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Stability Analysis of Discrete LPV Systems Subject to Rate-Bounded Parameters ⋆

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Abstract: This paper considers the stability analysis of the feedback connection of a discrete LTI system and time-varying parameters whose variation intervals and bounds of variation rates are assumed known. To tackle the problem, the robust $\mathcal{D}$-admissibility of uncertain descriptor systems is first analyzed. Based on this result, we derive a necessary and sufficient LMI condition for the existence of a parameter dependent Lyapunov function to ensure the robust stability of the considered LPV system. In view of the infinitely many LMIs involved due to the uncertainty description, three sufficient conditions in finite number of LMIs are derived by means of the vertex separator, the D-G scaling, and the SOS relaxation techniques. Finally, a simple example is used to illustrate the effectiveness of the proposed method.

1. INTRODUCTION

Linear parameter-varying (LPV) systems (Iwasaki and Shibata [2001]; Scherer [2001]) have received considerable attention because of their wide applicability in various fields such as gain scheduling (Rugh and Shamma [2000]) or model predictive control (Kothare et al. [1996]). Hence, there is a strong need for research on stability analysis of LPV systems.

In this paper, we consider the stability analysis problem of a special class of LPV systems described by the feedback connection of a linear time-invariant system and time-varying parameters. Both information of the variation intervals and the bounds for the variation rate of these parameters are assumed available. Our approach is similar to that taken by Iwasaki and Shibata [2001]. The difference is that we focus on discrete-time systems, while they focus on continuous-time systems.

From the considered LPV system, an augmented system is constructed so that the information of variation rate of each parameter can be exploited. The augmented system can be viewed as an uncertain descriptor system. This motivates us to consider the robust $\mathcal{D}$-admissibility analysis of uncertain descriptor systems, which can be reduced to robust admissibility problems for continuous/discrete-time descriptor systems by choosing suitable $\mathcal{D}$-regions. (The robust admissibility problems for continuous-time descriptor systems is studied in Iwasaki and Shibata [2001], and the $\mathcal{D}$-admissibility problem is discussed in Wei and Lee [2007].) Based on an equivalent characterization of robust $\mathcal{D}$-admissibility of the uncertain descriptor system, a sufficient condition in LMIs is first derived to ensure this property. To link with the stability issue of the discrete LPV system, this sufficient condition is shown further to imply the exponential stability of the uncertain descriptor system. This result leads directly to a set of new sufficient condition which implies the exponential stability of the system augmented from the considered discrete LPV system. By relaxing the positive definite requirement of P solved from the new sufficient condition, we show that the new condition is equivalent to the existence of a parameter dependent Lyapunov function for the discrete LPV system, which depends on the parameters in a linear fractional manner. Since the new sufficient condition involves infinitely many LMIs, three sufficient conditions in finite number of LMIs are derived by means of the vertex separator and the D-G scaling, proposed in Iwasaki and Shibata [2001], and the SOS relaxation, proposed in Scherer [2006], respectively. Finally, a simple example is used to compare our results with those from related studies (Amato [2006]; Daafouz and Bernussou [2001]; Oliveira et al. [1999]).

The following notations are used in the sequel. $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{S}^n$ denotes the set of symmetric matrices of dimensions $n \times n$. For a subset $\mathcal{D}$ in the complex plane, $\mathcal{D}^c$ denotes the complement of $\mathcal{D}$. For a matrix D, its transpose is denoted as D^T and, when it is full-column rank, $D^\dagger$ is used to denote any left inverse of it. For matrices M and N having the same number of columns, $[M;\ N]$ is used to mean $[M^T\ N^T]^T$. Finally, the symbol $\otimes$ denotes the Kronecker product between two matrices.

2. ROBUST $\mathcal{D}$-ADMISSIBILITY FOR RECTANGULAR DESCRIPTOR SYSTEMS

In this section, the robust $\mathcal{D}$-admissibility of rectangular descriptor systems is analyzed. The result will be applied in the next section to the stability analysis problem of

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discrete LPV systems subject to available parameter variation bound information. Consider a rectangular descriptor system

$$\begin{aligned} \delta x &= Ax + \hat{B}\hat{\xi} \\ 0 &= Cx + \hat{D}\hat{\xi} \end{aligned} \tag{1}$$

with

$$\hat{B} = \begin{bmatrix} B & 0 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D & J \end{bmatrix}$$

where x, $\hat{\xi}$ are descriptor variables, A, B, C, D are given real matrices, $\hat{D}$ is rectangular with J a real constant parameter matrix belonging to a known compact set $\mathbf{J}$, and each element of $\mathbf{J}$ is assumed to have full column rank. In system (1), $\delta x = \dot{x}(t)$ when the system dynamics is continuous and $\delta x = x(k+1)$ when the system dynamics is discrete, respectively.

When system (1) is solvable at some initial condition $[x(0_-);\ \hat{\xi}(0_-)]$, usually the solution is desired to be unique, impulse free, and stable. We briefly call system (1) *admissible* if all solutions of the system possess the desired properties, see Wei and Lee [2007] for a precise definition.

To address the robust $\mathcal{D}$-admissibility problem of system (1), let $\mathcal{D}$ be a specified region in the complex plane described by

$$\mathcal{D} \triangleq \left\{ \lambda \in \mathbb{C} : \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \begin{bmatrix} r & s \\ \bar{s} & q \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} < 0 \right\} \tag{2}$$

where $r \in \mathbb{R}$, $s \in \mathbb{C}$, $q \in \mathbb{R}$ are given scalars. The two most important $\mathcal{D}$-regions are the open left half plane ($r = 0$, $s = 1$, $q = 0$) and the open unit disk ($r = 1$, $s = 0$, $q = -1$), which are the stable regions for continuous/discrete-time systems, respectively. Therefore, in the sequel, we assume $r \geq 0$. In fact, under this assumption, $\mathcal{D}$ is a convex region and can represent any open half plane and any open disk in the complex plane by choosing r, s, and q appropriately. Based on Definition 3 of Wei and Lee [2007], we have the following definition.

Definition 1. (Robust $\mathcal{D}$-admissibility). The uncertain descriptor system (1) is said to be robustly $\mathcal{D}$-admissible if the following conditions are met:

a) $[D\ J]$ has full column rank for all $J \in \mathbf{J}$.
b) For each $J \in \mathbf{J}$, the unobservable hidden modes of $(\mathcal{F}, \mathcal{H})$ lie in the $\mathcal{D}$-region (2), where

$$\mathcal{F} \triangleq A - \hat{B}\hat{D}^{\dagger}C, \qquad \mathcal{H} \triangleq \left(I - \hat{D}\hat{D}^{\dagger}\right) C.$$

Note that condition a) ensures the existence of the left inverse $\hat{D}^{\dagger}$ of $\hat{D}$. The left inverse of $\hat{D}$ is not unique unless $\hat{D}$ is square. Proposition 1 of Wei and Lee [2007] proves that the unobservable hidden modes of $(\mathcal{F}, \mathcal{H})$ do not vary with the choice of $\hat{D}^{\dagger}$. The following lemma provides another equivalent characterization of robust $\mathcal{D}$-admissibility of system (1), whose proof is omitted for brevity.

Lemma 1. Consider the uncertain descriptor system (1). Let the $\mathcal{D}$-region be given by (2) and define

$$\mathcal{M}_\lambda \triangleq \begin{bmatrix} A - \lambda I & \hat{B} \\ C & \hat{D} \end{bmatrix}$$

where λ is a complex scalar. The uncertain descriptor system (1) is robustly $\mathcal{D}$-admissible if and only if, for all $J \in \mathbf{J}$, $\hat{D}$ is full column rank and, moreover, $\mathcal{M}_\lambda$ has full column rank for all $\lambda \in \mathcal{D}^c$.

Next, a sufficient condition is proposed to ensure the robust $\mathcal{D}$-admissibility of system (1).

Proposition 1. Let the $\mathcal{D}$-region be given by (2) with $r \geq 0$. The uncertain descriptor system (1) is robustly $\mathcal{D}$-admissible if there exist a positive definite matrix P and a symmetric matrix Θ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \left(\begin{bmatrix} r & s \\ \bar{s} & q \end{bmatrix} \otimes P \right) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \Theta \begin{bmatrix} C & D \end{bmatrix} < 0 \tag{3}$$

and

$$J^T \Theta J \geq 0 \tag{4}$$

hold for all $J \in \mathbf{J}$.

Proof. The (2,2)-block in (3) reads

$$rB^T PB + D^T \Theta D < 0$$

which implies $D^T \Theta D < 0$ due to $r \geq 0$ and $P > 0$. It is straightforward to verify that

$$D^T \Theta D < 0 \quad \text{and} \quad J^T \Theta J \geq 0$$

imply $[D\ J]$ has full column rank. Now, suppose $[D\ J]$ has full column rank for all $J \in \mathbf{J}$ and system (2) is not robustly $\mathcal{D}$-admissible. Then, by Lemma 1, there exist a scalar $\lambda \in \mathcal{D}^c$ and a parameter $J \in \mathbf{J}$ such that the matrix

$$\begin{bmatrix} A - \lambda I & B & 0 \\ C & D & J \end{bmatrix}$$

does not have full column rank. So, there exists a nonzero vector $[x;\ w;\ z]$ such that

$$(A - \lambda I)x + Bw = 0 \tag{5}$$

$$Cx + Dw + Jz = 0. \tag{6}$$

Note that x is nonzero. Otherwise, since $[D\ J]$ has full column rank, $[w;\ z]$ is zero and get a contradiction. Post- and pre-multiplying (3) by $[x;\ w]$ and its transpose, respectively, we obtain

$$\Phi + (Cx + Dw)^T \Theta (Cx + Dw) < 0 \tag{7}$$

where

$$\Phi = \begin{bmatrix} Ax + Bw \\ x \end{bmatrix}^T \left(\begin{bmatrix} r & s \\ \bar{s} & q \end{bmatrix} \otimes P \right) \begin{bmatrix} Ax + Bw \\ x \end{bmatrix}.$$

Inequality (7) implies $\Phi < 0$ due to (4), (6). With (5), $\Phi < 0$ can be written as

$$\begin{aligned} &\left(\begin{bmatrix} \lambda \\ 1 \end{bmatrix} \otimes x \right)^T \left(\begin{bmatrix} r & s \\ \bar{s} & q \end{bmatrix} \otimes P \right) \left(\begin{bmatrix} \lambda \\ 1 \end{bmatrix} \otimes x \right) \\ &= \left(\begin{bmatrix} \lambda \\ 1 \end{bmatrix}^T \begin{bmatrix} r & s \\ \bar{s} & q \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \right) \otimes \left(x^T P x \right) < 0 \end{aligned}$$

which contradicts the fact that $\lambda \in \mathcal{D}^c$. □

When there are no uncertainties involved in system (1) (J is absent), the robust $\mathcal{D}$-admissibility problem reduces to the $\mathcal{D}$-admissibility problem. In that case, the sufficient condition in Proposition 1 is also necessary, see Theorem 1 of Wei and Lee [2007].

Next, we show that the sufficient condition in Proposition 1 implies the existence of a quadratic Lyapunov candidate $V(x) = x^T Px$ for system (1). Hence, even if the parameter J is time-varying, the solution of system (1), if any, is proved to be exponentially stable. In fact, the quadratic stability ensures the exponential stability of system (1) against arbitrarily fast parameter variations.

Proposition 2. If there exist a positive definite matrix P and a symmetric matrix Θ such that (3) and $J^T(k)\Theta J(k) \geq 0$ hold for all $J(k) \in \mathbf{J}$, $k \in \mathbb{N}$, then the quadratic Lyapunov candidate $V(x) = x^T Px$ satisfies one of the following conditions:

1) $\dot{V}(x(t)) < 0$ for all $x(t)$ satisfying (1) if $(r, s, q) = (0, 1, 0)$.
2) $\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0$ for all $x(k)$ satisfying (1) if $(r, s, q) = (1, 0, -1)$.

Proof. Below, when we write J, we mean that $J(k)$ for some $k \in \mathbb{N}$. For any (x, w, z) satisfying (1), we have

$$\delta x = Ax + Bw, \quad 0 = Cx + Dw + Jz.$$

Post- and pre-multiplying (3) by $[x;\ w]$ and its transpose, respectively, we obtain (7), which implies $\Phi < 0$ due to (4), (6). Substituting $Ax + Bw = \delta x$ into $\Phi < 0$, the proposition is proved. $\square$

3. MAIN RESULT

In this section, the analysis results for the robust $\mathcal{D}$-admissibility of rectangular descriptor systems will be used to deal with the stability analysis of discrete LPV systems subject to rate-bounded parameters. Consider the discrete-time state-space system

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k) \\ z(k) &= Cx(k) + Dw(k) \\ w(k) &= \Delta(k)z(k) \end{aligned} \tag{8}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times p}$ are given matrices, and $\Delta(k)$ is time-varying and belongs to the following set for all $k \in \mathbb{N}$:

$$\begin{aligned} \boldsymbol{\Delta}(\gamma, \rho) = \{\Delta : \mathbb{N} \to \mathbb{R}^{p\times p} \mid \Delta(k) \in \boldsymbol{\Delta}_\gamma, \\ (\Delta(k+1) - \Delta(k)) \in \boldsymbol{\Delta}_\rho\} \end{aligned} \tag{9}$$

where γ, $\rho \in \mathbb{R}^\ell$ are given vectors with nonnegative entries, and, for $\alpha \triangleq \gamma$ or ρ

$$\boldsymbol{\Delta}_\alpha \triangleq \{\mathrm{diag}(\sigma_1 I_{p_1}, \ldots, \sigma_\ell I_{p_\ell}) \mid \sigma_i \in \mathbb{R},\ |\sigma_i| \leq \alpha_i\}. \tag{10}$$

Note that the bound on $|\Delta_i(k)|$ is given by γ_i, while that on $|\Delta_i(k+1)-\Delta_i(k)|$ is given by ρ_i, where $\Delta_i(k)$ is the ith block on the diagonal of $\Delta(k)$, $1 \leq i \leq \ell$. The dimension of the parameter is $p = \sum_{i=1}^{\ell} p_i$.

Advance the second and the third equations in (8), we obtain

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k) \\ z(k) &= Cx(k) + Dw(k) \\ z(k+1) &= CAx(k) + CBw(k) + Dw(k+1) \\ w(k) &= \Delta(k)z(k) \\ w(k+1) &= \Delta(k+1)z(k+1) \\ &= \Delta_d(k)z(k+1) + \Delta(k)z(k+1) \end{aligned} \tag{11}$$

where $\Delta_d(k) = \Delta(k+1) - \Delta(k)$ is introduced in order to use the variation rate information of the parameter $\Delta(k)$. Below, when we write x, w, z, δx, δw, δz, we mean that $x(k)$, $w(k)$, $z(k)$, $x(k+1)$, $w(k+1)$, $z(k+1)$, respectively. Now, system (11) can be written as an uncertain rectangular descriptor system

$$\begin{bmatrix} \delta x \\ \delta w \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{cc|ccccc} A & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ \hline C & D & 0 & 0 & I & 0 & 0 \\ CA & CB & D & 0 & 0 & I & 0 \\ CA & CB & D & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & \Delta & 0 & 0 \\ 0 & 0 & I & -I & 0 & \Delta & 0 \\ 0 & 0 & 0 & I & 0 & 0 & \Delta_d \end{array}\right] \left[\begin{array}{c} x \\ w \\ \hline \delta w \\ \varphi \\ -z \\ -\delta z \\ -\delta z \end{array}\right] \tag{12}$$

where Δ, Δ_d refer to $\Delta(k)$, $\Delta_d(k)$, respectively. For ease of notation, define

$$\zeta_1 = \begin{bmatrix} x;\ w \end{bmatrix},\ \zeta_2 = \begin{bmatrix} \delta w;\ \varphi;\ -z;\ -\delta z;\ -\delta z \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ I_p & 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & D \\ CA & CB \\ CA & CB \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & 0 \\ D & 0 \\ D & 0 \\ 0 & 0 \\ I & -I \\ 0 & I \end{bmatrix}, \tag{13}$$

$$J(k) = \begin{bmatrix} I;\ \nabla(k) \end{bmatrix},\ \nabla(k) = \mathrm{diag}(\Delta(k), \Delta(k), \Delta_d(k)). \tag{14}$$

Then, (12) can be written as

$$\begin{bmatrix} \delta\zeta_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} & 0 \\ \bar{C} & \bar{D} & J(k) \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}. \tag{15}$$

Note that (15) is in the form of (1). In view of condition 2) of Proposition 2, we have the following result.

Proposition 3. Consider system (15) where $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $J(k)$ are defined as in (13), (14). Define

$$\begin{aligned} \mathbf{J} = \{[I;\ \nabla] \mid \nabla = \mathrm{diag}(\Delta, \Delta, \Delta_d), \\ \Delta \in \boldsymbol{\Delta}_\gamma,\ \Delta_d \in \boldsymbol{\Delta}_\rho\} \end{aligned} \tag{16}$$

where $\boldsymbol{\Delta}_\gamma$, $\boldsymbol{\Delta}_\rho$ are given by (10). System (15) is exponentially stable if there exist a positive definite matrix P and a symmetric matrix Θ such that

$$\begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix}^T \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes P \right) \begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{D}^T \end{bmatrix} \Theta \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} < 0 \tag{17}$$

and

$$J^T(k)\Theta J(k) \geq 0 \tag{18}$$

hold for all $J(k) \in \mathbf{J}$, $k \in \mathbb{N}$.

With the additional Schur stable assumption on A, P need not be imposed positive definite as in Proposition 3 to ensure the exponential stability of system (8). Next, we show that in fact this weaker condition is equivalent to the existence of a parameter dependent Lyapunov matrix for system (8).

Theorem 1. Consider system (8) and let $\boldsymbol{\Delta}(\gamma,\rho)$ be given by (9). Suppose A is Schur stable. Let $P \in \mathbb{S}^{n+p}$ be given. Then

$$A_\Delta^T(k) P_\Delta(k+1) A_\Delta(k) - P_\Delta(k) < 0, \tag{19}$$

$$P_\Delta(k) > 0 \tag{20}$$

for all $\Delta(k) \in \boldsymbol{\Delta}(\gamma,\rho)$, $k \in \mathbb{N}$, where

$$N(k) = (I_p - \Delta(k)D)^{-1}\Delta(k)C,$$

$$A_\Delta(k) = A + BN(k), \quad P_\Delta(k) = \begin{bmatrix} I_n \\ N(k) \end{bmatrix}^T P \begin{bmatrix} I_n \\ N(k) \end{bmatrix}$$

if and only if there exists a symmetric matrix $\Theta \in \mathbb{S}^{6p}$ such that (17), (18) hold for all $J(k) \in \mathbf{J}$, $k \in \mathbb{N}$, where $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $J(k)$, $\mathbf{J}$ are defined as in (13), (14), (16).

Proof. Let $k \in \mathbb{N}$. In view of Lemma 10 of Iwasaki and Shibata [2001], there exists a symmetric matrix Θ such that (17), (18) hold for all $J(k) \in \mathbf{J}$ if and only if

$$\zeta^T \begin{bmatrix} \bar{A} & \bar{B} \\ I_{n+p} & 0 \end{bmatrix}^T \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes P \right) \begin{bmatrix} \bar{A} & \bar{B} \\ I_{n+p} & 0 \end{bmatrix} \zeta < 0 \tag{21}$$

for all nonzero ζ such that

$$\begin{bmatrix} \nabla(k) & -I \end{bmatrix} \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} \zeta = 0 \tag{22}$$

where $\Delta(k) \in \boldsymbol{\Delta}_\gamma$ and $\Delta_d(k) \in \boldsymbol{\Delta}_\rho$ are arbitrary. For notational simplicity, from now on, $\Delta(k)$, $\Delta_d(k)$ will be written as Δ, Δ_d, respectively. Substituting (13), (14) into (22), we have

$$\begin{bmatrix} \Delta & 0 & 0 & -I & 0 & 0 \\ 0 & \Delta & 0 & 0 & -I & 0 \\ 0 & 0 & \Delta_d & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} C & D & 0 & 0 \\ CA & CB & D & 0 \\ CA & CB & D & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix} \zeta$$

$$= \begin{bmatrix} \Delta C & \Delta D - I & 0 & 0 \\ \Delta CA & \Delta CB & \Delta D - I & I \\ \Delta_d CA & \Delta_d CB & \Delta_d D & -I \end{bmatrix} \begin{bmatrix} x \\ w \\ \eta \\ \varphi \end{bmatrix} = 0,$$

which implies

$$w = Nx, \quad N = (I - \Delta D)^{-1}\Delta C$$
$$\eta = (I - \Delta D)^{-1}\Delta C(Ax + Bw) + (I - \Delta D)^{-1}\varphi$$
$$\varphi = \Delta_d C(Ax + Bw) + \Delta_d D\eta.$$

The last two equalities give

$$\eta = (I - \Delta D)^{-1}\Delta C(Ax + Bw) + (I - \Delta D)^{-1}\left(\Delta_d C(Ax + Bw) + \Delta_d D\eta\right)$$

or

$$(I - \bar{\Delta}D)\eta = \bar{\Delta}C(Ax + Bw), \quad \bar{\Delta} \triangleq \Delta + \Delta_d$$

or

$$\eta = \bar{N}(A + BN)x, \quad \bar{N} \triangleq (I - \bar{\Delta}D)^{-1}\bar{\Delta}C.$$

Now, we have

$$\begin{bmatrix} x; & w; & \eta; & \varphi \end{bmatrix} = \begin{bmatrix} I; & N; & \bar{N}(A + BN); & \star \end{bmatrix} x \tag{23}$$

where $\star$ means don't care. Hence, all nonzero ζ satisfying (22) must have the form (23) with nonzero x. In this case, (21) becomes

$$\xi^T \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes P \right) \xi < 0 \tag{24}$$

where

$$\xi = \begin{bmatrix} \bar{A} & \bar{B} \\ I_{n+p} & 0 \end{bmatrix} \zeta = \begin{bmatrix} \begin{bmatrix} I \\ \bar{N} \end{bmatrix} (A + BN) \\ \begin{bmatrix} I \\ N \end{bmatrix} \end{bmatrix} x.$$

Hence, (24) becomes

$$x^T \left((A + BN)^T \begin{bmatrix} I \\ \bar{N} \end{bmatrix}^T P \begin{bmatrix} I \\ \bar{N} \end{bmatrix} (A + BN) - \begin{bmatrix} I \\ N \end{bmatrix}^T P \begin{bmatrix} I \\ N \end{bmatrix} \right) x < 0$$

for all nonzero $x \in \mathbb{R}^n$, which is equivalent to (19) by noting that

$$\bar{\Delta}(k) = \Delta(k) + \Delta_d(k) = \Delta(k+1),$$
$$\bar{N}(k) = (I - \bar{\Delta}D)^{-1}\bar{\Delta}C = N(k+1).$$

So far, we have shown that (19) is true for all $\Delta(k) \in \boldsymbol{\Delta}(\gamma,\rho)$ if and only if (17), (18) hold for some Θ for all $J(k) \in \mathbf{J}$. The rest of the proof is to show $P_\Delta > 0$ holds for all $\Delta \in \boldsymbol{\Delta}(\gamma,\rho)$. To prove this, we only need to show that $P_\Delta > 0$ holds for all $\Delta \in \boldsymbol{\Delta}(\gamma,0)$. Since $0 \in \boldsymbol{\Delta}_\rho$, we can assume (19) is true for all $\Delta \in \boldsymbol{\Delta}(\gamma,0)$. Note that the following facts are true.

1) Since $I - \Delta D$ is nonsingular for all $\Delta \in \boldsymbol{\Delta}(\gamma,0)$, P_Δ is continuous of Δ in $\boldsymbol{\Delta}(\gamma,0)$ and so are its eigenvalues. (One can use (17) to prove $I - \Delta D$ is nonsingular for all $\Delta \in \boldsymbol{\Delta}(\gamma,0)$.)
2) Since A is Schur stable, $P_\Delta > 0$ when $\Delta = 0$.

If we can show that P_Δ is nonsingular for all $\Delta \in \boldsymbol{\Delta}(\gamma,0)$, then, in view of facts 1), 2), P_Δ is positive definite for all $\Delta \in \boldsymbol{\Delta}(\gamma,0)$, and the proof is done. Suppose P_Δ is not nonsingular for all $\Delta \in \boldsymbol{\Delta}(\gamma,0)$. Then by facts 1), 2), there exists $\Delta_0 \in \boldsymbol{\Delta}(\gamma,0)$ such that the eigenvalues of P_{Δ_0} are $0, \lambda_1, \ldots, \lambda_{n-1}$ where $\lambda_1, \ldots, \lambda_{n-1}$ are positive real scalars. Let z be the eigenvector of P_{Δ_0} corresponding to the eigenvalue 0. Now, we have

$$z^T \left(A_{\Delta_0}^T P_{\Delta_0} A_{\Delta_0} - P_{\Delta_0} \right) z = z^T A_{\Delta_0}^T P_{\Delta_0} A_{\Delta_0} z \geq 0$$

which contradicts the fact that (19) is true for all $\Delta \in \boldsymbol{\Delta}(\gamma,0)$. □

To verify the LMI condition in Theorem 1, we have to check whether (18) holds for all $J(k) \in \mathbf{J}$. Since $\mathbf{J}$ is an infinite set, this involves infinitely many LMIs. To verify only a finite number of LMIs, we propose three sufficient conditions in the following based on the techniques: the vertex separator, the D-G scaling, and the matrix sum-of-squares relaxation, where the first two techniques are from Iwasaki and Shibata [2001] and the last one is from Scherer [2006]. The proofs are omitted for brevity, see the references for a detailed discussion.

Corollary 1. (Vertex Separator). Consider system (8). Let $\mathbf{\Delta}_\gamma$, $\mathbf{\Delta}_\rho$ be given by (10) and $\mathbf{\Delta}_\gamma^v$, $\mathbf{\Delta}_\rho^v$ be the vertices of $\mathbf{\Delta}_\gamma$, $\mathbf{\Delta}_\rho$, respectively. Define

$$\begin{aligned} \Delta &= \mathrm{diag}(\delta_1 I_{p_1}, \ldots, \delta_\ell I_{p_\ell}), \\ \Delta_d &= \mathrm{diag}(\sigma_1 I_{p_1}, \ldots, \sigma_\ell I_{p_\ell}), \\ \nabla &= \mathrm{diag}(\Delta, \Delta, \Delta_d). \end{aligned} \tag{25}$$

Let $U \in \mathbb{R}^{3p\times 3p}$ be a permutation matrix such that

$$U\nabla U^T = \mathrm{diag}\left(\delta_1 I_{2p_1}, \ldots, \delta_\ell I_{2p_\ell}, \sigma_1 I_{p_1}, \ldots, \sigma_\ell I_{p_\ell}\right).$$

Let

$$\begin{aligned} &Q_{ii} \in \mathbb{S}^{2p_i},\ i = 1, \ldots, \ell \\ &Q_{jj} \in \mathbb{S}^{p_i},\ j = \ell+1, \ldots, 2\ell,\ i = 1, \ldots, \ell \end{aligned}$$

be the matrices on the diagonal of a symmetric matrix $Q \in \mathbb{S}^{3p}$. Let $\mathbf{J}$ be given by (16). If there exist matrices $P \in \mathbb{S}^{n+p}$, $R \in \mathbb{S}^{3p}$, $S \in \mathbb{R}^{3p\times 3p}$, $Q \in \mathbb{S}^{3p}$ such that P and

$$\Theta \triangleq \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}^T \begin{bmatrix} R & S \\ S^T & Q \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$$

satisfy (17),

$$Q_{ii} \le 0,\ i = 1, \ldots, 2\ell,$$

$$\begin{bmatrix} I \\ \nabla \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \nabla \end{bmatrix} \ge 0,\ \nabla = \mathrm{diag}(\Delta, \Delta, \Delta_d)$$

for all $\Delta \in \mathbf{\Delta}_\gamma^v$ and all $\Delta_d \in \mathbf{\Delta}_\rho^v$, then the LMI condition in Theorem 1 holds.

Corollary 2. (D-G Scaling). Consider system (8). Let ∇ be given by (25). Define

$$\begin{aligned} \mathbf{D} &= \left\{D \in \mathbb{S}^{3p} : D\nabla = \nabla D,\ D = D^T > 0\right\}, \\ \mathbf{G} &= \left\{G \in \mathbb{R}^{3p\times 3p} : G\nabla = \nabla G,\ G + G^T = 0\right\}, \\ \Upsilon_\gamma &= \mathrm{diag}(\gamma_1 I_{p_1}, \ldots, \gamma_\ell I_{p_\ell}), \\ \Upsilon_\rho &= \mathrm{diag}(\rho_1 I_{p_1}, \ldots, \rho_\ell I_{p_\ell}), \\ \Upsilon &= \mathrm{diag}(\Upsilon_\gamma, \Upsilon_\gamma, \Upsilon_\rho). \end{aligned}$$

If there exist matrices $P \in \mathbb{S}^{n+p}$, $R \in \mathbf{D}$, $S \in \mathbf{G}$ such that P and

$$\Theta \triangleq \begin{bmatrix} \Upsilon R \Upsilon & S \\ S^T & -R \end{bmatrix}$$

satisfy (17), then the LMI condition in Theorem 1 holds.

Before presenting the third corollary, we need to introduce sum-of-squares (SOS) matrices. A $p\times p$ polynomial matrix $S(\delta)$ in $\delta \in \mathbb{R}^m$ is said to be an SOS if there exists a (not necessarily square and typically tall) polynomial matrix $T(\delta)$ such that $S(\delta) = T^T(\delta)T(\delta)$. A recently developed software, YALMIP, can be used to determine whether a polynomial matrix is SOS, see Löfberg [2004] for more details.

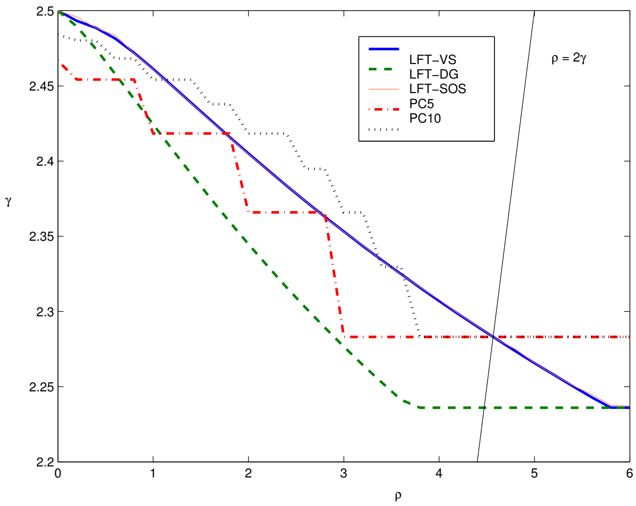

Fig. 1. Stability regions in ρ-γ plane.

Corollary 3. (SOS). Consider system (8). Let ∇ be given by (25). If there exist matrices $P \in \mathbb{S}^{n+p}$, $\Theta \in \mathbb{S}^{6p}$, $V_{ij} \in \mathbb{S}^{(2\ell+1)\times 3p}$, $W_{ij} \in \mathbb{S}^{(2\ell+1)\times 3p}$, $i = 1, 2, \ldots, \ell$, $j = 1, 2$, such that (17) holds, $V_{ij} \ge 0$, $W_{ij} \ge 0$, and

$$\begin{aligned} &\begin{bmatrix} I \\ \nabla \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \nabla \end{bmatrix} + \sum_{i=1}^{\ell} \big(g_{i1}(\delta_i) X_{i1}(\delta, \sigma) \\ &\quad + g_{i2}(\delta_i) X_{i2}(\delta, \sigma) + h_{i1}(\sigma_i) Y_{i1}(\delta, \sigma) + h_{i2}(\sigma_i) Y_{i2}(\delta, \sigma)\big) \end{aligned}$$

is SOS, where

$$\begin{aligned} g_{i1}(\delta_i) &= \delta_i - \gamma_i,\ g_{i2}(\delta_i) = -\delta_i - \gamma_i \\ h_{i1}(\sigma_i) &= \sigma_i - \rho_i,\ h_{i2}(\sigma_i) = -\sigma_i - \rho_i \\ X_{ij}(\delta, \sigma) &= L^T(\delta, \sigma) V_{ij} L(\delta, \sigma) \\ Y_{ij}(\delta, \sigma) &= L^T(\delta, \sigma) W_{ij} L(\delta, \sigma) \\ L(\delta, \sigma) &= \begin{bmatrix} 1 & \delta_1 & \cdots & \delta_\ell & \sigma_1 & \cdots & \sigma_\ell \end{bmatrix}^T \otimes I_{3p} \end{aligned}$$

then the LMI condition in Theorem 1 holds.

4. EXAMPLE

The following example is modified from the one in Oliveira et al. [1999]. Consider

$$A(\Delta(k)) = A + B\Delta(k)C \tag{26}$$

where

$$A = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.03 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ C^T = \begin{bmatrix} 0.8 \\ -0.5 \\ 0 \\ 1 \end{bmatrix}.$$

The problem is to

$$\begin{aligned} &\text{maximize:} && \gamma \\ &\text{subject to:} && (26) \text{ is stable for all } |\Delta(k)| \le \gamma, \\ & && |\Delta(k+1) - \Delta(k)| \le \rho, \end{aligned}$$

where ρ is a given scalar. Five methods together with a bisection algorithm are applied to solve the maximization problem:

LFT-VS	Corollary 1,
LFT-DG	Corollary 2,
LFT-SOS	Corollary 3,
PC5	Theorem 6.5 of Amato [2006] with $\nu = 5$,
PC10	Theorem 6.5 of Amato [2006] with $\nu = 10$.

The simulation results are shown in Fig. 1. Note that for all $\Delta(k)$ satisfies $|\Delta(k)| \leq \gamma$, we have $|\Delta(k+1) - \Delta(k)| \leq 2\gamma$. Hence, it is physically meaningless to select $\rho > 2\gamma$. We also plot the straight line $\rho = 2\gamma$ in Fig. 1. From Fig. 1, we see that LFT-VS is better than LFT-DG, and this fact can be seen from the conditions in Corollary 1 and Corollary 2. LFT-VS and LFT-SOS are almost identical in Fig. 1, but LFT-SOS requires more computational power than LFT-VS. It takes 10 seconds and 75 seconds for LFT-VS and LFT-SOS to solve the maximization problem, respectively. These tests are performed on a personal computer with a Pentium 4 processor (2 GHz) and 768 megabytes of RAM.

Theorem 6.5 of Amato [2006] uses a piecewise constant parameter dependent Lyapunov function (PC-PDLF) to ensure the exponential stability of system (8), which requires to divide the interval to which the parameter $\Delta(k)$ belongs into ν subintervals. Let the ν subintervals be denoted by Γ_i, $1 \leq i \leq \nu$. The PC-PDLF approach ensures the exponential stability of system (8) by letting the system quadratically stable in each subinterval Γ_i with respect to the constant Lyapunov matrix P_i, $1 \leq i \leq \nu$. It can be expected that the larger the number of subintervals, the less conservative the PC-PDLF approach, and the larger the computational burden.

Consider system (8) where $\Delta(k)$ belongs to the set given by (9). Note that ℓ is the number of parameters $\Delta_i(k)$, and p is the dimension of $\Delta(k)$. Denote the interval to which $\Delta_i(k)$ belongs by Ω_i, $1 \leq i \leq \ell$. Suppose Ω_i is partitioned into ν_i subintervals, and let μ_i be the maximum number of subintervals where $\Delta_i(k)$ can jump at the next time step. The following two tables summarize the number of variables and the total row size of LMIs for PC-PDLF and LFT-VS.

Method	Number of variables
PC-PDLF	$(\nu_1 \cdots \nu_\ell)n(n+1)/2$
LFT-VS	$(n+p)(n+p+1)/2 + 6p(6p+1)/2$

Method	Total row size of LMIs
PC-PDLF	$n(\nu_1 \cdots \nu_\ell)(\mu_1 \cdots \mu_\ell) \times 2^\ell$
LFT-VS	$(n+3p) + 3p \times 2^{2\ell} + 3p$

For this example, the number of variables and the total row size of LMIs of LFT-VS are less than those of PC5, but from Fig. 1 we see that LFT-VS performs better than PC5 when $\rho \leq 4.5671$. The maximum obtained by the classical quadratic stability method is 2.2361 which is the value to which both LFT-VS and LFT-DG converge when ρ is large enough.

If $\Delta(k)$ is constant rather than time-varying, then Theorem 2 of Oliveira et al. [1999] can be used to solve the maximization problem, and the resulting maximum is 2.5 which is the same as the values obtained by LFT-VS and LFT-DG at $\rho = 0$. Note that the values obtained by PC5 and PC10 at $\rho = 0$ are less than 2.5. We also use the PC-PDLF method with $\nu = 100$, $\rho = 0$ to test the maximization problem, the resulting maximum is 2.4984 which is still less than 2.5. This example reveals that the PC-PDLF method is conservative than both LFT-VS and LFT-DG at $\rho = 0$ even if a large ν is used. Theorem 3 of Daafouz and Bernussou [2001] is also applied to the maximization problem, but the information of variation rate is not used in their result. The resulting maximum is 2.2831.

5. CONCLUSION

In this paper, we deal with the stability analysis of discrete LPV systems with rate-bounded parameters. Three sufficient LMI conditions are proposed to ensure the existence of a parameter dependent Lyapunov function for the LPV system by means of the vertex separator, the *D-G* scaling, and the SOS relaxation techniques. Applying our results to gain scheduling and model predictive control is a future research topic.

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