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Feedforward Control Design for the Inviscid Burger Equation using Formal Power Series and Summation Methods

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Abstract: This article presents a flatness–based approach to the trajectory planning and feedforward control problem for the inviscid Burger equation with and without an additional quadratic nonlinearity. It uses the property of formal power series parameterizability of the underlying partial differential equation and uniform Euler–summability of the resulting power series to derive a parameterization of the system state and the system input in terms of a flat output. The article thereby extends the application of the formal power series approach from parabolic to first–order hyperbolic distributed–parameter systems.

Keywords: Distributed–parameter systems; nonlinear control systems; trajectory planning; feedforward control; formal power series; inviscid Burger equation.

1. INTRODUCTION

A commonly used approach in control theory to achieve good tracking performance is the two–degrees–of–freedom scheme (see, e.g., Horowitz (1963)). The approach separates the feedforward from the feedback part of the control design. The feedforward control design, which is considered in this article, consists of two major tasks: Determine an adequate trajectory for the system output, and calculate — at least approximately — the input that produces the desired system output in case of no disturbances and on the assumption of an exact mathematical model.

In the case of parabolic distributed–parameter systems like the heat equation and diffusion–convection–reaction systems, flatness–based concepts using formal power series have been successfully applied to achieve these targets as described in Laroche et al. (2000); Lynch and Rudolph (2002); Wagner et al. (2004); Meurer (2005). Here, the system state is represented as a formal power series in the spatial variable whose coefficients are functions of time. This allows a parameterization of the system state and input in terms of the flat output and its derivatives with respect to time. Motivated by these results, this article extends this approach to the inviscid Burger equation as a first step towards a general approach to the flatness–based feedforward control of nonlinear hyperbolic systems.

For linear hyperbolic systems, the general approach for flatness–based feedforward control designs consists of using Mikusiński's operational calculus (see Mikusiński (1983); Mikusiński and Boehme (1987)), which leads to solutions in terms of distributed shift operators acting on the flat output, and hence pre– and post–actuations. Using this approach, the feedforward control design has already been applied to the linear wave equation in Fliess et al. (1995); Mounier et al. (1995), the linear heat exchanger in Rudolph (2000), the heavy chain in Petit and Rouchon (2001), and a gantry crane in Thull et al. (2006). However — as opposed to the formal power series approach — this approach cannot be easily extended to nonlinear systems, and is therefore not considered in this contribution.

The inviscid Burger equation is chosen as an application example because it can be rigorously proven that the proposed approach yields the exact solution to the feedforward control problem, at least for certain trajectories. However, it is possible to invert the inviscid Burger equation based on the fact that the solution is constant along characteristic curves as described in Petit et al. (1998). Therefore, an example from fluid dynamics is considered, where a quadratic term is added to the equation. This creates an example that demonstrates the applicability of the approach to more complex systems, for which straightforward solution methods do not exist, i.e., for which the solution is not constant along characteristic curves.

The article is organized as follows. Section 2 formulates the problem considered in the article. In Section 3 and 4 the formal power series solution and corresponding simulation results are given for the classical as well as the modified inviscid Burger equation. Section 5 summarizes the results and presents a brief outlook on future research activities.

2. PROBLEM FORMULATION

Consider the partial differential equation

$$x_t(z,t) = x(z,t)x_z(z,t) + \alpha[x(z,t)]^2 \tag{1}$$

with $\alpha \in \mathbb{R}$ and initial and boundary conditions

$$x(z,t^-) = x_0(z), \tag{2}$$
$$x(1,t) = u(t), \tag{3}$$

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where t^- is the point of time at which a change in the input is necessary to produce a change in the output at the time $t = 0$. Therefore, t^- depends on the delay present in the system. Furthermore, $u(t)$ is the system input and

$$y(t) = x(0, t) \tag{4}$$

serves as the system output. The systen (1)–(4) describes the velocity distribution of particles in a one dimensional tube, where the individual particles do not interact with each other, but are subject to a force proportional to the square of their velocity. For $\alpha = 0$, it is often used in fluid dynamics as a simple model for describing shock waves.

The control objective considered in the following is to steer the output $y(t)$ along a desired trajectory $y_d(t)$. For this, we focus on desired trajectories that produce a setpoint change of the output from an initial to a final steady state. To achieve this, the simplest choice is given by

$$y_d(t) = \begin{cases} y_0 & , \quad t < 0 \\ y_0 + t & , \quad 0 \le t \le 1 \\ y_0 + 1 & , \quad 1 < t \end{cases} \tag{5}$$

for a rising trajectory, and

$$y_d(t) = \begin{cases} y_0 & , \quad t < 0 \\ y_0 - t & , \quad 0 \le t \le 1 \\ y_0 - 1 & , \quad 1 < t \end{cases} \tag{6}$$

for a falling trajectory, i.e., a polynomial of degree one during the setpoint change. However, the approach may be applied to any other kind of trajectory planning problems.

3. FEEDFORWARD CONTROL FOR THE CLASSICAL INVISCID BURGER EQUATION

In a first step, we consider the case $\alpha = 0$. This represents the classical inviscid Burger equation for which the exact solution is known.

3.1 Exact Solution

To determine the exact solution, the characteristic curves of (1)–(4) are required. The differential equation determining the characteristic curves reads as

$$\det \begin{pmatrix} 1 & -x(z,t) \\ \mathrm{d}t & \mathrm{d}z \end{pmatrix} = 0. \tag{7}$$

Transforming (1) into the characteristic coordinates

$$\zeta = z, \tag{8}$$

$$\tau = t + \frac{1}{x(z,t)} z, \tag{9}$$

yields the differential equation

$$\chi_\zeta(\zeta, \tau) = 0. \tag{10}$$

In consequence, the solution is constant along characteristic curves, which — in view of (9) — are given by

$$\gamma_{y_d,t_0} \;:\; t(z, t_0) = -\frac{1}{y_d(t_0)} z + t_0. \tag{11}$$

For simple trajectories like the ones given in (5) and (6), the exact solution can thus be determined.

Definition 1. Let $\mathcal{R}_1$ be the region of the (z,t)–plane bounded by the curves $z = 0$, $t = -1/y_0$ and $t = \gamma_{y_d,0}(z)$ as shown in Figure 1. Let $\mathcal{R}_2$ be the region bounded by the curves $z = 0$, $z = 1$, $t = \gamma_{y_d,0}(z)$, and $t = \gamma_{y_d,1}(z)$. Let $\mathcal{R}_3$ be the region bounded by the curves $z = 1$, $t = 1$ and $t = \gamma_{y_d,1}(z)$. Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$.

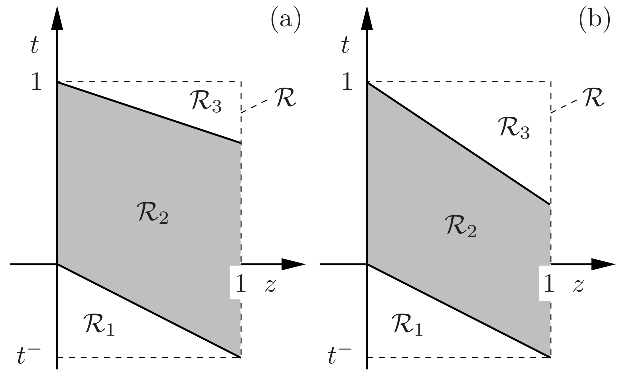

Fig. 1. Characteristic curves of the inviscid Burger equation in the (z,t)–plane passing through $(z,t) = (0,0)$ and $(z,t) = (0,1)$, respectively, and separation of the region $\mathcal{R}$ of interest into the subregions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$. (a) For a trajectory as defined in (5). (b) For a trajectory as defined in (6).

Proposition 2. Let the inviscid Burger equation be given as defined in (1)–(4) with $\alpha = 0$, $t^- = -1/y_0$, and $x_0(z) = y_0$, and let the desired trajectory be defined as given in (5) with $y_0 > 0$. Then, the solution to the feedforward control problem is given by

$$x(z,t) = \begin{cases} y_0 & , \quad (z,t) \in \mathcal{R}_1 \\ \frac{y_0+t}{2} + \sqrt{\left(\frac{y_0+t}{2}\right)^2 + z} & , \quad (z,t) \in \mathcal{R}_2 \\ y_0 + 1 & , \quad (z,t) \in \mathcal{R}_3, \end{cases} \tag{12}$$

with $u(t) = x(1, t)$.

Proof. Note that $x(z,t) = y_0$ and $x(z,t) = y_0 + 1$ are steady state solutions to (1) fulfilling the relevant initial and boundary conditions on $\mathcal{R}_1$ and $\mathcal{R}_3$, respectively, as well as $x(0,t) = y_d(t)$. Differentiating (12) with respect to z and t and inserting the results into (1) completes the proof.

Proposition 3. Let the inviscid Burger equation be given as defined in (1)–(4) with $\alpha = 0$, $t^- = -1/y_0$, and $x_0(z) = y_0$, and let the desired trajectory be defined as given in (6) with $y_0 > \frac{1}{2}\left(1 + \sqrt{5}\right)$. Then, the solution to the feedforward control problem is given by

$$x(z,t) = \begin{cases} y_0 & , \quad (z,t) \in \mathcal{R}_1 \\ \frac{y_0-t}{2} + \sqrt{\left(\frac{y_0-t}{2}\right)^2 - z} & , \quad (z,t) \in \mathcal{R}_2 \\ y_0 - 1 & , \quad (z,t) \in \mathcal{R}_3, \end{cases} \tag{13}$$

with $u(t) = x(1, t)$.

Proof. Analogous to the proof of Proposition 2.

The following section is dedicated to recovering these solutions by applying the formal power series approach and introducing an appropriate summation method.

3.2 Formal Power Series Solution

The formal power series approach consists of representing the system state as a formal power series and deriving a recurrence that describes the calculation of the series coefficients in terms of the flat system output. In order for the approach to work, the system needs to be formal power series parameterizable as defined in Wagner et al. (2004).

Definition 4. Consider a system as given in (1)–(4). Let the state $x(z,t)$ be represented as a formal power series

$$\hat{x}(z,t) = \sum_{n=0}^{\infty} \hat{x}_n(t) \frac{z^n}{n!}. \tag{14}$$

If formally differentiating the representation with respect to z and t and inserting the results into (1)–(4) yields a recurrence of the form

$$\hat{x}_0(t) = y(t), \tag{15}$$

$$\hat{x}_n(t) = \mathfrak{f}_n(\hat{x}_0, ..., \hat{x}_{n-1}, \dot{\hat{x}}_0, ..., \dot{\hat{x}}_{n-1}), \quad n \geq 1, \tag{16}$$

then the system is called *formal power series parameterizable* by the parameterizing output $y(t)$. The formal parameterized input is then given by

$$\hat{u}_d(t) := \hat{x}(1,t) = \sum_{n=0}^{\infty} \hat{x}_n(t) \frac{1}{n!}. \tag{17}$$

In order for the approach to be successful, the solution must be piecewise analytical in z. For hyperbolic systems like the inviscid Burger equation, the solution must then also be piecewise analytical in t, which makes the use of non–analytical Gevrey functions for the desired trajectory as done in Laroche et al. (2000); Lynch and Rudolph (2002) for parabolic distributed–parameter systems impossible and justifies the choice of piecewise polynomial trajectories as given in Section 2.

Since the solution on $\mathcal{R}_1$ and $\mathcal{R}_3$ is given by $x(z,t) = y_0$ and $x(z,t) = y_0 \pm 1$, respectively, and is therefore trivial, this section concentrates on the solution on $\mathcal{R}_2$. Inserting the formal power series approach (14) into the inviscid Burger equation (1)–(4) yields the differential recurrence

$$\hat{x}_0 = y, \tag{18}$$

$$\hat{x}_{n+1} = \frac{1}{\hat{x}_0} \left(\dot{\hat{x}}_n - \sum_{j=1}^{n} \binom{n}{j} \hat{x}_j \hat{x}_{n-j+1} \right), \ n \geq 0. \tag{19}$$

Since the conditions mentioned in Definition 4 are fulfilled, (1)–(4) is formal power series parameterizable by $y(t)$. However, the solution on $\mathcal{R}_2$ cannot be parameterized by $y_d(t)$ for $t < 0$, because the formal power series solution would have to be identical to the solution on $\mathcal{R}_1$, i.e. $\hat{x}(z,t) = y_0$, which is obviously incorrect. Therefore, in order to derive the solution on $\mathcal{R}_2$, we assume that there exists a solution which is analytical in z on $\mathcal{R}_2$ and which can be analytically continued on $\mathcal{R}_1$. To find this solution, the desired trajectory $y_d(t)$ is analytically continued from $[0,1]$ on $[t^-,1]$, producing a function $y_a(t)$. This is possible due to the choice of trajectories taken in Section 2. If such a solution exists, it can be parameterized by $y_a(t)$.

Proposition 5. Let the inviscid Burger equation be given as defined in (1)–(4) with $\alpha = 0$, $t^- = -1/y_0$, and $x_0(z) = y_0 > 0$, and let the desired trajectory be defined as given in (5), which is analytically continued from $[0,1]$ on $[t^-,1]$. Then

(1) the interval I, on which the formal power series $\hat{u}_d(t)$ needs to be calculated is given by

$$I = \left[-\frac{1}{y_0}, \frac{y_0}{y_0+1} \right], \tag{20}$$

(2) the coefficients $\hat{x}_n(t)$ of the formal power series for the analytical continuation $y_a(t)$ are given by

$$\hat{x}_0 = y_0 + t, \tag{21}$$

$$\hat{x}_1 = \frac{1}{y_0 + t}, \tag{22}$$

$$\hat{x}_n = 2 \frac{(-1)^{n+1}(2n-3)!}{(y_0+t)^{2n-1}(n-2)!}, \quad n \geq 2, \tag{23}$$

(3) for $y_0 > 1 + \sqrt{2}$, $\hat{x}(z,t)$ converges uniformly $\forall (z,t) \in \mathcal{R}_1 \cup \mathcal{R}_2$,

(4) for $y_0 < 1+\sqrt{2}$, $\exists\,(z,t) \in \mathcal{R}_2$ for which $\hat{x}(z,t)$ diverges,

(5) for $1 < y_0 < 1 + \sqrt{2}$, $\hat{x}(z,t)$ is not convergent $\forall (z,t) \in \mathcal{R}_1 \cup \mathcal{R}_2$, but $\exists\ q$ for which $\hat{x}(z,t)$ is uniformly (E,q)–summable $\forall (z,t) \in \mathcal{R}_1 \cup \mathcal{R}_2$.

Proof.

(1) The characteristic curves $\gamma_{y_a,\theta}(z)$ as given in (11) intersect the line $z = 1$ at

$$T(\theta) = \theta - \frac{1}{y_0 + \theta}, \tag{24}$$

which evaluates to

$$T(0) = -\frac{1}{y_0}, \quad T(1) = \frac{y_0}{y_0+1} \tag{25}$$

at the points $\theta = 0$ and $\theta = 1$, respectively. Since $\mathrm{d}T/\mathrm{d}\theta > 0\ \forall\ \theta \in (0,1)$, all $\gamma_{y_a,\theta}(z)$ with $\theta \in (0,1)$ intersect the line $z = 0$ in between those two values of T.

(2) The proof of the expression for $\hat{x}_n(t)$ is obtained by inserting (21)–(23) as an induction hypopthesis into (18)–(19) and using a first induction with respect to n and a second induction with respect to j.

(3) Using the convergence criterion of d'Alembert to determine the region of uniform convergence of $\hat{x}(z,t)$ on $\mathcal{R}_1 \cup \mathcal{R}_2$ yields

$$\lim_{n\to\infty} \left| \frac{\hat{x}_{n+1} n! z}{\hat{x}_n (n+1)!} \right| \leq \left| -\frac{4z}{(y_0+t)^2} \right|_{z=1+\delta, t=-\frac{1}{y_0}} \tag{26}$$

$$= \frac{4(1+\delta)}{\left(y_0 - \frac{1}{y_0}\right)^2} =: Q < 1 \tag{27}$$

for $y_0 > 1 + \sqrt{2}$ with adequate positive $\delta = \delta(y_0)$.

(4) Using the divergence criterion of d'Alembert at $z = 1$ and $t = -1/y_0$ yields the claimed result.

(5) Due to Theorem 13 given in Appendix A, it is sufficient to show summability at $z = 1+\delta$ with $\delta > 0$ and $t = -1/y_0$. With $a_n = \hat{x}_n(-1/y_0)/n!$, it follows, that the operator E acting on an arbitrary a_n for $n \geq 1$ is given by

$$E = \frac{a_{n+1}}{a_n} = \left[-4 + \frac{6}{n+1} \right] \left[\frac{1}{y_0 - \frac{1}{y_0}} \right]^2 (1+\delta). \tag{28}$$

Due to Theorem 10 given in Appendix A, the summability of $\sum a_n$ can be deduced from the summability of $\sum a_{n+1}$. Since $y_0 > 1$ the operator E can be bounded by

$$-L := -4 \left(\frac{1}{y_0 - \frac{1}{y_0}} \right)^2 (1+\delta) < E < 0 \tag{29}$$

for all $n \geq 1$, which — using Proposition 12 from Appendix A — completes the proof.

A similar result holds true for decreasing trajectories as shown in the following proposition.

Proposition 6. Let the inviscid Burger equation be given as defined in (1)–(4) with $\alpha = 0$, $t^- = -1/y_0$, and $x_0(z) = y_0$, and let the desired trajectory be defined as given in (6), which is analytically continued from $[0, 1]$ on $[t^-, 1]$, with $y_0 > 2$. Then

(1) the interval I, on which the formal power series $\hat{u}_d(t)$ needs to be calculated, is given by

$$I = \left[-\frac{1}{y_0}, \frac{y_0 - 2}{y_0 - 1} \right], \tag{30}$$

(2) the coefficients $\hat{x}_n(t)$ are given by

$$\hat{x}_0(t) = y_0 - t, \tag{31}$$

$$\hat{x}_1(t) = -\frac{1}{y_0 - t}, \tag{32}$$

$$\hat{x}_n(t) = 2\frac{(2n-3)!}{(y_0 - t)^{2n-1}(n-2)!}, \quad n \geq 2, \tag{33}$$

(3) $\hat{u}_d(t)$ is uniformly convergent everywhere in I.

Proof.

(1) Analogous to the proof of Proposition 5 we find

$$T(\theta) = \theta - \frac{1}{y_0 - \theta}, \tag{34}$$

which evaluates to the claimed values at $\theta = 0$ and $\theta = 1$, respectively. Also, $\mathrm{d}T/\mathrm{d}\theta > 0 \; \forall \, \theta \in (0, 1)$, i.e., all $\gamma_{y_a,\theta}(z)$ with $\theta \in (0, 1)$ intersect the line $z = 0$ in between those two values of T.

(2) Analogous to the proof of Proposition 5.

(3) Using the convergence criterion of d'Alembert, $\hat{x}(z, t)$ can be shown to converge for all $t < y_0 - 2$. Noting

$$\frac{y_0 - 2}{y_0 - 1} < y_0 - 2 \tag{35}$$

for $y_0 > 2$ completes the proof.

The formal power series therefore represents an analytical function on $\mathcal{R}_2$. It remains to show that this analytical function is the correct solution.

Proposition 7. The formal power series solutions as determined in Proposition 5 and 6 and the exact solutions as given in (5) and (6), respectively, are equivalent on $\mathcal{R}_2$.

Proof. The exact solutions are analytical functions on $\mathcal{R}_2$ which can be analytically continued on $\mathcal{R}_1$. Therefore, they can be expanded into their respective Taylor series at $z = 0$ on $\mathcal{R}_1 \cup \mathcal{R}_2$. By induction, it can be shown that the Taylor series coefficients and the power series coefficients are identical. Noting the summability property of the series completes the proof.

Since infinite series cannot in general be exactly evaluated, especially, when the coefficients are determined by a differential recurrence, a finite approximation is needed.

Definition 8. The terms $\hat{x}_\epsilon(z, t)$ and $\hat{u}_{d,\epsilon}(t)$ denote approximations of $\hat{x}(z, t)$ and $\hat{u}_d(t)$, respectively, and are defined by a finite (E, q)–sum using N coefficients and the summation parameter q as given in Definition 9 in Appendix A.

Using the knowledge of the exact solution, the maximum error $u_{e,\max}$ in the approximate input $\hat{u}_{d,\epsilon}(t)$ as compared to the exact input $u_d(t)$ obtained from (12) and (13) for $z = 1$, respectively, can be determined by

$$u_{e,\max} = \max_{t \in I} |\hat{u}_{d,\epsilon}(t) - u_d(t)| \, . \tag{36}$$

The results for a choice of summation parameters N and q as well as a choice of y_0 are given in Table 1. The results show that the summation parameter q should not be chosen too large and that excellent approximations are possible even with a moderate number of coefficients.

Table 1. Input error using the formal power series approach to calculate the input to (1)–(4) with $\alpha = 0$ for trajectories as defined in (5) evaluated on I in steps of Δt .

y_0	N	q	$u_{e,max}$	Δt
2	30	2	$< 10^{-4}$	0.001
2	50	2	$< 10^{-8}$	0.001
2	30	3	$< 10^{-3}$	0.001
2	50	3	$< 10^{-5}$	0.001
1.5	30	3	$< 10^{-3}$	0.001
1.5	40	3	$< 10^{-4}$	0.001

3.3 Simulation Results

For more complex systems and trajectories, for which the solution is not explicitly known, the quality of the approximate input determined by the formal power series approach must be verified by simulation. Therefore, the simulation is performed at first for the classical inviscid Burger equation with $\alpha = 0$. The results are obtained by applying a finite difference method with forward integration using the discretization steps $\Delta z = 0.005$ and $\Delta t = 0.001$ in space and time, respectively. They are given in Figure 2 for a trajectory as given in (5). These and later results are presented in the following way: In part (a), the reference solution for the necessary input $u_d(t)$ as stated in Proposition 2 and 3, respectively, and the finite formal power series solution $\hat{u}_{d,\epsilon}(t)$ using (E, q)–summation are given. Part (b) compares the desired trajectory $y_d(t)$ to the simulation result $y(t)$ which is obtained when using as an input to the system the finite formal power series solution $\hat{u}_{d,\epsilon}(t)$. In part (c), the solution $x(z, t)$ is given, which is obtained by the simulation model when using the finite formal power series solution $\hat{u}_{d,\epsilon}(t)$ as the input. Finally, part (d) shows the maximum error

$$y_{e,\max} := \max_{t \in [0,1]} |y(t) - y_d(t)| \tag{37}$$

of the simulated output $y(t)$, which uses $\hat{u}_{d,\epsilon}(t)$ as the input, compared to the desired output $y_d(t)$ as a function of the summation parameters.

The results displayed in Figure 2b confirm that the formal power series approach paired with Euler–summation and an adequate choice of summation parameters yields a highly satisfactory approximation $\hat{u}_{d,\epsilon}(t)$ for the input. Also, the variation of summation parameters shown in Figure 2d demonstrates the robustness of the solution method with respect to the choice of summation parameters and suggests that the parameter q should be chosen based on the number of coefficients N that can be calculated.

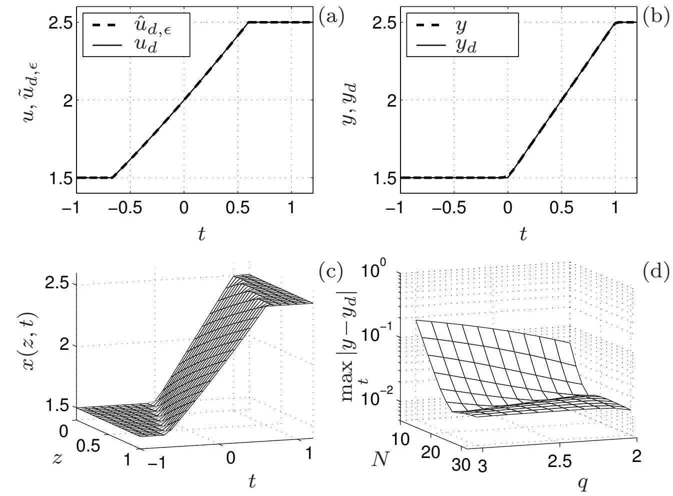

Fig. 2. Numerical simulation results for the inviscid Burger equation as defined in (1)–(4) with $\alpha = 0$ for the desired trajectory as defined in (5) with $y_0 = 1.5$, using the summation parameters $N = 30$ and $q = 3$. (a) Reference input and formal power series solution. (b) Desired and obtained output. (c) Solution on $\mathcal{R}$. (d) Maximum output error.

4. FEEDFORWARD CONTROL FOR THE MODIFIED INVISCID BURGER EQUATION

Since the exact solution to (1)–(4) can be found for trajectories as defined in (5) and (6) for the case $\alpha = 0$ without the presented formal power series approach, this section is dedicated to the case $\alpha \neq 0$. Previous methods that use certain properties of the Burger equation or apply Mikusiński's operational calculus demand that the partial differential equation or the corresponding ordinary differential equation can be solved in closed form. In contrast, the formal power series approach simply requires that the system is parameterizable and that the solution is piecewise analytical.

4.1 Formal Power Series Solution

For $\alpha \neq 0$, the differential recurrence reads as

$$\hat{x}_0 = y_d, \tag{38}$$

$$\hat{x}_{n+1} = \frac{\dot{\hat{x}}_n}{\hat{x}_0} - \sum_{j=1}^{n} \binom{n}{j} \frac{\hat{x}_j}{\hat{x}_0} \left[\hat{x}_{n-j+1} + \alpha \hat{x}_{n-j}\right] - \alpha \hat{x}_n, \tag{39}$$

which — similar to the case $\alpha = 0$ — shows that the system is still formal power series parameterizable. Although an analytical expression for the coefficients $a_n(t) = \hat{x}_n(t)/n!$ cannot be found, the numerical behavior of a finite number of coefficients can be determined for a given $y_d(t)$.

For this, in Figure 3a and 3b, the ratio of subsequent coefficients $a_{n+1}(t)/a_n(t)$ is given for various t as a function of the index n for an increasing and a decreasing trajectory as defined in (5) and (6), respectively. The behavior for the increasing trajectory shows that the ratio tends towards a negative value smaller than -1 for various values of t. This suggests that the series is not convergent for the considered values of t. However, since the ratio is negative, there is a good chance that the series is (E, q)–summable with an appropriate choice of the summation parameter q.

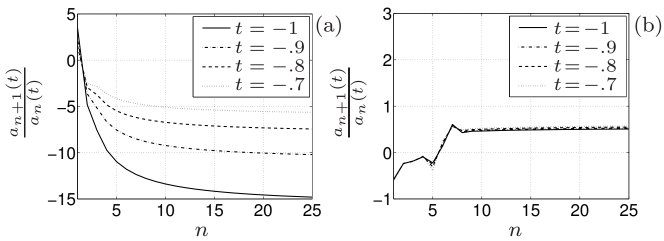

Fig. 3. Numerical evaluation of $a_{n+1}(t)/a_n(t)$ for various t as a function of the index n for the inviscid Burger equation with a quadratic nonlinearity using $\alpha = 0.5$ and $y_0 = 1.5$. (a) For a rising trajectory as defined in (5). (b) For a falling trajectory as defined in (6).

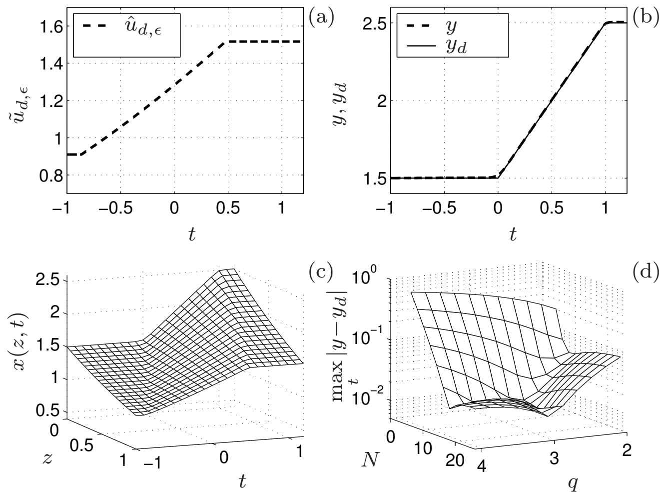

Fig. 4. Numerical simulation results for the modified inviscid Burger equation as defined in (1)–(4) with $\alpha = 0.5$ for the desired trajectory as defined in (5) with $y_0 = 1.5$, using the summation parameters $N = 24$ and $q = 4$. (a) Reference input and formal power series solution. (b) Desired and obtained output. (c) Solution on $\mathcal{R}$. (d) Maximum output error.

For the decreasing trajectory, Figure 3b shows that the ratio tends to a positive value with absolute value smaller than 1. This suggests that the series is convergent and that no special summation method is needed in the solution process; in fact, since the series does not seem to be alternating, a summation method would not be beneficial. This can be verified by simulation.

4.2 Simulation Results

The simulation results are depicted in Figure 4. They demonstrate that the heuristic approach of applying formal power series to the modified inviscid Burger equation produces excellent tracking behavior if the summation parameters are chosen adequately. As oppposed to the classical Burger equation, the steady state solutions are no longer constant, but functions of z, and a change in the amplitude of $u(t)$ of approximately 0.6 — as opposed to 1.0 — produces a change of 1.0 in the output $y(t)$. Figure 4d indicates that q should be chosen larger than 3 and that a minimum number of coefficients N is required.

5. SUMMARY

This article extends existing flatness–based power series methods for the trajectory planning and feedforward control from parabolic to hyperbolic distributed–parameter systems. It demonstrates that polynomials are the adequate choice for the desired trajectory during the setpoint change, and that the notion of convergence must be replaced by the notion of summability to maximize the set of trajectories that can be successfully planned and tracked. For the inviscid Burger equation, the applicability of the method is rigorously proven for a particular choice of trajectories, whereas for the inviscid Burger equation with a quadratic nonlinearity, the method is shown to be a heuristic method that yields excellent results as demonstrated by simulation. Therefore, a first step towards a comprehensive feedforward control design technique for nonlinear hyperbolic distributed–parameter systems has been completed. Future research will be directed towards the application of the approach to second order hyperbolic systems like the linear and nonlinear variants of the torsional rod, the heavy chain, and the heat exchanger.

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Appendix A. SUMMATION METHODS

In the following, only Proposition 12 and its proof constitute new results, whereas all remaining definitions and theorems are taken from Hardy (1964).

Definition 9. Let a series $\sum a_n$ be given and define

$$b_n := \frac{1}{(q+1)^{n+1}} \sum_{i=0}^{n} \binom{n}{i} q^{n-i} a_i. \tag{A.1}$$

If $\sum b_n$ converges towards a limit A for a given value of q, then the series $\sum a_n$ is called (E,q)*–summable* to A. $(E,1)$–summable series are said to be Euler–summable.

Theorem 10. If the series $\sum a_n$ is (E,q)–summable to A, then $\sum a_{n+1}$ is summable to $A - a_0$.

Definition 11. Let a series $\sum a_n$ be given. Then, the E–operator of the series is defined by

$$a_{n+1} = E\{a_n\}. \tag{A.2}$$

i.e., the operator acting on the set of series coefficients that transforms a coefficient a_n into its subsequent coefficient a_{n+1}.

Using the E–operator, the coefficients b_n occurring in the definition of (E,q)–summability can be rewritten as

$$b_n = \frac{(q+E)^n}{(q+1)^{n+1}}\{a_0\}, \tag{A.3}$$

where E^n is interpreted as n successive applications of E.

Proposition 12. Let a series $\sum a_n$ be given. If there exists an $\epsilon > 0$ and an $L > 0$ such that the operator E as given in Definition 11 can be bounded by

$$-L < E \leq 1 - \epsilon, \tag{A.4}$$

then $\exists\, q = q(L)$ for which $\sum a_n$ is (E,q)–summable.

Proof. Using (A.3) we can write

$$b_{n+1} = \frac{q+E}{q+1}\{b_n\}. \tag{A.5}$$

Taking the absolute value on both sides and using the operator norm $\|.\|$ yields

$$|b_{n+1}| = \left|\frac{q+E}{q+1}\{b_n\}\right| \leq \left\|\frac{q+E}{q+1}\right\| |b_n|, \tag{A.6}$$

which can be used to apply the convergence criterion of d'Alembert for the series $\sum b_n$. For $q = 2L$ this yields

$$\left|\frac{b_{n+1}}{b_n}\right| \leq \left\|\frac{q+E}{q+1}\right\| = \left\|\frac{2L+E}{2L+1}\right\| < 1 - \delta \tag{A.7}$$

with $\delta > 0$ depending only on L and ϵ, which completes the proof.

Theorem 13. Let a power series $\sum a_n z^n$ be given that is (E,q)–summable at a point $z = z_1$ for a given value of q. Then, the series is (E,q)–summable on the interval $[0, z_1]$ for the same value of q. If $0 < z_0 < z_1$, then $\sum a_n z^n$ is uniformly (E,q)–summable on $[0, z_0]$.