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Invariant Approximations of the Maximal Invariant Set or "Encircling the Square"

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Abstract: This paper offers a method for the computation of invariant approximations of the maximal invariant set for constrained linear discrete time systems subject to bounded, additive, disturbances. The main advantage of the method is that it generates invariant sets at any step of the underlying set iteration. Conditions under which the sequence of generated invariant sets is monotonically non–decreasing and converges to the maximal invariant set are provided. Explicit formulae for the estimates of the Hausdorff distance between the underlying iterates and the maximal invariant set are derived. *Copyright© 2008 IFAC.*

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1. INTRODUCTION

The set invariance theory has been subject to an extensive study over the last 50 years due to its close relationship with basic concepts of control theory, some of which are control synthesis under uncertainty, reachability analysis and stability theory. Indeed, utilization of set invariance concepts permits control synthesis for uncertain, constrained, control systems guaranteeing *a–priori* that the controlled dynamics exposed to the uncertainty are well behaved. A more detailed exposition of set invariance and its applications in control can be found in the monographs (Aubin, 1991; Blanchini and Miani, 2008). The main issues in set invariance are theoretical considerations and algorithmic procedures related to the maximal and the minimal invariant sets (Bertsekas, 1972; Aubin, 1991; Kolmanovsky and Gilbert, 1998; Raković *et al.*, 2005; Artstein and Raković, 2008; Raković, 2007; Blanchini and Miani, 2008). Since the computation of the maximal and the minimal invariant sets can be prohibitive in many cases, the characterization and the computation of invariant approximations of the maximal and the minimal invariant sets have been considered; see, for instance, (Kolmanovsky and Gilbert, 1998; B. D. O'Dell and E. A. Misawa, 2002; Raković *et al.*, 2005; Raković, 2007).

In this paper, we discuss a method that generates invariant sets at any step of the underlying set recursion. Conditions under which the corresponding sequence of invariant sets is monotonically non–decreasing and converges, in finite time, to the maximal invariant set are given. The proposed algorithm is "dual" to the method discussed in (Raković, 2007). Explicit formulae for the estimate of the Hausdorff distance between the corresponding set iterates and the maximal invariant set are derived.

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Paper Structure: Section 2 presents preliminaries. Sections 3 discusses the computation of invariant approximations of the maximal invariant set and analyzes corresponding convergence issues. Sections 4, 5 and 6 comment on computational considerations and provide a few illustrative examples and conclusion.

Basic Nomenclature and Definitions: The sets of non–negative, positive integers and non–negative real numbers are denoted, respectively, by N, N_+ and R_+, i.e. $N := \{0, 1, 2, \ldots\}$, $N_+ := \{1, 2, \ldots\}$ and $R_+ := \{x \in R \ : \ x \geq 0\}$. For two sets $X \subset R^n$ and $Y \subset R^n$ and a vector $x \in R^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y \ : \ x \in X, \ y \in Y\}$ and the Minkowski (Pontryagin) set difference is $X \ominus Y := \{z \in R^n \ : \ z \oplus Y \subseteq X\}$. Given the sequence of sets $\{X_i \subset R^n\}_{i=a}^{b}$, $a \in N$, $b \in N, b > a$, we denote $\bigoplus_{i=a}^{b} X_i := X_a \oplus \cdots \oplus X_b$. Given a set X and a real matrix M of compatible dimensions (possibly a scalar) *we denote by* MX *the image of* X *under* M so that $MX := \{Mx \ : \ x \in X\}$ and, similarly, *we denote by* $M^{-1}X$ *the inverse image of* X *under* M so that $M^{-1}X := \{x \ : \ Mx \in X\}$. Given a matrix $M \in R^{n \times n}$, $\rho(M)$ denotes the largest absolute value of its eigenvalues. A set $X \subset R^n$ is *a* C *set* if it is compact, convex, and contains the origin. A set $X \subset R^n$ is *a proper* C *set* if it is a C set and the origin is in its non–empty interior. A set $X \subseteq R^n$ is a symmetric set if $X = -X$. Given a non–empty closed subset $\mathcal{X}$ of R^n, the collection of non–empty compact subsets of $\mathcal{X}$ is denoted by $\text{Com}(\mathcal{X})$. The collection of C subsets of $\mathcal{X}$ is denoted by $\text{ComC}(\mathcal{X})$. The collection of proper C subsets of $\mathcal{X}$ is denoted by $\text{ComCP}(\mathcal{X})$. For $X \in \text{Com}(\mathcal{X})$ and $Y \in \text{Com}(\mathcal{X})$, the *Hausdorff semi–distance* and the *Hausdorff distance (metric)* of X and Y are, respectively, given by:

$$h_L(X, Y) := \min_{\alpha}\{\alpha \ : \ X \subseteq Y \oplus \alpha L, \ \alpha \geq 0\} \text{ and}$$
$$H_L(X, Y) := \max\{h_L(X, Y), h_L(Y, X)\},$$

where L is a given, symmetric, proper C set in R^n.

2. PRELIMINARIES & PROBLEM FORMULATION

Consider the following autonomous discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + w, \tag{2.1}$$

where $x \in R^n$ is the current state, x^+ is the successor state, $A \in R^{n\times n}$ is the state transition matrix and $w \in R^n$ is an unknown disturbance taking values in the set $W \subset R^n$. The variables x and w are subject to hard constraints:

$$x \in \mathcal{X} \text{ and } w \in W. \tag{2.2}$$

In this paper we invoke the following assumptions:

Assumption 1. The set W is a C set in R^n.

Assumption 2. The set $\mathcal{X}$ is a proper C set in R^n.

Assumption 3. The matrix A is strictly stable ($\rho(A) < 1$).

To discuss invariance related issues we follow the set–dynamics approach (Artstein and Raković, 2008) and associate the map $\mathcal{R}(\cdot) : \text{Com}(R^n) \to \text{Com}(R^n)$ with the system (2.1) and the disturbance set W given by:

$$\mathcal{R}(X) := AX \oplus W. \tag{2.3}$$

When $W \in \text{Com}(R^n)$, the function $\mathcal{R}(\cdot)$ maps, indeed, $\text{Com}(R^n)$ to itself. In addition, if $X \in \text{ComC}(R^n)$ and $W \in \text{ComC}(R^n)$ then $\mathcal{R}(X) \in \text{ComC}(R^n)$.

Definition 1. The set $\Omega \subset R^n$ is an *invariant* set for the system (2.1) and constraint set $(\mathcal{X}, W)$ if and only if $\Omega \subseteq \mathcal{X}$ and $Ax + w \in \Omega$ for all $x \in \Omega$ and all $w \in W$, i.e. iff $\Omega \subseteq \mathcal{X}$ and $\mathcal{R}(\Omega) \subseteq \Omega$ ($A\Omega \oplus W \subseteq \Omega$).

We use the term invariant set rather than robust positively invariant set (as is customary in the literature); no confusion should arise. We denote by $\text{ComInv}(A, W, \mathcal{X})$ the collection of all, compact, invariant subsets of $\mathcal{X}$:

$$\text{ComInv}(A, W, \mathcal{X}) := \{\Omega \ : \ \Omega \in \text{Com}(\mathcal{X}), \ \mathcal{R}(\Omega) \subseteq \Omega\}. \tag{2.4}$$

Definition 2. The set $\underline{\Omega} \subseteq R^n$ is *the minimal invariant set* for the system (2.1) and constraint set (R^n, W) over the collection of invariant sets $\text{ComInv}(A, W, R^n)$ if and only if $\underline{\Omega} \in \text{ComInv}(A, W, R^n)$ and $\underline{\Omega} \subset \Phi$ for all $\Phi \in \text{ComInv}(A, W, R^n)$ such that $\Phi \neq \underline{\Omega}$.

Definition 3. The set $\bar{\Omega} \subseteq R^n$ is *the maximal invariant set* for the system (2.1) and constraint set $(\mathcal{X}, W)$ over the collection of invariant sets $\text{ComInv}(A, W, \mathcal{X})$ if and only if $\bar{\Omega} \in \text{ComInv}(A, W, \mathcal{X})$ and $\Phi \subset \bar{\Omega}$ for all $\Phi \in \text{ComInv}(A, W, \mathcal{X})$ such that $\Phi \neq \bar{\Omega}$.

The minimal invariant set is, under Assumptions 1 and 3, unique and is given explicitly by:

$$X_\infty = \bigoplus_{i=0}^{\infty} A^i W, \tag{2.5}$$

and is, furthermore, a C set in R^n. The main results of (Kolmanovsky and Gilbert, 1998) yield that the collection of invariant sets $\text{ComInv}(A, W, \mathcal{X})$ is non–empty if and only if the minimal invariant set X_∞ and the state constraint set $\mathcal{X}$ are such that $X_\infty \subseteq \mathcal{X}$. In addition, the maximal invariant set is finitely determined when Assumptions 1–3 hold and $X_\infty \subseteq \text{interior}(\mathcal{X})$ (Kolmanovsky and Gilbert, 1998). Hence, we also invoke:

Assumption 4. The minimal invariant set X_∞ and the state constraint set $\mathcal{X}$ are such that $X_\infty \subseteq \alpha\mathcal{X}$ for some $\alpha \in [0, 1)$.

We denote by $\text{Com}(\mathcal{X}, X_\infty)$ the collection of all compact subsets of $\mathcal{X}$ that contain the minimal invariant set X_∞:

$$\text{Com}(\mathcal{X}, X_\infty) := \{X \in \text{Com}(\mathcal{X}) \ : \ X_\infty \subseteq X\} \tag{2.6}$$

and invoke the map $\mathcal{B}(\cdot) : \text{Com}(\mathcal{X}, X_\infty) \to \text{Com}(\mathcal{X}, X_\infty)$:

$$\mathcal{B}(X) := \left(A^{-1}(X \ominus W)\right) \cap \mathcal{X}. \tag{2.7}$$

Clearly, under Assumptions 1–4, the function $\mathcal{B}(\cdot)$ maps, indeed, $\text{Com}(\mathcal{X}, X_\infty)$ to itself. The standard viability algorithm (Bertsekas, 1972; Aubin, 1991; Kolmanovsky and Gilbert, 1998; Blanchini and Miani, 2008) for the computation of the maximal invariant set is given by:

$$\Omega_{k+1} := \mathcal{B}(\Omega_k), \ k \in N, \ \Omega_0 := \mathcal{X}. \tag{2.8}$$

The set sequence $\{\Omega_k\}_{k=0}^{\infty}$ generated by (2.8) is, under Assumptions 1–3, a monotonically non–increasing sequence of compact sets that is bounded below by the minimal invariant set X_∞ and hence it converges. Furthermore, its limit is the maximal invariant set Ω_∞ and, under Assumptions 1–4 there exists a finite $k^* \in N$ such that $\Omega_{k^*+1} = \Omega_{k^*}$ and, consequently, $\Omega_\infty = \Omega_{k^*}$ is a proper C set in R^n. The smallest integer $k^* \in N$ such that $\Omega_{k^*+1} = \Omega_{k^*}$ is referred to as *the determinedness index* and, in this case, we say that the maximal invariant set is finitely determined. However, even in the case when the maximal invariant set is finitely determined with the corresponding determinedness index k^*, none of the iterates $\Omega_k, \ k < k^*$, enjoys invariance property and, moreover, the determinedness index k^* can be reasonably large rendering the set iteration (2.8) computationally expensive and corresponding iterates relatively complex sets.

We recall a few elementary facts (Kuratowski, 1972; Schneider, 1993) that are of much help.

Lemma 1. Let X, Y and Z be three arbitrary non–empty subsets of R^n. Then:

$$(X \cap Y) \ominus Z = (X \ominus Z) \cap (Y \ominus Z) \text{ and} \tag{2.9a}$$

$$(X \cap Y) \oplus Z \subseteq (X \oplus Z) \cap (Y \oplus Z). \tag{2.9b}$$

Lemma 2. Let $f(\cdot) : \mathcal{X} \to \mathcal{Y}$ and let $f^{-1}(Y)$ denote the inverse image of the set $Y \subset \mathcal{Y}$, i.e. $f^{-1}(Y) = \{x \ : \ f(x) \in Y\}$. Let also $f(X)$ denote the image of the set $X \subseteq \mathcal{X}$ with respect to $f(\cdot)$, i.e. $f(X) = \{f(x) \ : \ x \in X\}$. Then:

$$f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2) \text{ and} \tag{2.10a}$$

$$f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2), \tag{2.10b}$$

for any Y_1 and Y_2 contained in $\mathcal{Y}$, and similarly, for any X_1 and X_2 contained in $\mathcal{X}$.

The following simple observation is also of help:

Lemma 3. Let X and Y be arbitrary C sets in R^n. Then:

$$Y \cap X \subseteq (\lambda_1 Y \cap X) \oplus (\lambda_2 Y \cap X), \tag{2.11}$$

for all λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \geq 0, \ \lambda_2 \geq 0$.

The main objectives of this paper are to:

(i) provide a modification of the viability algorithm (2.8) such that the iterates of the modified set recursion are invariant sets and converge, in finite time, to the maximal invariant set,
(ii) derive estimates of the Hausdorff distance between the iterates of the modified procedure and the maximal invariant set, and
(iii) discuss some special families of invariant sets resulting from the modified set recursion.

3. INVARIANT APPROXIMATIONS OF THE MAXIMAL INVARIANT SET

We follow the set–dynamics approach employed in (Artstein and Raković, 2008) and utilize set–dynamics induced by the mappings $\mathcal{R}(\cdot)$ and $\mathcal{B}(\cdot)$ restricted to appropriate spaces. The mapping $\mathcal{R}(\cdot)$ induces the set–dynamics:

$$X^+ = \mathcal{R}(X) = AX \oplus W. \tag{3.1}$$

As shown in (Artstein and Raković, 2008), under Assumptions 3 and when $W \in \text{Com}(R^n)$, the mapping $\mathcal{R}(\cdot)$ is a contraction in $\text{Com}(R^n)$ with respect to Hausdorff distance $H_L(\cdot,\cdot)$ and hence admits the unique fixed point, precisely the minimal invariant set X_∞. Under Assumptions 1–4, Proposition 4.3 of (Artstein and Raković, 2008) and invariance of Ω_∞ imply that the set dynamics (3.1) result in the trajectory $\{X_k\}_{k=0}^{\infty}$ such that $X_k \in \text{Com}(\Omega_\infty)$ and $H_L(X_k, X_\infty) \to 0$ as $k \to \infty$ for any arbitrary initial condition $X_0 \in \text{Com}(\Omega_\infty)$, where Ω_∞ and X_∞ are, respectively, the maximal and the minimal invariant sets. In fact, under Assumptions 1–4, the set X_∞ is the stable attractor for the set–dynamics (3.1) restricted to $\text{Com}(\Omega_\infty)$ and is, furthermore, the unique set which solves the set equation:

$$X = \mathcal{R}(X) \text{ i.e. } X = AX \oplus W. \tag{3.2}$$

Set–dynamics approach utilized in (Artstein and Raković, 2008) has, *inter alia*, resulted in the characterization of a family of outer invariant approximations of the minimal invariant set offering computational benefits in the linear–convex setting (Raković, 2007):

Proposition 1. Suppose Assumptions 1 and 3 hold. Then there exist a symmetric set $L \in \text{ComCP}(R^n)$ and a scalar $\lambda \in [0, 1)$ such that, for all $k \in N$,

$$A^k L \subseteq \lambda^k L. \tag{3.3}$$

Furthermore, for any symmetric set $L \in \text{ComCP}(R^n)$ and a scalar $\lambda \in [0, 1)$ such that $AL \subseteq \lambda L$, sets S_k given by:

$$S_k := \mathcal{R}^k(\{0\}) \oplus \lambda^k(1-\lambda)^{-1}\mu L, \tag{3.4}$$

where $\mathcal{R}^{k+1}(\{0\}) = \bigoplus_{i=0}^{k} A^i W$ and $\mathcal{R}^0(\{0\}) = \{0\}$, $\mu := H_L(W, \{0\}) = \min_\gamma\{\gamma : W \subseteq \gamma L\}$, are invariant sets for the system (2.1) and constraint set (R^n, W) for any $k \in N$, i.e. $\forall k \in N : \mathcal{R}(S_k) \subseteq S_k$.

We utilize the power of the set–dynamics approach by focusing on set–dynamics induced by the mapping $\mathcal{B}(\cdot)$ restricted to the collection of sets $\text{Com}(\mathcal{X}, X_\infty)$:

$$Y^+ = \mathcal{B}(Y) = \left(A^{-1}(Y \ominus W)\right) \cap \mathcal{X}. \tag{3.5}$$

Clearly, set–dynamics (3.5) produces the trajectory $\{Y_k\}_{k=0}^{\infty}$ such that $Y_k = \Omega_k$ for all $k \in N$ when $\{\Omega_k\}_{k=0}^{\infty}$ is the set sequence generated by (2.8) and the initial condition is $Y_0 = \mathcal{X}$. Hence, we discuss the possibility to utilize set–dynamics (3.5) in order to generate the sequence of improving inner invariant approximations of the maximal invariant set Ω_∞. *The underlying idea is to simply generate the trajectory* $\{Y_k\}_{k=0}^{\infty}$ *of the set–dynamics* (3.5) *such that* Y_k *is invariant and it approaches the maximal invariant set* Ω_∞ *from the inside. Off–hand intuition might suggest that, under Assumptions 1–4, the trajectory* $\{Y_k\}_{k=0}^{\infty}$ *of the set–dynamics* (3.5) *starting from an initial condition* Y_0 *which is invariant is a monotonically non–decreasing sequence of invariant sets converging to the maximal invariant set* Ω_∞*. However, this is not the case, as shown by our telling example:*

Example 1. Consider the following system:

$$x^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \beta \end{bmatrix} x + w,$$

where $\beta \in (0, 1)$ can be arbitrarily chosen. The disturbance and state constraint sets are given by:

$$W = [-1, 1] \times \{0\} \text{ and } \mathcal{X} = 3B_\infty,$$

where B_∞ denotes the closed unit ∞–norm ball. It is easy to see that the minimal invariant set X_∞ is, for this example, given by:

$X_\infty = [-2, 2] \times \{0\}$ and that $\mathcal{R}(X_\infty) = AX_\infty \oplus W = X_\infty$. By direct inspection, the maximal invariant set Ω_∞, for this example, satisfies $\Omega_\infty = \mathcal{X}$. Consequently, Assumptions 1–4 are all satisfied for this particular example. It is also clear that the maximal invariant set Ω_∞ is the fixed point of the mapping $\mathcal{B}(\cdot)$, i.e. the maximal invariant set Ω_∞ satisfies the set–equation $Y = \mathcal{B}(Y)$. However, the maximal invariant set Ω_∞ is not the unique set that solves the set–equation $Y = \mathcal{B}(Y)$ because the set $\hat{Y} = [-3, 3] \times \{0\}$ is an invariant set which is also a solution to the set equation $Y = \mathcal{B}(Y)$ as verified by noticing that $\hat{Y} \ominus W = [-3, 3] \times \{0\} \ominus [-1, 1] \times \{0\} = [-2, 2] \times \{0\}$ and verifying that:

$$\mathcal{B}(\hat{Y}) = \left(\begin{bmatrix} 2 & 0 \\ 0 & \beta^{-1} \end{bmatrix} ([-2, 2] \times \{0\})\right) \cap 3B_\infty = \hat{Y}.$$

Our telling example illustrates that the condition:

$$Y = \mathcal{B}(Y), \text{ i.e. } Y = \left(A^{-1}(Y \ominus W)\right) \cap \mathcal{X}, \tag{3.6}$$

is only a necessary condition for the set Y to be the maximal invariant set and, clearly, not a sufficient condition. In fact, there is no reason to expect that the mapping $\mathcal{B}(\cdot)$ admits the unique fixed point unless additional assumptions are invoked. The following observation is of help:

Proposition 2. Suppose Assumptions 1–4 hold. Then there exist a proper C set in R^n, say S, and a scalar $\theta \in [0, 1)$ such that:

$$\mathcal{R}(S) = AS \oplus W \subseteq \theta S \text{ and } X_\infty \subseteq S \subseteq \mathcal{X}. \tag{3.7}$$

We discuss in Section 4 some practical choices of the set S and the corresponding scalar $\theta \in [0, 1)$. When Assumptions 1–4 hold Proposition 2 justifies the following hypothesis that we utilize in our analysis.

Hypothesis 1. The set $S \in \text{ComCP}(\mathcal{X})$ and scalar $\theta \in (0, 1)$ are such that $AS \oplus W \subseteq \theta S$.

Lemmata 1 and 2 imply that the mapping $\mathcal{B}(\cdot)$ is additive with respect to the set intersection:

Lemma 4. Consider the mapping $\mathcal{B}(\cdot)$ given by (2.7). Let S_1 and S_2 be any two arbitrary elements of the collection of sets $\text{Com}(\mathcal{X}, X_\infty)$ given by (2.6). Then:

$$\mathcal{B}(S_1 \cap S_2) = \mathcal{B}(S_1) \cap \mathcal{B}(S_2) \tag{3.8}$$

and $\mathcal{B}(S_1 \cap S_2) \neq \emptyset$, $\mathcal{B}(S_1) \neq \emptyset$ and $\mathcal{B}(S_2) \neq \emptyset$.

The mapping $\mathcal{B}(\cdot)$ is an invariance preserving mapping:

Lemma 5. Consider the mappings $\mathcal{R}(\cdot)$ and $\mathcal{B}(\cdot)$ given, respectively, by (2.3) and (2.7). Let Y be any arbitrary element of the collection of sets $\text{ComInv}(A, W, \mathcal{X})$ given by (2.4). Then:

$$Y \subseteq \mathcal{B}(Y),\ \mathcal{R}(\mathcal{B}(Y)) \subseteq Y \text{ and } \mathcal{R}(\mathcal{B}(Y)) \subseteq \mathcal{B}(Y). \tag{3.9}$$

As the final ingredient, we introduce an auxiliary set sequence $\{\Gamma_k\}_{k=0}^{\infty}$ obtained by the simple, but adequate, scaling of an invariant set S satisfying Hypothesis 1. Let, for a given set S and scalar θ satifying Hypothesis 1,:

$$\underline{\gamma} := \sup_{\gamma}\{\gamma \ : \ \gamma S \subseteq \mathcal{X},\ \gamma \in R_+\}, \tag{3.10a}$$

$$\bar{\gamma} := \inf_{\gamma}\{\gamma \ : \ \mathcal{X} \subseteq \gamma S,\ \gamma \in R_+\} \text{ and} \tag{3.10b}$$

$$\Gamma_k := \theta^{-k}\underline{\gamma}S \text{ for all } k \in N. \tag{3.10c}$$

The following simple fact is of help.

Lemma 6. Suppose Assumptions 1–4 hold and consider the set sequence $\{\Gamma_k\}_{k=0}^{\infty}$ given by (3.10) where the set S and a scalar θ satisfy Hypothesis 1. Then for all $k \in N$: (i) $\Gamma_k \subset \Gamma_{k+1}$, (ii) $\mathcal{R}(\Gamma_{k+1}) \subseteq \Gamma_k$ and (iii) $\mathcal{R}(\Gamma_k) \subseteq \Gamma_k$.

The set sequence $\{\Gamma_k\}_{k=0}^{\infty}$ is utilized for generating improving inner invariant approximations of the maximal invariant set via the set sequence $\{\tilde{\Gamma}_k\}_{k=0}^{\infty}$ given by:

$$\tilde{\Gamma}_k := \Gamma_k \cap \Omega_k,\ k \in N, \tag{3.11}$$

where sets Γ_k and Ω_k are given, respectively, by (3.10) and (2.8). Sets Ω_k are iterates of the viability algorithm and are not necessarily invariant sets. Our next result, however, establishes invariance of sets $\tilde{\Gamma}_k$ for any $k \in N$.

Proposition 3. Suppose Assumptions 1–4 hold and consider the set sequence $\{\tilde{\Gamma}_k\}_{k=0}^{\infty}$ given by (3.11) where sets Ω_k and Γ_k are given, respectively, as in (2.8) and (3.10) and where the set S and a scalar θ satisfy Hypothesis 1. Then: (i) $\mathcal{R}(\tilde{\Gamma}_{k+1}) \subseteq \tilde{\Gamma}_k$ for all $k \in N$, (ii) $\tilde{\Gamma}_k \subseteq \tilde{\Gamma}_{k+1}$ for all $k \in N$ and (iii) sets $\tilde{\Gamma}_k$ are invariant sets for the system (2.1) and constraint set (2.2) for any $k \in N$.

Hence, by Proposition 3, the auxiliary set sequence $\{\Gamma_k\}_{k=0}^{\infty}$ is *an invariance and monotonicity correction sequence for iterates* Ω_k *of the viability algorithm* (2.8). In fact, the set sequence $\{\tilde{\Gamma}_k\}_{k=0}^{\infty}$ enables us to demonstrate that the set–dynamics (3.5) for an adequate initial condition result in the trajectory $\{Y_k\}_{k=0}^{\infty}$ which is a sequence of monotonically non–decreasing, invariant sets converging to the maximal invariant set Ω_∞. To simplify our statements we invoke:

Assumption 5. The set sequence $\{\Omega_k\}_{k=0}^{\infty}$ is generated by the standard viability algorithm (2.8). The set sequence $\{\tilde{\Gamma}_k\}_{k=0}^{\infty}$ is given by (3.11) where sets Ω_k and Γ_k are given, respectively, by (2.8) and (3.10) and where the set S and a scalar θ satisfy Hypothesis 1. The set sequence $\{Y_k\}_{k=0}^{\infty}$ is the trajectory of the set–dynamics (3.5) with initial condition $Y_0 = \tilde{\Gamma}_0$.

Proposition 4. Suppose Assumptions 1–5 hold. Then for all $k \in N$: (i) $\tilde{\Gamma}_k \subseteq Y_k \subseteq \Omega_k$, (ii) $Y_k \subseteq Y_{k+1} \subseteq \Omega_{k+1}$ and (iii) $\mathcal{R}(Y_k) \subseteq Y_k$, i.e. sets Y_k are invariant for any $k \in N$.

We now turn our attention to the convergence issues and the estimates of the Hausdorff distance between the terms of the set sequences $\{\tilde{\Gamma}_k\}_{k=0}^{\infty}$ and $\{Y_k\}_{k=0}^{\infty}$ and the maximal invariant set Ω_∞.

Proposition 5. Suppose Assumptions 1–5 hold. Then for all $k \in N$: (i) $\tilde{\Gamma}_k \subseteq Y_k \subseteq \Omega_\infty$, (ii) $H_L(\tilde{\Gamma}_k, \Omega_\infty) \leq H_L(\tilde{\Gamma}_k, \Omega_k) = h_L(\Omega_k, \tilde{\Gamma}_k)$, (iii) $H_L(Y_k, \Omega_\infty) \leq H_L(Y_k, \Omega_k) = h_L(\Omega_k, Y_k)$, and (iv) $H_L(Y_k, \Omega_\infty) \leq H_L(\tilde{\Gamma}_k, \Omega_\infty)$.

Remark 1. Clearly, by Proposition 5, we can derive an upper estimate of the Hausdorff distance between the terms of the set sequences $\{\tilde{\Gamma}_k\}_{k=0}^{\infty}$ and $\{Y_k\}_{k=0}^{\infty}$ and the maximal invariant set Ω_∞ without actually employing the maximal invariant set Ω_∞. The proposed upper estimates can be obtained by computing the Hausdorff semi–distance between iterates Ω_k of the viability algorithm (2.8) and the terms of the set sequences $\{\tilde{\Gamma}_k\}_{k=0}^{\infty}$ and $\{Y_k\}_{k=0}^{\infty}$. Computationally simpler upper estimates can be obtained by utilizing the fact that sets Ω_k, $\tilde{\Gamma}_k$ and Y_k are, under Assumptions 1–4, proper C sets in R^n. To this end, let, for any $k \in N$,:

$$\phi_k := \inf_{\phi}\{\phi \ : \ \Omega_k \subseteq \phi Y_k,\ \phi \in R_+\}, \tag{3.12a}$$

$$\varphi_k := \inf_{\varphi}\{\varphi \ : \ Y_k \subseteq \varphi L,\ \varphi \in R_+\}, \tag{3.12b}$$

$$\sigma_k := \inf_{\sigma}\{\sigma \ : \ \Omega_k \subseteq \sigma\tilde{\Gamma}_k,\ \sigma \in R_+\} \text{ and} \tag{3.12c}$$

$$\varsigma_k := \inf_{\varsigma}\{\varsigma \ : \ \tilde{\Gamma}_k \subseteq \varsigma L,\ \varsigma \in R_+\}. \tag{3.12d}$$

Utilizing Proposition 5 and (3.12) the guaranteed upper estimates of the Hausdorff distance between the terms of the set sequences $\{\tilde{\Gamma}_k\}_{k=0}^{\infty}$ and $\{Y_k\}_{k=0}^{\infty}$ and the maximal invariant set Ω_∞ are given, for all $k \in N$, by:

$$H_L(Y_k, \Omega_\infty) \leq (\phi_k - 1)\varphi_k \text{ and } H_L(\tilde{\Gamma}_k, \Omega_\infty) \leq (\sigma_k - 1)\varsigma_k.$$

Slightly weaker, but guaranteed and explicit, upper estimates as well as an explicit upper estimate for the determinedness index can be obtained by utilizing Lemma 3 as outlined next. By Proposition 4, our assumptions, and (3.10) we have the following relations for all $k \in N$:

$$\tilde{\Gamma}_k \subseteq Y_k \subseteq \Omega_\infty \subseteq \Omega_k \subseteq \mathcal{X} \subseteq \bar{\gamma}S. \tag{3.13}$$

Since, by (3.10) and (3.11), $\tilde{\Gamma}_k = \theta^{-k}\underline{\gamma}S \cap \Omega_k$ and by (3.13) $\Omega_k = \Omega_k \cap \bar{\gamma}S$, it follows that, for all $k \in N$,:

$$\theta^{-k}\underline{\gamma}S \cap \Omega_k \subseteq Y_k \subseteq \Omega_\infty \subseteq \Omega_k = \bar{\gamma}S \cap \Omega_k. \tag{3.14}$$

The relations (3.14) clearly become set equalities when $\bar{\gamma}S \cap \Omega_k \subseteq \theta^{-k}\underline{\gamma}S \cap \Omega_k$. Due to our assumptions and Hypothesis 1, S is a proper C set in R^n, $\theta \in (0, 1)$ and $\bar{\gamma} \geq \underline{\gamma} > 1$ are finite so that:

$$\bar{\gamma}S \subseteq \theta^{-k}\underline{\gamma}S \Rightarrow \bar{\gamma}S \cap \Omega_k \subseteq \theta^{-k}\underline{\gamma}S \cap \Omega_k. \tag{3.15}$$

The set inclusion $\bar{\gamma}S \subseteq \theta^{-k}\underline{\gamma}S$ is true if $\bar{\gamma} \leq \theta^{-k}\underline{\gamma}$ or, equivalently, for all $k \geq \bar{k}$ where $\bar{k}$ is given by:

$$\bar{k} := \min_{k}\{k \ : \ k \geq \frac{\ln \underline{\gamma} - \ln \bar{\gamma}}{\ln \theta},\ k \in N\}. \tag{3.16}$$

The formula (3.16) *provides an explicit upper estimate of the determinedness index* $\bar{k}$*, that can be evaluated directly, prior to viability computations* (2.8)*, by merely evaluating* (3.10) *and* (3.16).

Our main results are summarized by:

Theorem 1. Suppose Assumptions 1–5 hold. Then for all $k \in N$: (i) $\tilde{\Gamma}_k \subseteq \tilde{\Gamma}_{k+1}$ and $\tilde{\Gamma}_k$ is invariant set for the system (2.1) and constraint set (2.2), (ii) $Y_k \subseteq Y_{k+1}$ and Y_k is invariant set for the system (2.1) and constraint set (2.2), (iii) $H_L(\tilde{\Gamma}_k, \Omega_\infty) \leq \max\{0, (\bar{\gamma} - \theta^{-k}\underline{\gamma})\}h_L(S, \{0\})$, (iv) $H_L(Y_k, \Omega_\infty) \leq \max\{0, (\bar{\gamma} - \theta^{-k}\underline{\gamma})\}h_L(S, \{0\})$. Furthermore, for all $k \in N$ such that $k \geq \bar{k}$ where $\bar{k}$ is given by (3.16) it holds that (v) $H_L(\tilde{\Gamma}_k, \Omega_\infty) = 0$ and (vi) $H_L(Y_k, \Omega_\infty) = 0$.

4. COMPUTATIONAL REMARKS & SPECIAL CASES

Additional structure of disturbance and state constraint sets, W and $\mathcal{X}$, and the set S permits the characterization of some specific families of well behaved inner invariant approximations of the maximal invariant set Ω_∞.

The set S and a scalar $\theta \in (0,1)$ satisfying Hypothesis 1 can be potentially obtained by employing the standard convex optimization techniques. Restricting the set S to be an ellipsoid, say $S_\mathcal{E} := \{x \in R^n \ : \ x'Px \leq 1\}$, where $P \in R^{n\times n}$ is a positive definite symmetric matrix, the following set of constraints:

$$AS_\mathcal{E} \oplus W \subseteq \theta_\mathcal{E} S_\mathcal{E},\ S_\mathcal{E} \in \mathrm{ComCP}(\mathcal{X}),\ \theta_\mathcal{E} \in (0,1) \quad (4.1)$$

can be posed, under relatively mild assumptions on W, as an optimization problem involving linear matrix inequalities (Boyd *et al.*, 1994). The following facts are worth noticing and are relevant when Assumptions 1–4 hold and, in addition, W and $\mathcal{X}$ are polytopes and an ellipsoid $S_\mathcal{E}$ and a scalar $\theta_\mathcal{E} \in (0,1)$ satisfy (4.1):

Remark 2. The computation of the trajectory of set–dynamics (3.5) with the initial condition $Y_0 = \underline{\gamma} S_\mathcal{E}$ where $\underline{\gamma}$ is given as in (3.10) and when $S_\mathcal{E}$ (ellipsoidal set) and a scalar $\theta_\mathcal{E} \in (0,1)$ satisfy (4.1), i.e. the set sequence $\{Y_k\}_{k=0}^\infty$, is computationally expensive due to the necessity to compute the Minkowski (Pontryagin) set differences between sets $Y_k,\ k \in N$ and the polytope W. *However, the computation of the set sequence* $\{\tilde{\Gamma}_k\}_{k=0}^\infty$ *given by* (3.11) *is rather simple and direct since sets* $\Gamma_k = \theta_\mathcal{E}^{-k}\underline{\gamma} S_\mathcal{E}$ *specified by* (3.10) *are, in this case, proper* C *ellipsoidal sets in* R^n *and sets* Ω_k *are proper* C *polytopes in* R^n. *The inner invariant approximations of the maximal invariant set* Ω_∞, *sets* $\tilde{\Gamma}_k = \Gamma_k \cap \Omega_k,\ k \in N$ *are, in this case, given by the intersection of an ellipsoid and a polytope for a finite number of integers and converge, in finite time, to the maximal invariant set* Ω_∞ *which is polytopic. We refer to sets* $\tilde{\Gamma}_k$ *as semi–ellipsoidal invariant sets and provide an illustrative example in Section 5.*

Similarly as in (Raković, 2007), when an ellipsoidal set $S_\mathcal{E}$ and a scalar $\theta_\mathcal{E} \in (0,1)$ satisfy (4.1) there exist of a proper C polytope in R^n, say $S_\mathcal{P}$, and a scalar $\theta_\mathcal{P} \in (0,1)$ such that, for any given $\delta \in (0, 1-\theta_\mathcal{E})$,:

$$(\theta_\mathcal{E} + \delta) S_\mathcal{E} \subseteq S_\mathcal{P} \subseteq S_\mathcal{E} \text{ and} \quad (4.2a)$$

$$AS_\mathcal{P} \oplus W \subseteq \theta_\mathcal{P} S_\mathcal{P} \text{ with } \theta_\mathcal{P} = \frac{\theta_\mathcal{E}}{\theta_\mathcal{E}+\delta} \in (0,1). \quad (4.2b)$$

When Assumptions 1–4 hold and, in addition, W and $\mathcal{X}$ are polytopes and a polytopic set $S_\mathcal{P}$ and a scalar $\theta_\mathcal{P} \in (0,1)$ satisfy (4.2b), the following facts are worth noticing for the computations:

Remark 3. In this case, the computation of both the trajectory of set–dynamics (3.5), i.e. the set sequence $\{Y_k\}_{k=0}^\infty$, with the initial condition $Y_0 = \underline{\gamma} S_\mathcal{P}$ where $\underline{\gamma}$ is given as in (3.10), and the set sequence $\{\tilde{\Gamma}_k\}_{k=0}^\infty$ given by (3.11) is computationally feasible. *Furthermore, the sets* Y_k *and* $\tilde{\Gamma}_k$ *are polytopic sets for all* $k \in N$ *and are well behaved inner invariant approximations of the maximal invariant set* Ω_∞ *converging to it in finite time.*

We also remark that for computational reasons one can alternatively utilize results of Proposition 1 for the computation of the set S and a scalar $\theta \in (0,1)$ satisfying Hypothesis 1 in the case when the set of inequalities (4.1) is not feasible but Assumptions 1– 4 are satisfied.

Remark 4. A special, but interesting, case is when the disturbance set is $W = \{0\}$. Essentially, the complete theoretical analysis of Section 3 is applicable directly with obvious modifications. In this case, Assumption 4 is implied directly by Assumption 2 and Hypothesis 1 is replaced by the requirement that the proper C set S and a scalar $\theta \in (0,1)$ are such that $AS \subseteq \theta S$ and $S \subseteq \mathcal{X}$. For example, it is direct to verify, by utilizing Lemma 4 and the fact that the iterates of (2.8) satisfy $\Omega_\infty \subseteq \Omega_{k+1} \subseteq \Omega_k \subseteq \mathcal{X}$ for all $k \in N$, that sets $Y_k,\ k \in N$ are given by:

$$Y_k = \bar{Y}_k \cap \Omega_k,\ k \in N, \text{ where} \quad (4.3a)$$

$$\bar{Y}_{k+1} := A^{-1}\bar{Y}_k,\ k \in N \text{ with } \bar{Y}_0 = \underline{\gamma} S, \quad (4.3b)$$

where $\underline{\gamma}$ is given as in (3.10). When $\mathcal{X}$ and S are proper C polytopes in R^n, sets $\bar{Y}_k$ and Ω_k, and consequently Y_k are also proper C polytopes in R^n for $k \in N$. When $\mathcal{X}$ and S are, respectively, proper C polytope and ellipsoid in R^n and the state transition matrix A is a non–singular matrix, sets $\bar{Y}_k$ and Ω_k are, respectively, proper C ellipsoids and polytopes in R^n for $k \in N$ and hence, sets Y_k are proper C semi–ellipsoidal sets in R^n for a finite number of integers (when the matrix A is singular sets $Y_k,\ k \in N$ remain proper C sets in R^n but might not be semi–ellipsoidal sets in the proper sense.). In either case, the set sequence $\{Y_k\}_{k=0}^\infty$ converges, in finite time, to the maximal invariant set Ω_∞ which is a proper C polytope in R^n.

5. ILLUSTRATIVE EXAMPLES

We provide two illustrative examples.

Example 2. The first example is the system with:

$$A = 0.9\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 0.8916 & 0.1225 \\ -0.1225 & 0.8916 \end{bmatrix},$$

where $\theta = \frac{\pi}{23}$ and

$$W = 0.01B_\infty \text{ and } \mathcal{X} = 100B_\infty \cap \{x \in R^2 \ : \ x^2 \geq -20\}.$$

Hereafter B_∞ is the closed unit ∞–norm ball and x^i denotes the i^{th} coordinate of a vector x. We computed the set sequence $\{\tilde{\Gamma}_k\}_{k=0}^\infty$ given by (3.11) as indicated in Remark 2. The set $S_\mathcal{E}$ and a scalar $\theta_\mathcal{E}$ satisfying Hypoth-

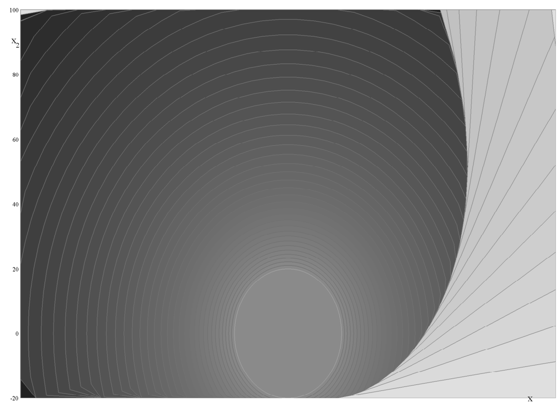

Fig. 1. The invariant sets $\tilde{\Gamma}_k$ and usual iterates Ω_k.

esis 1 are obtained by solving the linear matrix inequality problem specified in (4.1) and are utilized via (3.10) for the initial set $\tilde{\Gamma}_0$. The inner invariant approximations, sets $\tilde{\Gamma}_k$, of the maximal invariant set Ω_∞ are shown in darker gray–scale shading in Figure 1. The iterates Ω_k of (2.8)

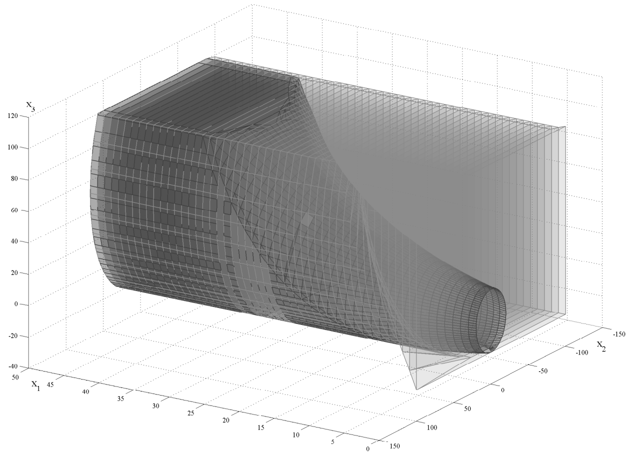

Fig. 2. The evolution of the "$\tilde{\Gamma}$" and "Ω" set–dynamics.

are depicted, in lighter gray–scale shading, in the same figure. Assertions of Propositions 3 and 4 are illustrated in Figure 1, where it is clear by inspection that $\tilde{\Gamma}_k \subseteq \Omega_k$ for all k. The finite time convergence of both sequences $\{\tilde{\Gamma}_k\}_{k=0}^{\infty}$ and $\{\Omega_k\}_{k=0}^{\infty}$ to the maximal invariant set Ω_∞ is also evident in Figures 1 and 2. Results of Propositions 3 and 4 and Theorem 1 and the fact that sets $\tilde{\Gamma}_k$ are proper C semi–ellipsoidal sets in R^2 for a finite number of integers are also illustrated in Figure 2 where the evolution of "$\tilde{\Gamma}$" and "Ω" set–dynamics is shown, respectively, in darker and lighter gray–scale shading.

Example 3. The second example is the system with:

$$A = 0.95 \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 0.9224 & 0.2273 \\ -0.2273 & 0.9224 \end{bmatrix},$$

where $\theta = \frac{\pi}{13}$ and $W = 0.1 B_\infty$, and

$$\mathcal{X} = \{x \in R^2 \ : \ -30 \le x^1 \le 70, \ -10 \le x^2 \le 20, \}.$$

The set $S_\mathcal{P}$ and a scalar $\theta_\mathcal{P}$ satisfying Hypothesis 1

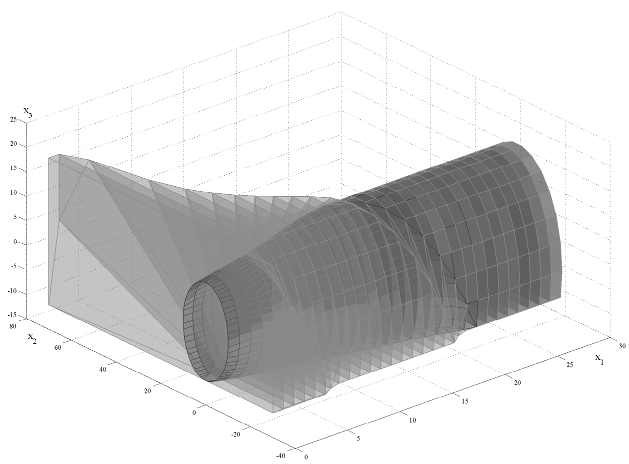

Fig. 3. The evolution of the "Y" and "Ω" set–dynamics.

are obtained by utilizing the minimal invariant set X_∞, given by (2.5), which is, for this particular example, computable explicitly and admits representation either with 52 inequalities or 52 extreme points. The inner invariant approximations, sets Y_k, $k \in N$, of the maximal invariant set Ω_∞ and sets Ω_k are shown, respectively, in darker and lighter gray–scale shading in Figure 3. Results of Propositions 3 and 4 and Theorem 1 are illustrated in Figure 3, where it is clear by inspection that $Y_k \subseteq \Omega_k$ for all k and that the set sequences $\{Y_k\}_{k=0}^{\infty}$ and $\{\Omega_k\}_{k=0}^{\infty}$ converge, in finite time, to the maximal invariant set Ω_∞. In this example sets Y_k and Ω_k are proper C polytopes in R^2 since $S_\mathcal{P}$ and $\mathcal{X}$ are also proper C polytopes in R^2.

6. CONCLUDING REMARKS

We offered a method for the computation of invariant approximations of the maximal invariant set for constrained linear discrete time systems subject to bounded, additive, disturbances. Under mild assumptions, inner invariant approximations converge, in finite time, to the maximal invariant set. We derived estimates of the Hausdorff distance between the underlying iterates and the maximal invariant set and the determinedness index.

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