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Control of a Pendulum-like System with Multiple Nonlinearities ⋆

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Abstract: This paper addresses a stability analysis problem and synthesis problem for pendulum-like systems with multiple nonlinearities. A method for analysing the Lagrange stability of a pendulum-like system with multiple nonlinearities is proposed. In order to study the synthesis problem, the paper develops an Extended Strict Bounded Real Lemma for unstable systems. A sufficient condition for Lagrange stabilization is proposed in terms of an algebraic Riccati equation with a sign infinite solution. An algorithm is given to solve the algebraic Riccati equation for a Lagrange stabilizing solution and thus gives a control law to stabilize the system in the sense of Lagrange stability.

Keywords: Nonlinear System, Multiple Equilibria, Lagrange Stability, Extended Strict Bounded Real Lemma, Multiple Nonlinearities, Repeated Nonlinearities, Algebraic Riccati Equation

1. INTRODUCTION

Nonlinear control theory has been an extremely active area of research motivated by the fact that in many practical control and estimation problems, the dynamics of the system are affected or even dominated by nonlinear effects. Also, nonlinearity can bring beneficial features to the system. For a large range of system configurations, a variety of research methods have been proposed to address stability analysis and controller design problems. This paper studies pendulum-like systems with multiple nonlinearities. The nonlinearities are restricted to a sector bound.

As described in Leonov et al. (1996), pendulum-like systems are a wide class of systems with infinite equilibria and a generalization of the mathematical pendulum system. Pendulum-like systems have many applications in phase locked loops and oscilliation theory as pointed out by Duan et al. (2007). Frequency-domain criteria for stability properties, such as Lagrange stability, dichotomy and gradient-like stability have been established by Leonov et al. (1996). Also, Duan et al. (2007),Wang et al. (2004),Yang et al. (2004) and Duan et al. (2004) studied the controller design problem and robustness analysis using the LMI methods. However, all of these papers studied systems with only a single nonlinearity. There are practical pendulum-like systems containing more than one nonlinearity. The theory of stability analysis and synthesis of such systems has not been studied to date. This paper will focus on these problems. We extend the existing results on Lagrange stability of pendulum-like systems with a single nonlinearity to systems with multiple nonlinearities. As a special case of nonlinear systems with multiple nonlinearities, systems with repeated nonlinearities have been studied by D'Amoto et al. (2001), Kulkarni and Safonov (2002), and Mancera and Safonov (2005) although these results have not involved Lagrange stability.

In Theorem 13.4 by Leonov et al. (1996), a frequency domain condition for the Lagrange stability of a pendulum-like system with a single nonlinearity has been proposed. This paper extends this result to the systems with multiple nonlinearities. A similar frequency-domain condition for a system with multiple nonlinearities being Lagrange stable is presented. When the multiple pendulum-like nonlinearities reduce to repeated nonlinearities, the conditions reduce to the form of the circle criterion.

The strict Bounded Real Lemma in Petersen et al. (1991) and Chen and Tu (1995) requires that the system matrix A be stable so that it is not applicable to unstable systems. This paper proposes an Extended Strict Bounded Real Lemma which only requires the pair (A, B) to be stabilizable.

The Lagrange stabilizability of nonlinear systems is defined and conditions for Lagrange stabilizability are also proposed. In our approach, the controller design problem is transformed into solving a Riccati equation while ensuring the closed-loop system matrix has $n-1$ eigenvalues with negative real parts. Built on the Extended Strict Bounded Real Lemma, the result is proved to be a sufficient condition for the system to be Lagrange stabilizable.

An algorithm is proposed to solve the Riccati equation arising in our approach. The algorithm ensures that the number of eigenvalues with positive real parts of the solution coincides to the number of eigenvalues with negative real parts of the closed-loop system matrix.

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2. LAGRANGE STABILITY OF PENDULUM-LIKE SYSTEMS WITH MULTIPLE NONLINEARITIES

The pendulum-like system considered here is a nonlinear system with two nonlinearities

$$\begin{aligned} \dot{x} &= Px + q_1\xi + q_2\tilde{\xi} \\ \sigma_1 &= r_1 x \\ \sigma_2 &= r_2 x \\ \xi &= \phi\left(t,\sigma_1\right) \\ \tilde{\xi} &= \tilde{\phi}\left(t,\sigma_2\right) \end{aligned} \tag{1}$$

where P is a constant $n \times n$ matrix, q_1, q_2, r_1^T and r_2^T are n-vectors and the functions $\phi, \tilde{\phi} : \mathcal{R}_+ \times \mathcal{R} \rightarrow \mathcal{R}$ are continuous and locally Lipschitz continuous in the second argument. In the sequel, we will show how this can be generalized to nonlinear systems with any number of nonlinearities. Also we assume

$$\det\ P = 0 \tag{2}$$

and

$$\begin{aligned} \phi\left(t,\sigma_1+\Delta\right) &= \phi\left(t,\sigma_1\right), t \in \mathcal{R}_+, \sigma_1 \in \mathcal{R}, \\ \tilde{\phi}\left(t,\sigma_2+\Delta\right) &= \tilde{\phi}\left(t,\sigma_2\right), t \in \mathcal{R}_+, \sigma_2 \in \mathcal{R}. \end{aligned} \tag{3}$$

We further assume that $\phi\left(\cdot\right), \tilde{\phi}\left(\cdot\right)$ satisfy the sector conditions,

$$\mu_1 \leq \frac{\phi\left(t,\sigma_1\right)}{\sigma_1} \leq \mu_2, t \in \mathcal{R}_+, \sigma_1 \neq 0 \tag{4}$$

$$\tilde{\mu}_1 \leq \frac{\tilde{\phi}\left(t,\sigma_2\right)}{\sigma_2} \leq \tilde{\mu}_2, t \in \mathcal{R}_+, \sigma_2 \neq 0 \tag{5}$$

where μ_1, μ_2, $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are given non-zero constants such that $\mu_1 \leq \mu_2$, $\tilde{\mu}_1 \leq \tilde{\mu}_2$ and we exclude the case of $\mu_1, \mu_2, \tilde{\mu}_1, \tilde{\mu}_2 = 0$. In this section, the Lagrange stability of the system (1) will be discussed.

Definition 1. (Leonov et al. (1996)) If all the solutions of system (1) are bounded, then the system (1) is said to be Lagrange stable.

Note that conditions (4) and (5) can be respectively rewritten as the conditions

$$\left(\mu_1^{-1}\phi\left(t,\sigma_1\right)-\sigma_1\right)^T\left(\mu_2^{-1}\phi\left(t,\sigma_1\right)-\sigma_1\right) \leq 0 \tag{6}$$

$$\left(\tilde{\mu}_1^{-1}\tilde{\phi}\left(t,\sigma_2\right)-\sigma_2\right)^T\left(\tilde{\mu}_2^{-1}\tilde{\phi}\left(t,\sigma_2\right)-\sigma_2\right) \leq 0 \tag{7}$$

To simplify the discussion, we further restrict our discussion to the case when

$$\mu = -\mu_1 = \mu_2, \tilde{\mu} = -\tilde{\mu}_1 = \tilde{\mu}_2. \tag{8}$$

Let $\sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$, $\bar{\xi} = \begin{bmatrix} \xi \\ \tilde{\xi} \end{bmatrix}$. If system (1) is re-written as

$$\begin{aligned} \dot{x} &= Px + \bar{q}\bar{\xi}; \\ \sigma &= \bar{r}x \end{aligned} \tag{9}$$

then we have $\sigma\left(s\right) = \chi\left(s\right)\bar{\xi}\left(s\right)$ where $\sigma\left(s\right)$ and $\bar{\xi}\left(s\right)$ are the Laplace Transforms of $\sigma\left(t\right)$ and $\bar{\xi}\left(t\right)$, respectively. Also,

$$\chi\left(s\right) = \bar{r}\left(sI-P\right)^{-1}\bar{q} \tag{10}$$

is the transfer function matrix of the system (9), where $\bar{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ and $\bar{q} = \left[\, q_1 \;\; q_2 \,\right]$. Given constants $\tau_1 > 0$, $\tau_2 > 0$ and define

$$\tilde{\chi}\left(s\right) = \begin{bmatrix} \tau_1^{\frac{1}{2}} & 0 \\ 0 & \tau_2^{\frac{1}{2}} \end{bmatrix} \chi\left(s\right) \begin{bmatrix} \tau_1^{-\frac{1}{2}} & 0 \\ 0 & \tau_2^{-\frac{1}{2}} \end{bmatrix}, \tag{11}$$

then the following theorem is presented:

Theorem 1. The system (1,2,3,4,5) is Lagrange stable if there exist constants $\lambda > 0$, $\tau_1 > 0$ and $\tau_2 > 0$ satisfying the following conditions:

(1) The matrix $P+\lambda I$ has $n-1$ eigenvalues with negative real parts;
(2) The following frequency domain inequality holds:

$$\tilde{\chi}^T\left(j\omega-\lambda\right)\tilde{\chi}\left(j\omega-\lambda\right) \leq \begin{bmatrix} \mu^{-2}I & 0 \\ 0 & \tilde{\mu}^{-2}I \end{bmatrix} \tag{12}$$

for all $\omega \geq 0$.

In order to prove Theorem 1, the definition of positively invariant set and two lemmata are given first.

Proof of Theorem 1: Define

$$\begin{aligned} \mathfrak{G}\left(\sigma,\bar{\xi}\right) \triangleq\ & \tau_1\left(\mu_1^{-1}\xi-\sigma_1\right)^T\left(\mu_2^{-1}\xi-\sigma_1\right) \\ & +\tau_2\left(\tilde{\mu}_1^{-1}\tilde{\xi}-\sigma_2\right)^T\left(\tilde{\mu}_2^{-1}\tilde{\xi}-\sigma_2\right) \end{aligned} \tag{13}$$

for system (1).

We write (13) as a quadratic form

$$\mathfrak{G}\left(\sigma,\bar{\xi}\right) = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \xi \\ \tilde{\xi} \end{bmatrix}^T \begin{bmatrix} \tau_1 & 0 & -\tau_1\alpha_1 & 0 \\ 0 & \tau_2 & 0 & -\tau_2\alpha_2 \\ -\tau_1\alpha_1 & 0 & \tau_1\beta_1 & 0 \\ 0 & -\tau_2\alpha_2 & 0 & \tau_2\beta_2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \xi \\ \tilde{\xi} \end{bmatrix} \tag{14}$$

where $\alpha_1 = \frac{\mu_1^{-1}+\mu_2^{-1}}{2}$, $\alpha_2 = \frac{\tilde{\mu}_1^{-1}+\tilde{\mu}_2^{-1}}{2}$, $\beta_1 = \mu_1^{-1}\mu_2^{-1}$ and $\beta_2 = \tilde{\mu}_1^{-1}\tilde{\mu}_2^{-1}$.

Let $M = \begin{bmatrix} \tau_1 & 0 & -\tau_1\alpha_1 & 0 \\ 0 & \tau_2 & 0 & -\tau_2\alpha_2 \\ -\tau_1\alpha_1 & 0 & \tau_1\beta_1 & 0 \\ 0 & -\tau_2\alpha_2 & 0 & \tau_2\beta_2 \end{bmatrix}$.

The matrix M can be partitioned as $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ where $M_{11} = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}$, $M_{22} = \begin{bmatrix} \tau_1\mu_1^{-1}\mu_2^{-1} & 0 \\ 0 & \tau_2\tilde{\mu}_1^{-1}\tilde{\mu}_2^{-1} \end{bmatrix}$, $M_{12} = M_{21} = \begin{bmatrix} -\tau_1\frac{\mu_1^{-1}+\mu_2^{-1}}{2} & 0 \\ 0 & -\tau_2\frac{\tilde{\mu}_1^{-1}+\tilde{\mu}_2^{-1}}{2} \end{bmatrix}$. Applying (8) gives $M_{12} = M_{21} = 0$, $M_{22} = \begin{bmatrix} -\tau_1\mu^{-2} & 0 \\ 0 & -\tau_2\tilde{\mu}^{-2} \end{bmatrix}$.

Using the fact that σ is related to $\bar{\xi}$ as in (9), (10), (11), we can write

$$\begin{aligned} & \mathfrak{G}\left(\chi\left(s\right)\bar{\xi}\left(s\right),\bar{\xi}\left(s\right)\right) \\ & = \begin{bmatrix} \chi\left(s\right)\bar{\xi}\left(s\right) \\ \bar{\xi}\left(s\right) \end{bmatrix}^T \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \chi\left(s\right)\bar{\xi}\left(s\right) \\ \bar{\xi}\left(s\right) \end{bmatrix} \end{aligned} \tag{15}$$

It follows that

$$\mathfrak{G}\left(\chi(s)\bar{\xi}(s), \bar{\xi}(s)\right) = \bar{\xi}^T(s)\chi^T(s)M_{11}\chi(s)\bar{\xi}(s) + \bar{\xi}^T(s)M_{22}\bar{\xi}(s) \quad (16)$$

This can be further written as

$$\mathfrak{G}\left(\chi(s)\bar{\xi}(s), \bar{\xi}(s)\right) = \bar{\xi}^T(s) \quad (17)$$
$$\cdot\left(\chi^T(s)\begin{bmatrix}\tau_1 I & 0\\ 0 & \tau_2 I\end{bmatrix}\chi(s) - \begin{bmatrix}\tau_1\mu^{-2}I & 0\\ 0 & \tau_2\tilde{\mu}^{-2}I\end{bmatrix}\right)\bar{\xi}(s)$$

From (12) and the definition of $\tilde{\chi}(s)$, it follows that

$$\begin{bmatrix}\tau_1^{-\frac{1}{2}} & 0\\ 0 & \tau_2^{-\frac{1}{2}}\end{bmatrix}\chi^T(j\omega-\lambda)\begin{bmatrix}\tau_1 & 0\\ 0 & \tau_2\end{bmatrix}\chi(j\omega-\lambda)$$
$$\cdot\begin{bmatrix}\tau_1^{-\frac{1}{2}} & 0\\ 0 & \tau_2^{-\frac{1}{2}}\end{bmatrix} \le \begin{bmatrix}\mu^{-2}I & 0\\ 0 & \tilde{\mu}^{-2}I\end{bmatrix} \quad (18)$$

Pre- and post-multiplying by $\begin{bmatrix}\tau_1^{-\frac{1}{2}} & 0\\ 0 & \tau_2^{-\frac{1}{2}}\end{bmatrix}^{-1}$ in (18) gives

$$\chi^T(j\omega-\lambda)\begin{bmatrix}\tau_1 I & 0\\ 0 & \tau_2 I\end{bmatrix}\chi(j\omega-\lambda) \le \begin{bmatrix}\tau_1\mu^{-2}I & 0\\ 0 & \tau_2\tilde{\mu}^{-2}I\end{bmatrix}. \quad (19)$$

Let $s = j\omega$. Applying (19) to (17) implies

$$\mathfrak{G}\left(\chi(j\omega-\lambda)\bar{\xi}(j\omega), \bar{\xi}(j\omega)\right) \le 0. \quad (20)$$

Thus, the condition of Theorem 1.10.1 in Leonov et al. (1996)) is satisfied where the matrix P is replaced by $P + \lambda I$. Applying (Yakubovich-Kalman Theorem 1.10.1 in Leonov et al. (1996)) to (20) implies that there exists a Hermitian $n \times n$ matrix H such that

$$\Im\left(x, \xi, \tilde{\xi}\right) = 2x^T H\left[(P+\lambda I)x + \bar{q}\bar{\xi}\right] + \mathfrak{G}\left(\sigma, \bar{\xi}\right) \le 0. \quad (21)$$

It follows that the set $\{x|x^T Hx < 0\}$ is positively invariant (See page 145 of Leonov et al. (1996)) for the system (1).

Suppose $x(t, t_0, x_0)$ is a solution of (1). Let d be an eigenvector of P corresponding to its zero eigenvalue, such that $\begin{bmatrix}r_1\\ r_2\end{bmatrix} d = \Delta$. Since (1) is pendulum-like with respect to $\Upsilon = id, i \in \mathcal{Z}, d \neq 0$, it follows that

$$x(t, t_0, x_0) - id = x(t, t_0, x_0 - id), \quad t \ge t_0, \quad i \in \mathcal{Z} \quad (22)$$

Since the set $\{x|x^T Hx < 0\}$ is positively invariant for (1), the interior

$$\Omega_i \triangleq \{x|(x-id)^T H(x-id) < 0\}$$

of a quadratic cone $\{x|(x-id)^T H(x-id) \le 0\}$ is positively invariant for (1). For an arbitrary $x_0 \in \Omega$, it follows that $x_0 - id \in \Omega_0$. Then by virtue of the positive invariance of Ω_0, we have

$$x(t, t_0, x_0 - id) \in \Omega_0 \quad (\forall t \ge t_0, t_0 \in \mathcal{R}).$$

Using (22), we then have that

$$[x(t, t_0, x_0) - id]^T H[x(t, t_0, x_0) - id] < 0 \quad (\forall t \ge t_0, t_0 \in \mathcal{R}).$$

Since $\mathfrak{G}(x, \bar{\xi})$ is non-positive for all $x \in \mathcal{R}^n$, $\bar{\xi} \in \mathcal{R}$, we obtain

$$2x^T H[P + \lambda I]x \le -(\bar{r}x)^2$$

if taking $\bar{\xi} = 0$ in

$$2x^T H(Px + Q\bar{\xi}) \le -2\lambda x^T Hx - \mathfrak{G}(\sigma, \bar{\xi}). \quad (23)$$

Note that since $\chi(s)$ is non-degenerate, the pair $(P + \lambda I, \bar{r})$ is observable. Then, it follows from Lemma 2.6.2 in Leonov et al. (1996)) that H has one negative and $n-1$ positive eigenvalues. Now, let $x = d$, $\bar{\xi} = 0$ in (23). Hence $d^T Hd < 0$.

For an arbitrary $i \in \mathcal{Z}$, define the set

$$\Upsilon_i \triangleq \Omega_i \cap \Omega_{-i}.$$

The set Υ_i is positively invariant as both Ω_i and Ω_{-i} are positively invariant.

Because of our condition on the spectrum of H, there exists a vector $h \neq 0$ such that

$$\{x|h^T x = 0, x \neq 0\} \subset \{x^T Hx > 0\}. \quad (24)$$

Now, we need to establish this fact and prove that $x(t, t_0, x_0)$ is bounded. The rest of this proof can follow the proof of Theorem 2.6.1 of Leonov et al. (1996). For simplicity, it is omitted here. Thus the theorem is proved. □

Remark 1. This theorem extends the result obtained in Theorem 2.6.1 (Leonov et al. (1996)) to the case of pendulum-like nonlinear systems with multiple nonlinearities.

Remark 2. If $\mu = \tilde{\mu}$, then condition (12) has a form $\tilde{\chi}^T(j\omega-\lambda)\tilde{\chi}(j\omega-\lambda) \le \mu^{-2}I$. This appears in the form of the circle criterion, so this result can be considered as extending the circle criterion to the case of pendulum-like nonlinear systems with repeated nonlinearities.

Remark 3. It is easy to extend the result to the case of m ($m \ge 1$) nonlinearities and obtain the following corollary:

Corollary 1. Let $\sigma = \begin{bmatrix}\sigma_1^T & \cdots & \sigma_m^T\end{bmatrix}^T$, $\bar{\xi} = \begin{bmatrix}\xi_1^T & \cdots & \xi_m^T\end{bmatrix}^T$, $Q = \begin{bmatrix}q_1 & \cdots & q_m\end{bmatrix}$, $\chi(j\omega) = R(j\omega - P)^{-1}Q$, $R = \begin{bmatrix}r_1\\ \vdots\\ r_m\end{bmatrix}$ and $\tilde{\chi}(j\omega) = \begin{bmatrix}\tau_1^{\frac{1}{2}} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \tau_m^{\frac{1}{2}}\end{bmatrix}\chi(j\omega)\begin{bmatrix}-\tau_1^{\frac{1}{2}} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & -\tau_m^{\frac{1}{2}}\end{bmatrix}$, where $\tau_1, \tau_2, \cdots, \tau_m$ are given positive contants. Suppose there exists a constant $\lambda > 0$ satisfying the following conditions:

(1) The matrix $P+\lambda I$ has $n-1$ eigenvalues with negative real parts;
(2) The frequency domain inequality
$$\tilde{\chi}^T(j\omega-\lambda I)\tilde{\chi}(j\omega-\lambda I) \le \begin{bmatrix}(\mu_1)^{-2}I & & 0\\ & \ddots & \\ 0 & & (\mu_m)^{-2}I\end{bmatrix}$$
is satisfied.

Then the following pendulum-like nonlinear system is Lagrange stable

$$\begin{aligned}\dot{x} &= Px + q_1\xi_1 + \cdots + q_m\xi_m \\ \sigma_i &= r_i x,\ i = 1, \cdots, m \\ \xi_i &= \phi_i(t, \sigma_i),\ i = 1, \cdots, m \end{aligned} \tag{25}$$

where the components of $q_1 \cdots q_m$ and $r_1 \cdots r_m$ have compatible dimensions and the nonlinearities $\phi_1, \cdots, \phi_m$: $\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ are continuous and locally Lipschitz continuous in the second argument, satisfying the sector constraints $-\mu_i \leq \frac{\phi_i(t,\sigma_i)}{\sigma_i} \leq \mu_i, i = 1, \cdots, m$. Also, we assume $det\ P = 0$ and $\phi(t, \sigma_i + \Delta) = \phi(t, \sigma_i), t \in \mathcal{R}_+, \sigma_i \in \mathcal{R}, i = 1, \cdots, m$.

3. EXTENDED STRICT BOUNDED REAL LEMMA

As presented in Petersen et al. (1991), the strict bounded real lemma relates an H-infinity condition in the frequency domain to the existence of a solution to an Algebraic Riccati Equation. This idea is widely applied to controller design for linear control systems. This paper will also use this idea when the controller design for pendulum-like systems is considered. However, in Theorem 2.1 by Petersen et al. (1991) and Chen and Tu (1995), the strict bounded real lemma requires that the system matrix A is stable, while the systems considered in the paper are normally unstable. In preparation for discussing the synthesis problem for pendulum-like systems, we prove an Extended Strict Bounded Real Lemma, in which matrix A is not required to be stable.

Consider the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ z(t) &= Cx(t)\end{aligned} \tag{26}$$

where the pair (A, B) is stabilizable.

We define the transfer function of (26) as $G(s) = C(sI - A)^{-1}B$ and denote $G^{\sim}(j\omega) \triangleq G(-j\omega)^T$.

Theorem 2. Suppose that A has no eigenvalue on the $j\omega$-axis and the pair (A, B) is stabilizable, then the Riccati Equation

$$A^*X + XA - XBB^*X = 0 \tag{27}$$

has a solution $X \geq 0$ such that $A - BB^*X$ is stable.

Proof of Theorem 2: As the Riccati equation (27) is a special case of the Riccati equation in Theorem 2.1 of Ran and Vreugdenhil (1988) with $C = 0$ and $\tilde{R} = I$, (27) has solution $X \geq 0$ and all the eigenvalues of the matrix $A - BB^*X$ are in the closed left half of the complex plane. Now, we further proof that the eigenvalues of the matrix $A - BB^*X$ are stable. We rewrite (27) as

$$(A - BB^*X)^*X + X(A - BB^*X) = -XBB^*X \tag{28}$$

Suppose $\tilde{\lambda}$ and x are eigenvalue and the corresponding eigenvector of $A - BB^*X$, respectively, i.e.,

$$(A - BB^*X)x = \tilde{\lambda}x.$$

Pre-and post-multiplying equation (28) by x^* and x, respectively gives

$$\left(\bar{\tilde{\lambda}} + \tilde{\lambda}\right) x^*Xx = -x^*XBB^*Xx$$

As all the eigenvalues of $A - BB^*X$ are in the closed left half of the complex plane, the left side of this equation is positive semidefinite. It follows that $B^*Xx = 0$. Furthermore, post-multiplying $A - BB^*X$ by x gives

$$(A - BB^*X)x = Ax + BB^*Xx = Ax.$$

Hence, because A has no eigenvalue on the $j\omega$-axis, $A - BB^*X$ has no eigenvalue on the $j\omega$-axis, either. That is, $A - BB^*X$ is stable. This completes the proof. □

Theorem 3. (*Extended Strict Bounded Real Lemma*) If A is an unstable matrix without any eigenvalue on the $j\omega$-axis and the pair (A, B) is stabilizable, then the following statements are equivalent

(1) $G(-j\omega)^T G(j\omega) \leq 1$;
(2) The algebraic Riccati equation

$$A^TH + HA + HBB^TH + C^TC = 0 \tag{29}$$

has a solution $H = H^T$;

Proof of Theorem 3: (1)⇒ (2): As A has no eigenvalue on the $j\omega$-axis, Theorem 13.34 by Zhou and Doyle (1998) is applicable to the system (26) where we let $s = j\omega$. Following Theorem 13.34 by Zhou and Doyle (1998), we have $G(j\omega)G^{\sim}(j\omega) = \hat{G}(j\omega)\hat{G}^{\sim}(j\omega)$ where $\hat{G}(s) = N(s)$.Here, $N(s))$ is defined as in Theorem 13.34 of Zhou and Doyle (1998). Therefore, $G(j\omega)G^{\sim}(j\omega) \leq I$ implies $\hat{G}(j\omega)\hat{G}^{\sim}(j\omega) \leq I$. As there exist a matrix X such that $A - BB^TX$ is stable, it follows from Strict Bounded Real Lemma (See Petersen et al. (1991)), $\hat{G}(j\omega)\hat{G}^{\sim}(j\omega) \leq I$ is equivalent to the fact that the following algebraic Riccati equation has a stabilizing solution $\hat{H} \geq 0$:

$$\begin{aligned}&(A - BB^TX)^T\hat{H} + \hat{H}(A - BB^TX) + \hat{H}BB^T\hat{H} \\ &+C^TC = 0.\end{aligned} \tag{30}$$

Let $H = \hat{H} - X$. Then substituting into (30) gives that

$$\begin{aligned}&(A - BB^TX)^T(H + X) + (H + X)(A - BB^TX) \\ &\quad +(H + X)BB^T(H + X) + C^TC = 0.\end{aligned} \tag{31}$$

Expanding the left side of the equality (31) gives that

$$\begin{aligned}&(A - BB^TX)^T(H + X) + (H + X)(A - BB^TX) \\ &\quad +(H + X)BB^T(H + X) + C^TC \\ &= A^TX + XA - X^TBB^TX + A^TH + HA \\ &\quad +HBB^TH + C^TC = 0.\end{aligned} \tag{32}$$

Also from Theorem 13.34 by Zhou and Doyle (1998), there exists a matrix X such that (27) holds. Therefore, the above equality then implies that

$$A^TH + HA + HBB^TH + C^TC = 0.$$

(2)⇒ (1): Define $\hat{H} = X + H$ then

$$\begin{aligned}&(A - BB^TX)^T\hat{H} + \hat{H}(A - BB^TX) + \hat{H}BB^T\hat{H} \\ &\quad +C^TC \\ &= (A - BB^TX)^T(X + H) + (X + H)(A - BB^TX) \\ &\quad + (X + H)BB^T(X + H) + C^TC \\ &= A^TX + XA - X^TBB^TX \\ &\quad +A^TH + HA + PBB^TH + C^TC.\end{aligned} \tag{33}$$

As the pair (A, B) is stabilizable, it follows from Theorem 2 that the Riccati equation (27) holds and $A - BB^T X$ is stable. As (29) holds, it follows that

$$(A - BB^T X)^T \hat{H} + \hat{H}(A - BB^T X) + \hat{H}BB^T\hat{H} + C^T C = 0. \quad (34)$$

Now, Applying the result of the Strict Bounded Real Lemma in Petersen et al. (1991), it follows that

$$\hat{G}(-j\omega)^T \hat{G}(j\omega) \leq 1.$$

From $G(j\omega)G^{\sim}(j\omega) = \hat{G}(j\omega)\hat{G}^{\sim}(j\omega)$, it follows that item (1) holds. This completes the proof. □

Theorem 3 is applicable to the systems where the pair (A, B) is stabilizable. As controllability of the pair (A, B) implies its stabilizability, Theorem 3 is applicable to systems with controllable (A, B). If the pair (A, B) is uncontrollable, using the Kalman decomposition,(see Antsaklis and Michel (2005)), an uncontrollable system can be split into the controllable part and uncontrollable part. For an uncontrollable (A, B), we have the following corollary of Theorem 3:

Corollary 2. Suppose A is an unstable matrix without any eigenvalue on the $j\omega$-axis and the pair (A, B) is uncontrollable. If the algebraic Riccati equation

$$A^T H + HA + HBB^T H + C^T C = 0 \quad (35)$$

has a solution $H = H^T$, then the following frequency domain condition holds:

$$G^T(-j\omega)\, G(j\omega) \leq 1.$$

Proof: If Riccati equation (35) holds, the matrices A, B have the partitions as in the Kalman decomposition, e.g., see (Antsaklis and Michel (2005)) and C, H are correspondingly partitioned as $C = [\,C_1\ C_2\,]$, $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}$. Then we have

$$A_1^T H_{11} + H_{11}A_1 + H_{11}B_1B_1^T H_{11} + C_1^T C_1 = 0.$$

As the pair (A_1, B_1) is controllable, the conditions of Theorem 3 are satisfied for the sub-system (A_1, B_1, C_1). So, we have $\bar{G}(-j\omega)^T\bar{G}(j\omega) \leq 1$ where $\bar{G}(j\omega) = C_1(j\omega I - A_1)^{-1}B_1$. For the standard form in Kalman decomposition, e.g., see (Antsaklis and Michel (2005)), $\bar{G}(j\omega) = G(j\omega)$. Therefore, $G(-j\omega)^T G(j\omega) \leq 1$. This completes the proof. □

4. CONTROLLER DESIGN FOR PENDULUM-LIKE SYSTEMS WITH MULTIPLE NONLINEARITIES

In this section, controller design for pendulum-like system with multiple nonlinearities will be considered. Also, we will consider repeated nonlinearities as a special case where $\mu_i = \mu, i = 1, \cdots, m$. In this section, the system to be controlled is described by the state equations

$$\begin{aligned} \dot{x} &= Px + Bu + Q\hat{\xi}; \\ z &= Ex + u \end{aligned} \quad (36)$$

where $\det P = 0$ and $\hat{\xi} = [\,\xi_1,\ \xi_2,\ \cdots,\ \xi_m\,]^T$ and hence there must be at least one of the system poles on the origin.

Note that the system (36) can be transformed into the standard uncontrollable form in the Kalman decomposition, e.g., see (Antsaklis and Michel (2005)).

Hence, there exists a non-singular matrix Y which tranforms the system (36) into the form

$$\begin{aligned} \dot{x} &= \tilde{P}x + \tilde{B}u(t) + \tilde{Q}\hat{\xi}; \\ z &= \tilde{E}x + u(t) \end{aligned} \quad (37)$$

where

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_{12} \\ 0 & \tilde{P}_2 \end{bmatrix}, \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, \tilde{Q} = \begin{bmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \end{bmatrix}, \tilde{E} = \begin{bmatrix} \tilde{E}_1^T \\ \tilde{E}_2^T \end{bmatrix}^T \quad (38)$$

and $(\tilde{P}_1, \tilde{B}_1)$ is controllable. We further assume that

A1. The components of the matrices in (38) are such that $\tilde{P}_1 \in \mathcal{R}^{(n-1)\times(n-1)}$, $\tilde{B}_1 \in \mathcal{R}^{(n-1)\times 1}$, $\tilde{P}_2 = 0$. That is, the system pole at the origin is uncontrollable.

In order to aid the discussion of the controller design for the system (37), we consider the following modified system:

$$\begin{aligned} \dot{x} &= (\tilde{P} + \lambda I)x + \tilde{B}u(t) + \tilde{Q}\hat{\xi} \\ z &= \tilde{E}x + u(t) \end{aligned} \quad (39)$$

where $\lambda > 0$, $x \in \mathcal{R}^n$ is the new state, $u \in \mathcal{R}^l$, $\hat{\xi} \in \mathcal{R}^p$ is the vector of nonlinearity inputs, and $z \in \mathcal{R}^l$ is the controlled output. By adding a term λI to the matrix $\tilde{P}$, the poles on the imaginary axis are moved to the right side of the complex plane.

We make the following assumption on system (37) and (39):

A2. $\tilde{P} + \lambda I - \tilde{B}\tilde{E}$ has no purely imaginary eigenvalues.

Definition 2. The system (37) is said to be Lagrange stabilizable if there exists a matrix K such that the closed-loop system is Lagrange stable with control law $u = Kx$.

Using Theorem 1, it follows that a sufficient condition for the system (37) to be Lagrange stabilizable is as follows:

(1) There exists a constant $\lambda > 0$ and a matrix K such that the matrix $\tilde{P} + \lambda I + \tilde{B}K$ has $n - 1$ eigenvalues with negative real parts;
(2) The following frequency-domain condition holds:
$$\hat{\chi}(-j\omega)^T\hat{\chi}(j\omega) \leq I \quad (40)$$
where $\hat{\chi}(j\omega) = (\tilde{E} + K)(j\omega - \tilde{P} - \lambda I - \tilde{B}K)^{-1}\bar{Q}$ with $\bar{Q} = V^{-1}\cdot\tilde{Q}$ with $V = \begin{bmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_m \end{bmatrix}$

Theorem 4. Suppose assumptions (*A1-A2*) are satisfied and the Riccati equation

$$(\tilde{P} + \lambda I - \tilde{B}\tilde{E})^T H + H(\tilde{P} + \lambda I - \tilde{B}\tilde{E}) - H\tilde{B}^T\tilde{B}H + H\bar{Q}\bar{Q}^T H = 0 \quad (41)$$

has a nonsingular solution $H = H^T$ such that $H_{11} = H_{11}^T > 0$, where $H_{11} \in \mathcal{R}^{(n-1)\times(n-1)}$ is the $(1, 1)$ block of the matrix $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}$. Then, the closed-loop system of

(37) is Lagrange stable with the required control law is given by $u(t) = -(\tilde{B}^T H + \tilde{E})x(t)$.

Remark 4. When H is singular, (41) is not useful for judging the Lagrange stability of system (37). Selecting a different value of λ can lead to a nonsingular H.

Proof: From (41), it follows that

$$\left(\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}\right)^T H_{11} + H_{11}\left(\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}\right) + H_{11}\tilde{B}_1\tilde{B}_1^T H_{11} + (H_{11}\bar{Q}_1 + H_{12}\bar{Q}_2)(H_{11}\bar{Q}_1 + H_{12}\bar{Q}_2)^T = 0 \quad (42)$$

As $(H_{11}\bar{Q}_1 + H_{11}\bar{Q}_2)(H_{11}\bar{Q}_1 + H_{12}\bar{Q}_2)^T \geq 0$, (42) implies that

$$\left(\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}\right)^T H_{11} + H_{11}\left(\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}\right) \leq -H_{11}\tilde{B}_1\tilde{B}_1^T H_{11} \quad (43)$$

Pre-multiplying (43) by $x^* H_{11}^{-1}$ and post-multipling $H_{11}^{-1}x$ gives

$$x^* H_{11}^{-1}\left(\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}\right)^T x + x^*\left(\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}\right) H_{11}^{-1}x \leq -x^*\tilde{B}_1\tilde{B}_1^T x \quad (44)$$

The pair $(\tilde{P}_1, \tilde{B}_1)$ being controllable implies that $(\tilde{P}_1 + \lambda I, \tilde{B}_1)$ is controllable. Let $\tilde{\lambda}$ be an eigenvalue of the matrix $H_{11}^{-1}\left(\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}\right)^T$. Then we have $\left(\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}\right)^T x = H_{11}\tilde{\lambda}x$. As H_{11} is non-singular and $\tilde{\lambda}$ must not equal zero, the matrix $\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}$ is not singular, either. Therefore, $\begin{pmatrix} \tilde{B}_1^T \\ \tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11} \end{pmatrix}$ has full rank and hence the pair $\left(\tilde{B}_1^T, \tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}\right)$ is observable according to Corollary 4.3.4 in Lancaster and Rodman (1995). Following Lemma 2.6.2 in Leonov et al. (1996)), it follows that the number of positive eigenvalues of H_{11} coincides with the number of eigenvalues of $\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T$ with negative real parts. As H_{11} is a positive definite matrix, $\tilde{P}_1 + \lambda I - \tilde{B}_1\tilde{E}_1 - \tilde{B}_1\tilde{B}_1^T H_{11}$ is a negative definite matrix, i.e., its eigenvalues are all less than zero. Therefore, the eigenvalues of $\tilde{P} + \lambda I + \tilde{B}K$ has $n-1$ eigenvalues with negative real parts while the other eigenvalue $\lambda > 0$.

As the Riccati equation (41) can be written as

$$(\tilde{P} + \lambda I - \tilde{B}\tilde{E} - \tilde{B}^T H)^T H + H(\tilde{P} + \lambda I - \tilde{B}\tilde{E} - \tilde{B}^T H) + H\tilde{B}\tilde{B}^T H + H\bar{Q}\bar{Q}^T H = 0 \quad (45)$$

Applying Corollary 2 to (45) and taking $-\tilde{B}^T H$ as C in (35) implies

$$\left(-B^T H(j\omega - \tilde{P} - \lambda I - \tilde{B}E - \tilde{B}\tilde{B}^T H)^{-1}\bar{Q}\right)^T \cdot \left(-B^T H(j\omega - \tilde{P} - \lambda I - \tilde{B}E - \tilde{B}\tilde{B}^T H)^{-1}\bar{Q}\right) \leq I \quad (46)$$

That is, the inequality (40) holds. Therefore, the closed-loop of system (39) is Lagrange stable. This completes the proof. □

Remark 5. The Riccati equation (41) can be written as a Lyapunov equation $\bar{H}(\tilde{P} + \lambda I - \tilde{B}\tilde{E})^T + (\tilde{P} + \lambda I - \tilde{B}\tilde{E})\bar{H} - \tilde{B}^T\tilde{B} + \bar{Q}\bar{Q}^T = 0$, on the condition that H is nonsigular.This fact will help solve the Riccati equation.

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