY VOLTERRA CONTROL

In order to put the present problem into proper perspective, it is useful to briefly outline two other problems: (i) the problem of optimal control of ordinary differential equations with discrete-valued controls, and (ii) the ordinary problem of control of Volterra integral equations. The information of this section is offered as background for the new results that are stated in the next section.

A controlled ODE system with discrete-valued controls has the form

$$\frac{dx(t)}{dt} = f(t,x(t),u(t)),\ x(0) = x_0; \\ u(t) \in \mathbf{A} \equiv \{\alpha_1, \alpha_2, \cdots, \alpha_m\} \tag{4}$$

and the objective is to minimize a functional

$$J_1 := \int_0^T \Phi_1(t,x(t),u(t))\,dt + \Phi_0(x(T))\ . \tag{5}$$

This problem can be handled either by the method of Pontryagin's maximum principle or by the method of dynamic programming. In the method of dynamic programming, we define the value function

$$V(s,\xi) := \\ = \inf_{u(\cdot)} \left\{ \int_s^T \Phi_1(t,x_{s\xi}(t),u(t))\,dt + \Phi_0(x_{s\xi}(T) \right\} \tag{6}$$

where $x_{s\xi}$ is the solution of the ODE with side conditions $x_{s\xi}(s) = \xi$. Then V satisfies (in the appropriate sense of generalized solutions) the Hamilton-Jacobi equation

$$\frac{\partial V(s,\xi)}{\partial s} + \min_{\alpha \in \mathbf{A}} \left\{ \frac{\partial V(s,\xi)}{\partial \xi} f(s,\xi,\alpha) + \Phi_1(s,\xi,\alpha) \right\}; \\ V(T,\xi) = \Phi_0(\xi) \tag{7}$$

This method cannot be used for the problem we are studying in this paper.

The standard case of controlled Volterra integral equations concerns a model of the form

$$x(t) = x_0(t) + \int_0^t g(t,s,x(s),u(s))\,ds$$

and the optimization objective is the minimization of a cost functional

$$J_1 := \int_0^T \Phi_1(t,x(t),u(t))\,dt + \Phi_0(x(T))\ . \tag{8}$$

One important difference between this classical case of Volterra control and the problem of binary Volterra control we study in this paper is that, in the classical case, the control is a function of one variable, whereas in the problem of this paper the control is a function of two variables; this difference, in turn, creates huge differences in the variational analyses of the two types of problems. The classical case of Volterra control has been studied primarily in Schmidt (1982) and Vinokurov (1969); some other related works are Belbas (2007, 2007a). The approach based on the ideas of Pontryagin's maximum principle uses a Hamiltonian functional

$$H(t,x,x(T),u,\psi(\cdot)) := \Phi_1(t,x,u) + \\ + \Phi_{0,X}(x(T))f(T,t,x,u) + \int_t^T \psi(s)f(s,t,x,u)\,ds \quad . \tag{9}$$

The variable X stands for $x(T)$. The Hamiltonian is a functional in the co-state, but a function of the value of the control, and, apart from the dependence on $x(T)$, a function of the current value x of the state. This is an important difference between controlled ODEs and controlled Volterra integral equations. The co-state ψ satisfies the Hamiltonian integral equation (Volterra integral equation in reverse time)

$$\psi(t) = \frac{\partial}{\partial x} H(t,x,x(T),u,\psi(\cdot))\ . \tag{10}$$

Under suitable conditions, if $u^*(\cdot)$ is an optimal control and $x^*(\cdot)$, $y^*(\cdot)$ are the corresponding state trajectory and co-state trajectory, $u^*(\cdot)$ satisfies, for almost all $t \in [0,T]$,

$$H(t,x^*(t),x^*(T),u^*(t),\psi^*(\cdot)) \le \\ \le H(t,x^*(t),x^*(T),u,\psi^*(\cdot)) \tag{11}$$

for every admissible value of u.

This method cannot be used for the binary Volterra control problem of this paper, because the extremal principle above does not hold when the control function depends on two variables.

4. THE RESTRICTED PROBLEM

In order to obtain reasonable analytical results, we restrict the class of admissible control functions as follows: admissible control functions for the restricted problem are defined via two collections of interval-valued functions (in other words, interval-valued multi-functions),

We consider a collection of time-varying intervals $(a_i(t), a_{i+1}(t))$, $(b_j(t), b_{j+1}(t))$ which cover the interval $(0, t)$. We assume that the functions $a_i(\cdot), b_j(\cdot)$ are continuously differentiable. We use the following notation:

$$\begin{aligned}
&\widetilde{A}(t) = \{(a_i(t), a_{i+1}(t)) : i \in \mathbf{I}_A\},\\
&\widetilde{B}(t) = \{(b_j(t), b_{j+1}(t)) : j \in \mathbf{I}_B\};\\
&\hat{A}(t) = \bigcup_{i \in \mathbf{I}_A} (a_i(t), a_{i+1}(t));\\
&\hat{B}(t) = \bigcup_{j \in \mathbf{I}_B} (b_j(t), b_{j+1}(t)).
\end{aligned} \tag{12}$$

The definition of a control policy requires, among other things, a measurable selection from D_T into $\mathbf{A} \equiv \{0,1\}$.

A control function $u(t,\cdot)$ is determined by

$$\begin{aligned}
&u(t,s) = 0 \ \ for \ s \in A(t) \setminus \hat{B}(t),\ A(t) \in \widetilde{A}(t);\\
&u(t,s) = 1 \ for \ s \in B(t) \setminus \hat{A}(t),\ B(t) \in \widetilde{B}(t);\\
&u(t,s) = \sigma(t,s) \ \ for \ s \in A(t) \cap B(t),\\
&A(t) \in \widetilde{A}(t),\ B(t) \in \widetilde{B}(t).
\end{aligned} \tag{13}$$

The restricted optimal control problem concerns the minimization of the functional J over control policies of the type defined in this section.

It should be noted that, although for theoretical reasons we have called this "restricted problem", this model is quite general for practical implementations, and anything more general would be difficult to interpret in terms of practical implementation.

We define

$$\begin{aligned}
&S_{0,\pm}(a_i) := \{A(t) \setminus \hat{B}(t) : A(t) \in \widetilde{A}(t),\\
&\ a_i \text{ is a left endpoint } (-) \text{ or}\\
&\ \text{a right endpoint } (+) \text{ of } A(t) \setminus \hat{B}(t)\};\\
&S_{1,\pm}(a_i) := \{B(t) \setminus \hat{A}(t) : B(t) \in \widetilde{B}(t),\\
&\ a_i \text{ is a left endpoint } (-) \text{ or}\\
&\text{a right endpoint } (+) \text{ of } B(t) \setminus \hat{A}(t)\};\\
&S_{\sigma,\pm}(a_i) := \{A(t) \cap B(t) : A(t) \in \widetilde{A}(t), B(t) \in \widetilde{B}(t),\\
&\ a_i \text{ is a left endpoint } (-) \text{ or}\\
&\ \text{a right endpoint } (+) \ of \ A(t) \cap B(t)\}.
\end{aligned} \tag{14}$$

Similar definitions are used for $S_{0,\pm}(b_j), S_{1,\pm}(b_j), S_{\sigma,\pm}(\beta_j)$.

Preparatory to the variational analysis of our problem, we introduce some notation concerning different types of variations.

For $E \in S_{k,\pm}(a_i)$ or $E \in S_{k,\pm}(b_j)$, where $k \in \{0,1,\sigma\}$, we define

$$\begin{aligned}
&\delta_{a_i} \int_E f(t,s,x(s),u(t,s))\, ds =\\
&= (-1)^{k_{(1)}} f(t, a_i(t), x(a_i(t)), k_{(2)})\, \delta a_i(t);\\
&\delta_{b_j} \int_E f(t,s,x(s),u(t,s))\, ds =\\
&= (-1)^{k_{(1)}} f(t, b_j(t), x(b_j(t)), k_{(3)})\, \delta b_j(t);\\
&k_{(1)} = 1 \ \ for \ \ subscript \ \ "+" \ \ in \ \ S_{k,\pm};\\
&k_{(1)} = 0 \ \ for \ \ subscript \ \ "-" \ \ in \ \ S_{k,\pm};\\
&k_{(2)} = k \ ,\ if \ \ k \in \{0,1\},\ k_{(2)} = \sigma(t, a_i(t)) \ \ if \ \ k = \sigma;\\
&k_{(3)} = k \ ,\ if \ \ k \in \{0,1\},\ k_{(3)} = \sigma(t, b_j(t)) \ \ if \ \ k = \sigma.
\end{aligned} \tag{15}$$

Then the variation of the state trajectory, under a variation of the multi-functions that define a control policy, satisfies

$$\begin{aligned}
\delta x(t) = &\int_0^t f_x(t,s,x(s),u(t,s))\, \delta x(s)\, ds +\\
&+ \sum_{i,k} \sum_{E \in S_{k,\pm}(a_i)} \delta_{a_i} \int_E f(t,s,x(s),u(t,s))\, ds +\\
&+ \sum_{j,k} \sum_{E \in S_{k,\pm}(b_j)} \delta_{b_j} \int_E f(t,s,x(s),u(t,s))\, ds.
\end{aligned} \tag{16}$$

In view of the definitions of δ_{a_i} and δ_{b_j}, the last equation can be written in compact form as

$$\begin{aligned}
\delta x(t) = &\int_0^t f_x(t,s,x(s),u(t,s))\, \delta x(s)\, ds +\\
&+ \sum_i \varphi_i(t, a_i(t), x(a_i(t)))\, \delta a_i(t) +\\
&+ \sum_j \psi_j(t, b_j(t), x(b_j(t)))\, \delta b_j(t).
\end{aligned} \tag{17}$$

In an analogous way, the variation of the cost functional J can be expressed as

$$\begin{aligned}
\delta J = &\int_0^T \int_0^T \widetilde{F}_{x_2}(t,s,x(t),x(s),\widetilde{u}(t,s))\, \delta x(s)\, ds\, dt +\\
&+ \int_0^T \left\{ \sum_i \Phi_i(t, x(t), a_i(t), x(a_i(t)))\, \delta a_i(t) + \right.\\
&\left. + \sum_j \Psi_j(t, x(t), b_j(t), x(b_j(t)))\, \delta b_j(t) \right\} dt.
\end{aligned} \tag{18}$$

The functions $\widetilde{F}$, Φ_i, Ψ_j can be expressed in terms of the functions F_2, F_1, F_0, f, and the terms that appear in the definitions of the operators δ_{a_i} and δ_{b_j}; in this paper, we omit the explicit expressions for these functions, since they are very long and complicated, and they do not contribute to a conceptual understanding of the remaining of the variational analysis. The subscript x_2 denotes partial differentiation with respect to the second slot that contains x among the variables displayed inside $\widetilde{F}$. The symbol $\widetilde{u}$ denotes the symmetric extension of u, i.e.

$$\widetilde{u}(t,s) = u(t,s) \text{ if } t \ge s, \ \widetilde{u}(t,s) = u(s,t) \text{ if } t \le s. \qquad (19)$$

The above integral equation for δx is linear in δx, and therefore it possesses a resolvent kernel $R(t,s)$. Consequently, we define a co-state function $q(t)$ by

$$\begin{aligned} q(t) &= \int_0^T \widetilde{F}_{x_2}(\tau,t,\cdots)\,d\tau + \\ &+ \int_0^T \int_t^T \widetilde{F}_{x_2}(\tau,s,\cdots) R(s,t)\,d\tau\,ds. \end{aligned} \qquad (20)$$

Then, by using the duality theory for Volterra integral equations, we conclude that $q(t)$ satisfies

$$\begin{aligned} q(t) &= \int_0^T \widetilde{F}_{x_2}(\tau,t,\cdots)\,d\tau + \\ &+ \int_t^T q(s) f_x(s,t,x(t),u(s,t))\,ds. \end{aligned} \qquad (21)$$

This leads to the definition of the Hamiltonian-Volterra functional

$$\begin{aligned} &\mathbf{H}(t,x_2,x(\cdot),u(\cdot,t),q(\cdot)) := \\ &= \int_0^T \widetilde{F}(\tau,t,x(\tau),x_2,\widetilde{u}(t,\tau))\,d\tau + \\ &+ \int_s^T q(s) f(s,t,x_2,u(s,t))\,ds \end{aligned} \qquad (22)$$

so that the co-state q satisfies

$$q(t) = \frac{\partial}{\partial x_2}\mathbf{H}(t,x_2,x(\cdot),u(\cdot,t),q(\cdot)). \qquad (23)$$

Thus, in our problem, the Hamiltonian is a functional (rather than a function) in both the state and the co-state, whereas in the standard Volterra control problem the Hamiltonian is a functional of the co-state but a function in the state. Because, for our problem, the Hamiltonian has memory with respect to both state and co-state, we have termed this phenomenon "amplified memory effect".

Now, in both controlled ODEs and the classical case of control of Volterra integral equations, the increments of the Hamiltonian can be used to evaluate the variation of the cost functional. However, for the problem of this paper, a new set of Hamiltonians (utilizing, nevertheless, the same co-state) is needed to evaluate the variation of the cost functional. We have termed this phenomenon "multiplicity of Hamiltonians".

The previously obtained form of the variation of J leads us to define the Hamiltonians

$$\begin{aligned} \mathbf{K}_i &:= q(t)\varphi_i(t,\cdots) + \Phi_i(t,\cdots), \\ \mathbf{L}_j &:= q(t)\psi_j(t,\cdots) + \Psi_j(t,\cdots) \end{aligned}. \qquad (24)$$

With these definitions, the variation of J can be expressed as

$$\delta J = \sum_i \int_0^T \mathbf{K}_i\, \delta a_i(t)\,dt + \sum_j \int_0^T \mathbf{L}_j\, \delta b_j(t)\,dt \qquad (25)$$

and a necessary condition for optimality can be given in variational form as

$$\delta J \ge 0 \qquad (26)$$

for all admissible variations $\{\delta a_i(\cdot)\}$, $\{\delta b_j(\cdot)\}$.

We note that the Hamiltonian equations in our case are not merely Volterra integral equations in reverse time, but rather a species of general functional equations (functional-Volterra equations), because of the presence of terms like $x(a_i(t))$, $x(b_j(t))$ in the Hamiltonians.

Other, stronger, form of the optimality conditions seem possible, but the technical details will be the subject of future work. Finally, these results can be extended to more general sets (not just binary) of admissible values of the control.

REFERENCES

Belbas, S. A. (2007). A new method for optimal control of Volterra integral equations, *Applied Math. and Comp.*, **189**, 1902-1915.

Belbas, S. A. (2007a). A reduction method for optimal control of Volterra integral equations, accepted in *Applied Math. and Comp.*

Schmidt, W. H. (1982). Durch Integralgleichungen beschriebene optimale Prozesse in Banachräumen - notwendige Optimalitätsbedingungen, *Z. angew. Mathematik u. Mechanik*, **62**, 65-75.

Vinokurov, V. R. (1969). Optimal control of processes described by integral equations, Parts I, II, and III, *SIAM Journal of Control*, **7**, 324-355.