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Observability of affine discrete-time asynchronous switched systems.

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Abstract: This paper deals with observability of affine discrete-time asynchronous switched systems, that is affine switched systems whose switching times may be different from sampling instants. Two observability notions are studied: pathwise observability and mode observability. We show that there exist some sampling frequencies that preserve pathwise observability if any subsystem is observable. Necessary and sufficient conditions are given for mode observability in the autonomous case. The theoretical results are illustrated through an example.

Keywords: Switched discrete and hybrid systems

1. INTRODUCTION

Hybrid dynamical systems are systems whose behavior is simultaneously described by continuous and discrete variables. Each modality of the discrete variables corresponds to a mode. For a given mode, the hybrid system's evolution is described by a continuous system. Modes transitions obey conditions on the discrete inputs and/or on the states. In this paper, systems whose mode transitions depend on unknown discrete inputs will be referred as switched systems.

Observability of switched systems has already been studied in several papers in the linear and nonlinear cases. In Vidal et al. (2002), the authors deal with the observability of linear discrete-time switched systems. In their contribution, the system is supposed to remain in each mode during a period which is at least equal to twice its joint-observability index. The work exposed in Vidal et al. (2003) is the continuous-time counterpart of the previous article. In Babaali and Egerstedt (2004), the authors study the observability for autonomous and non-autonomous switched linear systems which may switch at each sampling instant. In that paper, the pathwise observability and the forward mode observability are shown to be decidable for autonomous linear discrete-time switched systems. The continuous-time version of this work is presented in Babaali and Pappas (2005). Generally speaking, observability of switched systems is easier to study when considering continuous-time systems rather than discrete-time systems. In the first case, it is only needed to study consecutive output derivatives obtained with the same mode, whereas in the second case, observability criterion is based on the consecutive outputs at different instants, which may be obtained with different modes.

The observability of piecewise linear systems was also characterized in Benali et al. (2004) (continous-time) and Birouche et al. (2006) (discrete-time). In Bemporad et al. (2000), the authors deal with observability of MLD (mixed logic dynamical) systems which are a kind of pathwise affine systems.

Some works were also published concerning observability of nonlinear hybrid systems. In Boutat et al. (2004), sufficient geometrical conditions are given to analyze the observability of continuous-time piecewise systems. This approach is based on observability canonical forms. A study concerning mode and state observability for nonlinear discrete-time switched systems can be found in Kajdan et al. (2007).

In the previously cited papers dealing with the discrete-time case, the authors consider only "pure discrete-time" hybrid systems which switch only at sampling instants. The particularity of the presented work is that we consider some sampled hybrid systems whose switches may occur between two sampling times. Such hybrid systems are said to be asynchronous.

In this paper, we will first present the considered class of hybrid systems. Once the collection of the consecutive outputs on a temporal window is expressed as a function of the state at the beginning of the window, the mode evolution and the consecutive inputs, then pathwise observability and mode observability will be characterized. Finally, the theoretical results will be illustrated through an example.

2. PROBLEM STATEMENT

The following class of switched systems will be considered in this paper:

$$\dot{x}(t) = A_{q(t)}x(t) + B_{q(t)}u(t) + \phi_{q(t)} \tag{1a}$$

$$y_k = C_{q(kT)}x(kT) + D_{q(kT)}u(kT) + \gamma_{q(kT)} \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ denote the state, the input and the output vectors respectively. T is the sampling period and $k \in \mathbb{N}$. $q(t) \in \mathcal{Q} = \{1, \ldots, s\} \subset \mathbb{N}$ is the mode of the system at time instant t. Each element $q \in \mathcal{Q}$ represents a kind of specific dynamic given by the matrices A_q, B_q, C_q, D_q and the vectors ϕ_q and γ_q. A switch between two modes occurs at time t_s if the unknown discrete input vector $d(t)$ varies at time t_s, i.e. $d(t_s^+) \neq d(t_s^-)$. For a given mode, the vectors ϕ_q and γ_q are constant. They may represent the influence of some discrete inputs or they can be used to model a nonlinear system which is linearized using affine functions. In the following, $\mathcal{S}(q) \subset \mathcal{Q} \setminus \{q\}$ will be the subset of modes which can be reached in one switch from a given mode q.

The following assumptions are assumed to be verified:

Assumptions 1.

- The mode $q(t)$ is a piecewise constant, right continuous function.
- The minimum time between two consecutive switching times t_j and t_{j+1} is greater or equal to T (then only one switch may occur at most on any time interval $[kT;(k+1)T[$.
- The input is constant on any time interval $[kT;(k+1)T[$: $u(t) = u_k,\ \forall t \in [kT;(k+1)T[$

Because of right continuity of $q(t)$, a switch is said to occur at time t_s if $q(t_s) \neq q(t_s^-)$. For more clarity, the mode and the state at time kT will be denoted q_k and x_k respectively.

3. SYSTEM EQUATIONS REWRITTEN ON A TEMPORAL WINDOW

If the mode and the inputs are constant over the interval $[t_0;t[$, then the solution of the differential equation (1a) with initial condition $x(t_0)$ is:

$$x(t) = f_{\left(q(t_0),t-t_0\right),u(t_0)}(x(t_0))$$

where

$$f_{(q,t),u}(x) = e^{tA_q}x + \left(\int_0^t e^{(t-\tau)A_q}d\tau\right)(B_q u + \phi_q) \qquad (2)$$

Then, the state x_{k+1} at time $(k+1)T$ can be written as a function of both the previous state x_k at time kT and the path $Q_{[k;k+1[}$ on $[kT;(k+1)T[$:

$$x_{k+1} = f_{Q_{[k;k+1[},u_k}(x_k) \qquad (3)$$

- If no switch occurs on interval $]kT;(k+1)T[$:
$$f_{Q_{[k;k+1[},u_k} = f_{(q_k,T),u_k}$$
- If a switch occurs at time $t_s \in]kT;(k+1)T[$:
$$f_{Q_{[k;k+1[},u_k} = f_{\left(q(t_s),(k+1)T-t_s\right),u_k} \circ f_{\left(q_k,t_s-kT\right),u_k}$$

The path $Q_{[k;\bar{k}[}$ on the time interval $[kT;\bar{k}T[$, $\bar{k} \geq k+1$ is the mode evolution on this interval. It is represented by a tuple which is composed by the different modes on $[kT;\bar{k}T[$, and the corresponding durations for each of these modes on $[kT;\bar{k}T[$. For instance, if switches occur at times t_1, t_2,..., t_N on the interval $]kT;\bar{k}T[$, then:

$$Q_{[k;\bar{k}[} = \left(q_k, t_1 - kT, q(t_1), t_2 - t_1, \ldots, q(t_N), \bar{k}T - t_N\right)$$

where $t_1 - kT > 0$, $t_{j+1} - t_j > 0$ and $\bar{k}T - t_N > 0$.

Using equation (3), the consecutive states $x_{\bar{k}}$, may be expressed as follows

$$x_{\bar{k}} = f_{Q_{[k;\bar{k}[},U_{[k;\bar{k}-1]}}(x_k)$$

where the input sequence $U_{[k;\bar{k}-1]}$, is given by

$$U_{[k;\bar{k}-1]} = \left[u_k^T\ u_{k+1}^T\ \ldots\ u_{\bar{k}-1}^T\right]^T$$

and where the function $f_{Q_{[k;\bar{k}[},U_{[k;\bar{k}-1]}}$, is defined recursively by:

$$f_{Q_{[k;\bar{k}[},U_{[k;\bar{k}-1]}} = f_{Q_{[\bar{k}-1;\bar{k}[},u_{\bar{k}-1}} \circ f_{Q_{[k;\bar{k}-1[},U_{[k;\bar{k}-2]}}$$

For more clarity, let

$$h_{q,u}(x) = C_q x + D_q u + \gamma_q$$

be the observation function. The output collection $Y_{[k;\bar{k}]}$ on the interval $[kT;\bar{k}T]$ may be expressed as a function of the state x_k, the input collection $U_{[k;\bar{k}]}$ and the path $Q_{[k;\bar{k}]}$ on the interval $[kT;\bar{k}T]$:

$$Y_{[k;\bar{k}]} = \left[y_k^T\ y_{k+1}^T\ \ldots\ y_{\bar{k}}^T\right]^T = H_{Q_{[k;\bar{k}]},U_{[k;\bar{k}]}}(x_k)$$

where

$$H_{Q_{[k;\bar{k}]},U_{[k;\bar{k}]}}(x_k) = \left[h_{q_k,u_k}^T(x_k)\ \ldots\ \left(h_{q_{\bar{k}},u_{\bar{k}}} \circ f_{Q_{[k;\bar{k}[},U_{[k;\bar{k}]}}\right)^T(x_k)\right]^T$$

The path $Q_{[k;\bar{k}]}$ on $[kT;\bar{k}T]$ is represented by a tuple. For instance if switches occur at times t_1, t_2,..., t_N on the interval $]kT;\bar{k}T]$, then:

$$Q_{[k;\bar{k}]} = \left(q_k, t_1 - kT, q(t_1), t_2 - t_1, \ldots, q(t_N), \bar{k}T - t_N\right)$$

where $t_1 - kT > 0$, $t_{j+1} - t_j > 0$ and $\bar{k}T - t_N \geq 0$. The main difference with $Q_{[k;\bar{k}[}$ is that $Q_{[k;\bar{k}]}$ takes into account a switch which may occur at time $t_N = \bar{k}T$, and in this case, $\bar{k}T - t_N = 0$.

If no switch occurs during interval $]kT;\bar{k}T]$ the path on the interval $[kT;\bar{k}T]$ is $Q_{[k;\bar{k}T]} = (q_k, (\bar{k}-k)T)$. Then, letting $L = \bar{k} - k$:

$$H_{(q_k,LT),U_{[k;k+L]}}(x) = \Omega_{(q_k,LT)}x + \Gamma_{(q_k,LT)}U_{[k;k+L]} + \Lambda_{(q_k,LT)} \qquad (4)$$

where the observability matrix $\Omega_{(q,LT)}$, the matrix $\Gamma_{(q,LT)}$ and the vector $\Lambda_{(q,LT)}$ are given by:

$$\Omega_{(q,LT)} = \begin{bmatrix} C_q \\ C_q e^{TA_q} \\ \vdots \\ C_q e^{LTA_q} \end{bmatrix}, \Lambda_{(q,LT)} = \begin{bmatrix} \gamma_q \\ C_q\Theta_{(q,T)}\phi_q + \gamma_q \\ \vdots \\ C_q\Theta_{(q,LT)}\phi_q + \gamma_q \end{bmatrix}$$

$$\Gamma_{(q,LT)} = \begin{bmatrix} D_q & 0 & \ldots & 0 & 0 \\ C_q\Xi_{(q,T)}B_q & D_q & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ C_q\Xi_{(q,LT)}B_q & C_q\Xi_{(q,(L-1)T)}B_q & \ldots & C_q\Xi_{(q,T)}B_q & D_q \end{bmatrix}$$

with $\Theta_{(q,t)} = \int_0^t e^{(t-\tau)A_q}d\tau$ and $\Xi_{(q,t)} = \int_0^T e^{(t-\tau)A_q}d\tau$.

Now, we will study what happens if some switches occur at times $t_1, \ldots, t_N$ on interval $]kT;\bar{k}T]$. Let k_j, $j \in \{1, \ldots, N\}$ be the integers such that $t_j \in]k_jT;(k_j+1)T]$

$$H_{Q_{[k;\bar{k}]},U_{[k;\bar{k}]}}(x) = \begin{bmatrix} H_{\left(q_k,(k_1-k)T\right),U_{[k;k_1]}}(x) \\ H_{\left(q(t_1),(k_2-k_1-1)T\right),U_{[k_1+1;k_2]}} \circ f_{Q_{[k;k_1+1[},U_{[k;k_1]}}(x) \\ \vdots \\ H_{\left(q(t_N),(\bar{k}-k_N-1)T\right),U_{[k_N+1;\bar{k}]}} \circ f_{Q_{[k;k_N+1[},U_{[k;k_N]}}(x) \end{bmatrix}$$

Since function f is affine, the general expression of $H_{Q_{[k;\bar{k}]},U_{[k;\bar{k}]}}(x)$, for any possible path $Q_{[k;\bar{k}[}$ is given by:

$$H_{Q_{[k;\bar{k}]},U_{[k;\bar{k}]}}(x) = \Omega_{Q_{[k;\bar{k}]}} x + \Gamma_{Q_{[k;\bar{k}]}} U_{[k;\bar{k}]} + \Lambda_{Q_{[k;\bar{k}]}} \tag{5}$$

4. PATHWISE OBSERVABILITY

We will denote $\mathcal{L}_L$ the set of all possible paths on any time interval with length LT, i.e. the set of every path $Q_{[k;k+L]}$, $k \in \mathbb{N}$.

Definition 1. (Pathwise observability). System (1) is said to be pathwise observable if there exists an integer L such that for any pair of possible states $(x, \bar{x}) \in (\mathbb{R}^n)^2$ and for every path $Q \in \mathcal{L}_L$:

$$x \neq \bar{x} \Longrightarrow H_{Q,U}(x) \neq H_{Q,U}(\bar{x})$$

The smallest integer L is the index of pathwise observability.

This definition means that system (1) is pathwise observable if and only if there exists an integer L such that, if the mode is known on an interval $[k; k+L]$, then the state at time kT is observable. Using the expression of $H_{Q,U}(x)$ given in (5), the following theorem is easily obtained:

Theorem 1. (Pathwise observability). System (1) is pathwise observable if and only if there exists an integer L such that for every path $Q \in \mathcal{L}_L$:

$$\operatorname{rank}(\Omega_Q) = n$$

Then pathwise observability only depends on observability matrices Ω_Q but not on inputs.

The following theorem gives sufficient conditions for pathwise observability of asynchronous affine systems:

Theorem 2. If there exists an integer ν such that

(i) for any mode $q \in \mathcal{Q}$, $\operatorname{rank}\left(\Omega_{(q,\nu T)}\right) = n$,

(ii) any two consecutive switching-times t_j and t_{j+1} are such that $t_{j+1} - t_j \geq (\nu+1)T$,

then, system (1) is pathwise observable, and the index of pathwise observability is lower or equal to 2ν.

Proof. Let consider a path $Q_{[0;2\nu]} \in \mathcal{L}_{2\nu}$ and assume that assumptions (i)-(ii) are verified:

- If no switch occurs on $]0; \nu T]$, then $\operatorname{rank}\left(\Omega_{Q_{[0;\nu]}}\right) = \operatorname{rank}\left(\Omega_{(q_0,\nu)}\right) = n$ and therefore $\operatorname{rank}\left(\Omega_{Q_{[0;2\nu]}}\right) = n$.
- If a switch occurs at time $t_s \in]k_sT; (k_s+1)T] \subset]0; \nu T]$, where $k_s \in \mathbb{N}$, then $k_s + \nu + 1 \leq 2\nu$ and
$$\Omega_{Q_{[0;k_s+\nu+1]}} = \begin{bmatrix} \Omega_{(q_0,k_sT)} \\ \Omega_{\left(q(t_s),\nu T\right)} e^{\left((k_s+1)T - t_s\right)A_{q(t_s)}} e^{t_s A_{q_0}} \end{bmatrix}$$
Since $e^{\left((k_s+1)T-t_s\right)A_{q(t_s)}}$ and $e^{t_sA_{q_0}}$ are square invertible matrices: $\operatorname{rank}\left(\Omega_{\left(q(t_s),\nu T\right)} e^{\left((k_s+1)T-t_s\right)A_{q(t_s)}} e^{t_sA_{q_0}}\right) = n$.
Consequently, $\operatorname{rank}\left(\Omega_{Q_{[0;2\nu]}}\right) = n$.

□

In order to apply theorem 2, the system needs to switch "sufficiently slowly", if compared to the sampling period. Furthermore, the sampling period must preserve observability of any subsystem (i.e. affine system corresponding to a given value q of the mode).

Let us recall the following result whose proof may be found in (Sontag, 1998, p 275):

Theorem 3. An observable continuous-time invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

remains observable provided that the sampling period T is smaller than $\frac{\pi}{|\operatorname{Im}\lambda|}$, for any eigenvalue of A.

This result permits to obtain a sampling period which guarantees observability of (1):

Theorem 4. If any subsystems of the switched system (1) is continuous-time observable, that is:

$$\operatorname{rank}\left(\left[C_q^T \;\; A_q^T C_q^T \;\; \ldots \;\; \left(A_q^{n-1}\right)^T C_q^T\right]^T\right) = n \quad \forall q \in \mathcal{Q}$$

and if there exists a minimum time $\Delta t_{s_{min}} > 0$ between two consecutive switches, then the switched system (1) is pathwise observable if the sampling period is such that:

$$T < \frac{\pi}{|\operatorname{Im}\lambda|} \qquad \text{and} \qquad T \leq \frac{\Delta t_{s_{min}}}{n}$$

for any eigenvalue λ of any matrix A_q.

Proof. If the sampling period is such that $T < \frac{\pi}{|\operatorname{Im}\lambda|}$ for any eigenvalue λ of matrix A_q, then according to theorem 3, observability of any subsystem is preserved, i.e.

$$\operatorname{rank}\left(\Omega_{(q,(n-1)T)}\right) = n, \quad \forall q \in \mathcal{Q}$$

Consequently, according to theorem 2 (with $\nu = n-1$), the system (1) is pathwise observable if furthermore $T \leq \frac{\Delta t_{s_{min}}}{n}$.

□

It can be noticed that sampled pathwise observability has been studied for linear systems with switched measurements in Babaali and Egersted (2004), but with no-switched state evolution.

5. MODE OBSERVABILITY

In the following, we will study mode observability only in the autonomous case, that is for switched systems given by:

$$\dot{x}(t) = A_{q(t)}x(t) + \phi_{q(t)} \tag{6a}$$

$$y_k = C_{q(kT)}x(kT) + \gamma_{q(kT)} \tag{6b}$$

Definition 2. System (6) is said to be mode observable if there exists an integer L such that for any pair of possible states $(x, \bar{x}) \in (\mathbb{R}^n)^2$ and for every pair of paths $\left(Q, \bar{Q}\right) = \left(Q_{[0;L]}, \bar{Q}_{[0;L]}\right) \in \mathcal{L}_L^2$:

$$Q_{[0;1[} \neq \bar{Q}_{[0;1[} \Longrightarrow H_Q(x) \neq H_{\bar{Q}}(\bar{x})$$

The smallest integer L is the index of mode observability.

This definition means that if the system (6) is mode observable with an index L_{MO}, and if two different paths $Q_{[k;\bar{k}[}$ and $\bar{Q}_{[k;\bar{k}[}$ are considered on an interval $[kT; \bar{k}T[$ with $\bar{k} > k$, then the two corresponding output collections on interval $[kT; (\bar{k} + L_{MO} - 1)T]$ will be different.

Let $\Omega_{Q,\bar{Q}} = \left[\Omega_Q \ \Omega_{\bar{Q}}\right]$ and $\Lambda_{Q,\bar{Q}} = \Lambda_Q - \Lambda_{\bar{Q}}$, necessary and sufficient conditions for mode observability are given in the following theorem:

Theorem 5. System (6) is mode observable if and only if there exists an integer L such that for every pair of paths $(Q, \bar{Q}) = (Q_{[0;L]}, \bar{Q}_{[0;L]}) \in \mathcal{L}_L^2$:

$$Q_{[0;1[} \neq \bar{Q}_{[0;1[} \Longrightarrow \text{rank}\left(\Omega_{Q,\bar{Q}}\right) \neq \text{rank}\left(\left[\Omega_{Q,\bar{Q}} \ \Lambda_{Q,\bar{Q}}\right]\right)$$

Proof. $H_Q(x)$ and $H_{\bar{Q}}(\bar{x})$ are given by

$$H_Q(x) = \Omega_Q x + \Lambda_Q \text{ and } H_{\bar{Q}}(\bar{x}) = \Omega_{\bar{Q}}\bar{x} + \Lambda_{\bar{Q}}$$

then

$$H_Q(x) \neq H_Q(\bar{x}), \forall x, \bar{x} \in \mathbb{R}^n \Longleftrightarrow \Lambda_{Q,\bar{Q}} \notin \text{Im}\left(\Omega_{Q,\bar{Q}}\right)$$

Consequently,

$$H_Q(x) \neq H_Q(\bar{x}), \forall x, \bar{x} \in \mathbb{R}^n$$
$$\Longleftrightarrow \text{rank}\left(\Omega_{Q,\bar{Q}}\right) \neq \text{rank}\left(\left[\Omega_{Q,\bar{Q}} \ \Lambda_{Q,\bar{Q}}\right]\right)$$

□

Given an integer L, computing the rank of $\Omega_{Q,\bar{Q}}$ and $\left[\Omega_{Q,\bar{Q}} \ \Lambda_{Q,\bar{Q}}\right]$ for any possible paths $Q, \bar{Q} \in \mathcal{L}_L$ may be difficult in general case. Moreover, an infinite number of values of L must be considered to prove that the system is not mode observable. Consequently, we propose the following theorem for mode observability under stronger conditions:

Theorem 6. Assume that there exists an integer μ, such that:

(i) any two consecutive switching-times t_j and t_{j+1} are such that $t_{j+1} - t_j \geq (2\mu + 1)T$,

(ii) any path $(q, \mu T)$ is observable:

$$x \neq \bar{x} \Rightarrow H_{(q,\mu T)}(x) \neq H_{(q,\mu T)}(\bar{x}),$$

(iii) any two different paths $Q = (q, \mu T)$, and $\bar{Q} = (\bar{q}, \mu T)$ are discernable:

$$q \neq \bar{q} \Rightarrow H_{(q,\mu T)}(x) \neq H_{(\bar{q},\mu T)}(\bar{x}), \ \forall x, \bar{x} \in \mathbb{R}^n.$$

Then, system (6) is mode observable, if and only if

(iv) $\begin{bmatrix} h_q(x) \\ f_P(x) \end{bmatrix} \neq \begin{bmatrix} h_{\bar{q}}(\bar{x}) \\ f_{\bar{P}}(\bar{x}) \end{bmatrix}, \forall x, \bar{x} \in \mathbb{R}^n$

(v) $P_{[0;1[} \neq \bar{\bar{P}}_{[0;1[} \Longrightarrow \forall x, \bar{x} \in \mathbb{R}^n, \begin{bmatrix} h_q(x) \\ f_P(x) \end{bmatrix} \neq \begin{bmatrix} h_{\bar{\bar{q}}}(\bar{x}) \\ f_{\bar{\bar{P}}}(\bar{x}) \end{bmatrix}$

for any paths

$P_{[0;1]} = (q, t, \bar{q}, T - t), \bar{P}_{[0;1]} = (\bar{q}, T), \bar{\bar{P}}_{[0;1]} = (\bar{\bar{q}}, \bar{t}, \bar{q}, T - \bar{t})$ with $\bar{q} \in \mathcal{S}(q) \cap \mathcal{S}(\bar{\bar{q}})$ and $t, \bar{t} \in]0; T]$.

Moreover, if all the previous conditions are verified, the index of mode observability is lower or equal to 3μ.

Proof.
The proof of this theorem is given in Appendix A. □

Let us clarify the meaning of the different assumptions of this theorem. For this, consider two paths $Q_{[k;k+3\mu]}$, $\bar{Q}_{[k;k+3\mu]}$ and two state vectors x_k and $\bar{x}_k$. If $Q_{[k;k+\mu]}$ and $\bar{Q}_{[k;k+\mu]}$ are two paths with no switch, i.e. $Q_{[k;k+\mu]} = (q_k, \mu T)$ and $\bar{Q}_{[k;k+\mu]} = (\bar{q}_k, \mu T)$, then assumption (iii) implies that the outputs $H_{Q_{[k;k+3\mu]}}(x_k)$ and $H_{\bar{Q}_{[k;k+3\mu]}}(\bar{x}_k)$ are different when $q_k \neq \bar{q}_k$. However, this assumption is not sufficient even if the system switches slowly (which corresponds to assumption (i)). If, for instance, $Q_{[k+1;k+3\mu]} = \bar{Q}_{[k+1;k+3\mu]}$, we need that when $Q_{[k;k+1[} \neq \bar{Q}_{[k;k+1[}$ the state evolutions or the outputs obtained with the two paths are different on the interval $[k; k+1[$ that is : $h_{q_k}(x_k) \neq h_{\bar{q}_k}(\bar{x}_k)$ or $f_{Q_{[k;k+1[}}(x_k) \neq f_{\bar{Q}_{[k;k+1[}}(\bar{x}_k)$). This explains the necessity of assumptions (iv) and (v) (for more details, the reader can refer to the subsection A.1 of the proof). Moreover, assumption (ii) ensures that when the evolutions on $[k; k+1[$ are different, then the two output collections on $[k+1; k+\mu+1]$ which are given by $H_{Q_{[k+1;k+\mu+1]}} \circ f_{Q_{[k;k+1[}}(x_k)$ and $H_{Q_{[k+1;k+\mu+1]}} \circ f_{\bar{Q}_{[k;k+1[}}(\bar{x}_k)$ are different. In subsection A.2 of the proof, the assumptions (i)-(v) are proved to be sufficient for mode observability by considering all the possible values for the paths $Q_{[k;k+3\mu]}$ and $\bar{Q}_{[k;k+3\mu]}$.

The following remark may be helpful in order to prove mode observability:

Remark 1. Notice that, by letting $z = f_{(q,t)}(x)$, the conditions (iv) and (v) are equivalent to:

$$\left(h_q \circ f_{(q,t)}^{-1} - h_{\bar{q}} \circ f_{(\bar{q},t)}^{-1}\right)(z) \neq 0$$

and

$$(q, t) \neq (\bar{\bar{q}}, \bar{t}) \Rightarrow \left(h_q \circ f_{(q,t)}^{-1} - h_{\bar{\bar{q}}} \circ f_{(\bar{\bar{q}},\bar{t})}^{-1} \circ f_{(\bar{q},t-\bar{t})}^{-1}\right)(z) \neq 0$$

for any modes $q, \bar{q}, \bar{\bar{q}} \in \mathcal{Q}$ such that $\bar{q} \in \mathcal{S}(q) \cap \mathcal{S}(\bar{\bar{q}})$, for every state $z \in \mathbb{R}^n$, for any $t \in]0; T]$ and for any $\bar{t} \in]0; t]$.

6. EXAMPLE

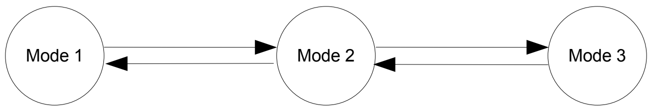

Fig. 1. Allowed switches in example system

Let us consider the autonomous switched system (6) with the three following modes:

$$A_1 = A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}; C_1 = C_2 = C_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\phi_1 = \phi_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \phi_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}; \gamma_1 = \gamma_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \gamma_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Notice that (ii) is satisfied for any integer $\mu \geq 0$ and that the system is pathwise observable for any $T > 0$. The permitted switches are presented in figure 1.

Using (2), the functions f_q, $q \in \{1; \ldots; 3\}$ are given by:

$$f_{(1,t)}(x) = f_{(3,t)}(x) = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} x$$

$$f_{(2,t)}(x) = \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{bmatrix} x + \begin{bmatrix} e^{5t} - 1 \\ e^{3t} - 1 \end{bmatrix}$$

First, we want to know if there exists an integer μ such that assumption (iii) is true. According to (4), the functions $H_{(q,T)}$, $q \in \{1; \ldots; 3\}$ are given by:

$$H_{(1,T)}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e^{2T} & 0 \\ 0 & e^{3T} \end{bmatrix} x, H_{(3,T)}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e^{2T} & 0 \\ 0 & e^{3T} \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$H_{(2,T)}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e^{5T} & 0 \\ 0 & e^{3T} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ e^{5T}-1 \\ e^{3T}-1 \end{bmatrix}$$

For each pair $(q, \bar{q})$ such that $q \neq \bar{q}$, we compute the difference $H_{(q,T)}(x) - H_{(\bar{q},T)}(\bar{x})$:

$$H_{(1,T)}(x) - H_{(2,T)}(\bar{x}) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ e^{2T} & 0 & e^{5T} & 0 \\ 0 & e^{3T} & 0 & e^{3T} \end{bmatrix} \begin{bmatrix} x \\ -\bar{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1-e^{5T} \\ 1-e^{3T} \end{bmatrix}$$

$$H_{(1,T)}(x) - H_{(3,T)}(\bar{x}) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ e^{2T} & 0 & e^{2T} & 0 \\ 0 & e^{3T} & 0 & e^{3T} \end{bmatrix} \begin{bmatrix} x \\ -\bar{x} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$H_{(2,T)}(x) - H_{(3,T)}(\bar{x}) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ e^{5T} & 0 & e^{2T} & 0 \\ 0 & e^{3T} & 0 & e^{3T} \end{bmatrix} \begin{bmatrix} x \\ -\bar{x} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ e^{5T} \\ e^{3T}-1 \end{bmatrix}$$

All these three quantities are different from 0, for any $x, \bar{x} \in \mathbb{R}^n$, and for any $T > 0$, thus assumption (iii) is true for $\mu = 1$.

The following expressions:

$$\left(h_1 \circ f^{-1}_{(1,t)} - h_2 \circ f^{-1}_{(2,t)}\right)(z) = \begin{bmatrix} e^{-2t} - e^{-5t} & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1-e^{-5t} \\ 1-e^{-3t} \end{bmatrix}$$

$$\left(h_2 \circ f^{-1}_{(2,t)} - h_3 \circ f^{-1}_{(3,t)}\right)(z) = \begin{bmatrix} e^{-5t} - e^{-2t} & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} e^{-5t} \\ e^{-3t}-1 \end{bmatrix}$$

are different from 0 for any vector $z \in \mathbb{R}^n$, and for any $t \neq 0$. This means that condition (iv) is true.

Now we want to prove that condition (v) is true.

$$\left(h_1 \circ f^{-1}_{(1,t)} - h_3 \circ f^{-1}_{(3,\bar{t})} \circ f^{-1}_{(2,t-\bar{t})}\right)(z) =$$
$$\begin{bmatrix} e^{-2t}\left(1-e^{3(\bar{t}-t)}\right) & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} e^{-2\bar{t}}\left(1-e^{5(\bar{t}-t)}\right)+1 \\ e^{-3\bar{t}}\left(1-e^{3(\bar{t}-t)}\right) \end{bmatrix} \quad (7)$$

is different from 0 for any t, $\bar{t}$ and for any $z \in \mathbb{R}^n$.

$$\left(h_2 \circ f^{-1}_{(2,t)} - h_2 \circ f^{-1}_{(2,\bar{t})} \circ f^{-1}_{(1,t-\bar{t})}\right)(z) =$$
$$\begin{bmatrix} e^{-5t}\left(1-e^{3(t-\bar{t})}\right) & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} e^{-5t}-e^{-5\bar{t}} \\ e^{-3t}-e^{-3\bar{t}} \end{bmatrix} \quad (8)$$

is different from 0 for any t and $\bar{t}$ such that $t \neq \bar{t}$ and for any $z \in \mathbb{R}^n$. Using the fact that $f_{(3,t)}(x) = f_{(1,t)}(x)$ and $h_3(x) = h_1(x) + \gamma_3$, for any $x \in \mathbb{R}^n$, the expression of $\left(h_q \circ f^{-1}_{(q,t)} - h_{\bar{\bar{q}}} \circ f^{-1}_{(\bar{\bar{q}},\bar{t})} \circ f^{-1}_{(\bar{q},t-\bar{t})}\right)(z)$ can be easily obtained from (7) and (8), and we can prove that:

$$(q,t) \neq (\bar{\bar{q}}, \bar{t}) \Rightarrow \left(h_q \circ f^{-1}_{(q,t)} - h_{\bar{\bar{q}}} \circ f^{-1}_{(\bar{\bar{q}},\bar{t})} \circ f^{-1}_{(\bar{q},t-\bar{t})}\right)(z) \neq 0$$

for any modes $q, \bar{q}, \bar{\bar{q}} \in \mathcal{Q}$ such that $\bar{q} \in \mathcal{S}(q) \cap \mathcal{S}(\bar{\bar{q}})$, for any state $z \in \mathbb{R}^n$, and for any t and $\bar{t}$ verifying $t \geq \bar{t} > 0$

Finally, using theorem 6, it can be deduced that this system is mode observable, for any sampling period T such that $0 < T \leq \frac{\Delta t_{s_{min}}}{2\mu+1} = \frac{\Delta t_{s_{min}}}{3}$. Moreover, the index of mode observability is lower or equal to $3\mu = 3$.

Now assume that the system can switch from mode 1 to mode 3. Then, since $f_{(3,t)}(x) = f_{(1,t)}(x)$:

$$\left(h_1 \circ f^{-1}_{(1,t)} - h_1 \circ f^{-1}_{(1,\bar{t})} \circ f^{-1}_{(3,t-\bar{t})}\right)(z) = 0$$

for any state $z \in \mathbb{R}^n$, for any t, and for any $\bar{t}$. Consequently, condition (v) is not satisfied and the system becomes mode unobservable.

The unobservability of the mode is due to the the impossibility to retrieve the switching-time if the system goes from mode 1 to mode 3. In order to illustrate this, consider two paths $Q_{[0;\infty[}$ and $\bar{Q}_{[0;\infty[}$ such that $Q_{[1;\infty[} = \bar{Q}_{[1;\infty[}$, $Q_{[0;1]} = (1, t, 3, T-t)$ and $\bar{Q}_{[0;1]} = (1, \bar{t}, 3, T-\bar{t})$, where $0 < \bar{t} < t \leq T$. These paths only differ by their first switching-time. Since, $f_{(1,t)}(x) = f_{(3,t)}(x)$, $\forall t \geq 0$, $\forall x \in \mathbb{R}^n$, the corresponding outputs obtained with the same initial state vector x_0 are equal:

$$H_{Q_{[0;+\infty[}}(x_0) = H_{\bar{Q}_{[0;+\infty[}}(x_0) = \begin{bmatrix} h_1(x_0) \\ H_{Q_{[1;+\infty[}} \circ f_{(1,T)}(x_0) \end{bmatrix}$$

7. CONCLUSION

In this paper, necessary and sufficient conditions for mode and pathwise observability of asynchronous switched systems were given. An academic example has shown the applicability of the theoretical results presented in the paper. Influence of inputs on mode observability will be the topic of a future paper. Next works will also consist in the design of observer for asynchronous switched systems.

REFERENCES

M. Babaali and M. Egersted. Pathwise observability through arithmetic progressions and non-pathological sampling. In *Proceeding of the 2004 American Control Conference*, Boston, Massachusetts, July 2004.

M. Babaali and M. Egerstedt. Observability of switched linear systems. In *Hybrid Systems: Computation and Control 2004*, pages 48–63, Philadelphia, PA, USA, 2004. Springer-Verlag.

M. Babaali and G.J. Pappas. Observability of switched linear systems in continuous time. In *Hybrid Systems: Computation and Control 2005*. Springer-Verlag, 2005.

A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Transactions on Automatic Control*, 45 (10):1864–1876, 2000.

A. Benali, D. Boutat, and J.P. Barbot. Une condition algébrique pour l'observabilité d'une classe de systèmes hybrides. In *Conférence Internationale Francophone d'Automatique*, Douz, Tunisia, November 2004. (In french).

A. Birouche, J. Daafouz, and C. Iung. Synthèse d'observateur hybride pour une classe de systèmes linéaires par morceaux. In *Conférence Internationale Francophone d'Automatique*, Bordeaux, France, 2006. (In french).

D. Boutat, A. Benali, and J.-P. Barbot. About the observability of piecewise dynamical systems. In *Symposium* *on Nonlinear Control Systems, Nolcos*, Stuttgart, Germany, 2004.

R. Kajdan, D. Aubry, and F. Kratz. About the observability of nonlinear switching systems. In *7th IFAC Symposium on Nonlinear Control Systems*, Pretoria, South Africa, 2007.

E. D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Textbooks in Applied Mathematics. Springer, New York, second edition, 1998.

R. Vidal, A. Chiuso., and S. Soatto. Observability and identifiability of jump linear systems. In *Proceedings of the 41st IEEE Conference on Decison and Control*, Las Vegas, Nevada USA, 2002.

R. Vidal, A. Chiuso, S. Soatto, and S. Sastry. Observability of linear hybrid systems. In *Hybrid Systems: Computation and Control*, Prague, the Czech Republic, April 2003.

Appendix A. PROOF OF THEOREM 6

In this proof, it will be considered that

$$Q = Q_{[0;\infty[} = (q_0, t_1, q(t_1), t_2 - t_1, \dots)$$
$$\bar{Q} = \bar{Q}_{[0;\infty[} = (\bar{q}_0, \bar{t}_1, \bar{q}(\bar{t}_1), \bar{t}_2 - \bar{t}_1, \dots)$$

Moreover, without loss of generality, t_1 and $\bar{t}_1$ are assumed to be such that $\bar{t}_1 \leq t_1$. k_j and $\bar{k}_j$ will denote the integers such that $t_j \in]k_jT; (k_j+1)T]$ and $\bar{t}_j \in]\bar{k}_jT; (\bar{k}_j+1)T]$

A.1 Necessity of assumptions (iv) and (v)

Assume that $\bar{k}_1 = 0$ and $Q_{[1;\infty[} = \bar{Q}_{[1;\infty[}$. This implies that: $\bar{Q}_{[0;1]} = (\bar{q}_0, \bar{t}_1, \bar{q}(\bar{t}_1), T - \bar{t}_1)$ and $q_1 = \bar{q}(\bar{t}_1)$ where $\bar{t}_1 \in]0; T]$ and $\bar{q}(\bar{t}_1) \in \mathcal{S}(\bar{q}_0)$. There are two cases for the expression of $Q_{[0;1[}$:

- If $k_1 = 0$, $q(t_1) = q_1 = \bar{q}(\bar{t}_1) \in \mathcal{S}(q_0) \cap \mathcal{S}(\bar{q}_0)$ and $Q_{[0;1]} = (q_0, t_1, \bar{q}(\bar{t}_1), T - t_1)$.
- If $k_1 \geq 1$, $q_0 = q_1 = \bar{q}(\bar{t}_1) \in \mathcal{S}(\bar{q}_0)$ and $Q_{[0;1]} = (\bar{q}(\bar{t}_1), T)$.

Since,

$$H_Q(x) = H_{\bar{Q}}(\bar{x})$$
$$\Longrightarrow \begin{cases} h_{q_0}(x) = h_{\bar{q}_0}(\bar{x}) \\ H_{Q_{[1;+\infty[}} \circ f_{Q_{[0;1[}}(x) = H_{Q_{[1;+\infty[}} \circ f_{\bar{Q}_{[0;1[}}(\bar{x}) \end{cases}$$
$$\Longrightarrow \begin{cases} h_{q_0}(x) = h_{\bar{q}_0}(\bar{x}) \\ f_{Q_{[0;1[}}(x) = f_{\bar{Q}_{[0;1[}}(\bar{x}) \end{cases}$$

it can be deduced by considering the two possible expressions for $Q_{[0;1[}$ that mode observability implies assumptions (iv) and (v).

A.2 Sufficiency of assumptions (iv) and (v)

The values of k_1 and $\bar{k}_1$ can be described using three cases:

(1) $\bar{k}_1 \geq \mu$,
(2) $k_1 - \bar{k}_1 \geq \mu + 1$ and $\bar{k}_1 < \mu$,
(3) $k_1 - \bar{k}_1 \leq \mu$ and $\bar{k}_1 < \mu$

We will prove for each case and under the assumptions of theorem 6, including assumptions (iv) and (v), that:

$$Q_{[0;1[} \neq \bar{Q}_{[0;1[} \Longrightarrow H_{Q_{[0;3\mu]}}(x) \neq H_{\bar{Q}_{[0;3\mu]}}(\bar{x})$$

Case 1

$$\begin{aligned} H_{Q_{[0;3\mu]}}(x) = H_{\bar{Q}_{[0;3\mu]}}(\bar{x}) &\Rightarrow H_{Q_{[0;\mu]}}(x) = H_{\bar{Q}_{[0;\mu]}}(\bar{x}) \\ &\Rightarrow H_{(q_0,\mu)}(x) = H_{(\bar{q}_0,\mu)}(\bar{x}) \\ &\Rightarrow q_0 = \bar{q}_0 \text{ (by (iii))} \end{aligned}$$

Case 2

It can be deduced from (i) that $\bar{k}_1 + \mu + 1 \leq \min\big(k_1, \bar{k}_2, 2\mu\big)$, then:

$$H_{Q_{[0;3\mu]}}(x) = H_{\bar{Q}_{[0;3\mu]}}(\bar{x})$$
$$\Rightarrow H_{Q_{[\bar{k}_1;\bar{k}_1+1+\mu]}}(x) = H_{\bar{Q}_{[\bar{k}_1;\bar{k}_1+1+\mu]}}(\bar{x})$$
$$\Rightarrow \begin{cases} h_{q_0} \circ f_{Q_{[0;\bar{k}_1[}}(x) = h_{\bar{q}_0} \circ f_{\bar{Q}_{[0;\bar{k}_1[}}(\bar{x}) \\ H_{(q_0,\mu)} \circ f_{Q_{[0;\bar{k}_1+1[}}(x) = H_{(\bar{q}(\bar{t}_1),\mu)} \circ f_{\bar{Q}_{[0;\bar{k}_1+1[}}(\bar{x}) \end{cases}$$
$$\Rightarrow \begin{cases} h_{q_0} \circ f_{Q_{[0;\bar{k}_1[}}(x) = h_{\bar{q}_0} \circ f_{\bar{Q}_{[0;\bar{k}_1[}}(\bar{x}) \\ q_0 = \bar{q}(\bar{t}_1) \text{ (by (iii))} \\ f_{Q_{[0;\bar{k}_1+1[}}(x) = f_{\bar{Q}_{[0;\bar{k}_1+1[}}(\bar{x}) \text{ (by (ii))} \end{cases}$$

If $q_0 = \bar{q}(\bar{t}_1)$, then $q_0 \in \mathcal{S}(\bar{q}_0)$ and

$$Q_{[\bar{k}_1;\bar{k}_1+1]} = (q_0, T)$$
$$\bar{Q}_{[\bar{k}_1;\bar{k}_1+1]} = \big(\bar{q}_0, \bar{t}_1 - \bar{k}_1T, q_0, (\bar{k}_1+1)T - \bar{t}_1\big)$$

Consequently, by applying (iv):

$$\begin{bmatrix} h_{q_0} \circ f_{Q_{[0;\bar{k}_1[}}(x) \\ f_{Q_{[0;\bar{k}_1+1[}}(x) \end{bmatrix} \neq \begin{bmatrix} h_{\bar{q}_0} \circ f_{\bar{Q}_{[0;\bar{k}_1[}}(\bar{x}) \\ f_{\bar{Q}_{[0;\bar{k}_1+1[}}(\bar{x}) \end{bmatrix} \forall x, \bar{x} \in \mathbb{R}^n$$

which means that $H_{Q_{[0;3\mu]}}(x) \neq H_{\bar{Q}_{[0;3\mu]}}(\bar{x})$, $\forall x, \bar{x} \in \mathbb{R}^n$.

Case 3

In this case, $k_1 + \mu + 1 \leq \min\big(k_2, \bar{k}_2, 3\mu\big)$, then

$$H_{Q_{[0;3\mu]}}(x) = H_{\bar{Q}_{[0;3\mu]}}(\bar{x})$$
$$\Rightarrow H_{Q_{[k_1;k_1+\mu+1]}}(x) = H_{\bar{Q}_{[k_1;k_1+\mu+1]}}(\bar{x})$$
$$\Rightarrow \begin{cases} h_{q_0} \circ f_{Q_{[0;k_1[}}(x) = h_{\bar{q}_{k_1}} \circ f_{\bar{Q}_{[0;k_1[}}(\bar{x}) \\ H_{(q(t_1),\mu)} \circ f_{Q_{[0;k_1+1[}}(x) = H_{(\bar{q}(\bar{t}_1),\mu)} \circ f_{\bar{Q}_{[0;k_1+1[}}(\bar{x}) \end{cases}$$
$$\Rightarrow \begin{cases} h_{q_0} \circ f_{Q_{[0;k_1[}}(x) = h_{\bar{q}_{k_1}} \circ f_{\bar{Q}_{[0;k_1[}}(\bar{x}) \\ q(t_1) = \bar{q}(\bar{t}_1) \text{ (by (iii))} \\ f_{Q_{[0;k_1+1[}}(x) = f_{\bar{Q}_{[0;k_1+1[}}(\bar{x}) \text{ (by (ii))} \end{cases}$$

where $Q_{[k_1;k_1+1]} = \big(q_0, t_1 - k_1T, q(t_1), (k_1+1)T - t_1\big)$

- If $\bar{k}_1 < k_1$ and $q(t_1) = \bar{q}(\bar{t}_1)$, then $\bar{Q}_{[k_1;k_1+1]} = \big(q(t_1), T\big)$ and by applying (iv):
$$\begin{bmatrix} h_{q_0} \circ f_{Q_{[0;k_1[}}(x) \\ f_{Q_{[0;k_1+1[}}(x) \end{bmatrix} \neq \begin{bmatrix} h_{\bar{q}_{k_1}} \circ f_{\bar{Q}_{[0;k_1[}}(\bar{x}) \\ f_{\bar{Q}_{[0;k_1+1[}}(\bar{x}) \end{bmatrix} \forall x, \bar{x} \in \mathbb{R}^n$$
- If $\bar{k}_1 = k_1$ and $q(t_1) = \bar{q}(\bar{t}_1)$, then $\bar{Q}_{[k_1;k_1+1]} = \big(\bar{q}_0, \bar{t}_1 - k_1T, q(t_1), (k_1+1)T - \bar{t}_1\big)$. Consequently by applying (v):
$$\begin{bmatrix} h_{q_0} \circ f_{Q_{[0;k_1[}}(x) \\ f_{Q_{[0;k_1+1[}}(x) \end{bmatrix} = \begin{bmatrix} h_{\bar{q}_{k_1}} \circ f_{\bar{Q}_{[0;k_1[}}(\bar{x}) \\ f_{\bar{Q}_{[0;k_1+1[}}(\bar{x}) \end{bmatrix}$$
$$\Rightarrow Q_{[k_1;k_1+1[} = \bar{Q}_{[k_1;k_1+1[}$$
$$\Rightarrow \begin{cases} Q_{[0;1[} = \bar{Q}_{[0;1[} & \text{if } k_1 = 0 \\ q_0 = \bar{q}_0 & \text{if } k_1 > 0 \end{cases} \Rightarrow Q_{[0;1[} = \bar{Q}_{[0;1[}$$

Therefore, $Q_{[0;1[} \neq \bar{Q}_{[0;1[} \Longrightarrow H_{Q_{[0;3\nu]}}(x) \neq H_{\bar{Q}_{[0;3\nu]}}(\bar{x})$ in case 3.

□