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Robust Tracking Controller Backstepping Design for SISO Uncertain Nonlinear System★

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Abstract: Controller design problem is dealt with for a class of plants with time-varying nonlinear uncertainties and unmodeled dynamics. A new method based on signal compensation is proposed to design a robust controller. A controller designed by this method consists of a nominal controller and a robust compensator. It is shown that robust tracking property and robust tracking transient performance can be achieved simultaneously. A salient feature of our results, shown in the present paper, is that the controller is linear and time-invariant one and we can tell the users how to tune on-line the parameters of the controller with the proposed robust and transient performance.

1. INTRODUCTION

Recently, global stabilization of nonlinear strict-feedback form systems has attracted considerable attention. In this paper, our attention is focused on the problem of robust controller backstepping design for a class of SISO uncertain nonlinear strict-feedback form systems with time-varying uncertainties and stable dynamic uncertainties. Many researches have been done on this problem, but it is still not solved completely.

Over the last decade, various robust stabilization techniques have been developed in the literature. Following the development of exact linearization techniques via smooth state feedback and diffeomorphic transformation (Isidori, 1989), researches on certain triangular structure systems attracted considerable interest. One of the recent breakthroughs in nonlinear control is the introduction of backstepping algorithms for feedback linearizable systems. The relative-degree constraint, overparameterization, and growth condition are removed by allowing the controlled plant to be nonlinearly dependent on structure uncertainty, such as unknown parameters or unmodeled time-varying disturbances. Although the idea of integrator backstepping may be implicit in some earlier works, its use as a design tool was initiated by Tsinias(1989,1991), Byrnes and Isidori(1989), Sontag and Sussmann(1988), and Saberi et al.(1990). In Kanellakopoulos, Kokotovic, and Morse(1991), the cases with unknown parameters were investigated. In Freeman and Kokotovic (1992,1993), Marino and Tomei(1993), the cases with disturbance were considered. In Taylor, Kokotovic and kanellokopoulos(1989), Jiang and Mareels(1997), Jiang and Hill(1999), the cases with unmodeled Dynamics were treated.

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In this paper, we consider robust control problem for a class of nonlinear strict-feedback form systems with uncertain parameters, bounded uncertainties and unmodeled dynamics. The uncertainties are norm-bounded, and the unmodeled dynamics is input-to-state stable as described in other papers. A new method is proposed based on signal compensation (Zhong, 2002) to design a robust controller. It will be shown that the states of the closed-loop system are ensured to be bounded and the global output tracking to a reference signal are achieved. And desired transient tracking property can also be assured The tracking error can be driven into an desired small neighborhood of the origin at an exponential rate. A controller designed by this method consists of a nominal controller and a robust compensator. The controller is linear and time invariant.

This paper is organized as follows. In section 2, the plant description, the reference model, the assumptions on the uncertainties, reference input and unmodeled dynamics are presented. In section 3, controller design method and main results are stated. Section 4 gives the statement and the proof of the main results. An example is shown in section 5. Conclusions are stated in section 6.

2. PROBLEM DESCRIPTION

Consider a nonlinear plant described by the following equations

$$\Sigma_x \begin{cases} \dot{x}_1(t) = x_2(t) + \phi_1(x,\eta,d,t) \\ \dot{x}_2(t) = x_3(t) + \phi_2(x,\eta,d,t) \\ \quad\vdots \\ \dot{x}_{n-1}(t) = x_n(t) + \phi_{n-1}(x,\eta,d,t) \\ \dot{x}_n(t) = u(t) + \phi_n(x,\eta,d,t) \end{cases} \tag{1}$$

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$$\sum_{\eta} \qquad \dot{\eta}(t) = \phi_{\eta}(\eta, x, d, t) \tag{2}$$
$$y_p(t) = x_1(t)$$

where x_i are the states, y_p is the output, d is a bounded external disturbance vector, $\phi_i\ (i = 1, 2, \cdots, n)$ are regarded as nonlinear time-varying uncertainties and ϕ_η is a vector field describing dynamic uncertainties.

Assumption A

A1) There are known positive constant vector $\xi_{i[i]} = [\xi_{i1} \quad \xi_{i2} \quad \cdots \quad \xi_{ii}]$ and positive valued function ς_i such that

$$\left|\phi_i(x, \eta, d, t)\right| \le \xi_{i[i]} \left|x_{[i]}(t)\right|_\infty + \varsigma_i\left(\left\|d(t)\right\|_\infty, \left\|\eta(t_0)\right\|\right)$$

where $\varsigma_i(0,0) = 0\ (i = 1, 2, \cdots, n)$ 。

A2) The system given by (2) is bounded-inputs (x and d) bounded-state (η) stable.

It is required to design a controller which produces a control input u to drive the output y_p of the plant to track a reference signal, denoted by y_m , which is given by the following reference model:

$$\dot{y}_m(t) = -\alpha_1 y_m(t) + \beta_1 r(t) \tag{3}$$

where α_1 and β_1 are given positive constants, r is a given reference input.

Assumption B

$r \in C^1$ and there exist known and positive constants η_r and $\eta_{\dot{r}}$ such that

$$|r|_\infty \le \eta_r, \quad |\dot{r}|_\infty \le \eta_{\dot{r}} \tag{4}$$

3. CONTROLLER DESIGN

Firstly, consider the subsystem

$$\dot{x}_1(t) = x_2(t) + \phi_1(x_1, \eta, d, t)$$

For the above system, a virtual controller is constructed as

$$\hat{x}_2(t) = u_1(t) + f_1 w_1(t) \tag{5}$$

where $u_1(t)$ is a nominal control input given by

$$u_1(t) = -\alpha_1 x_1(t) + \beta_1 r(t) \tag{6}$$

$w_1(t)$ is a robust control input to be designed, and f_1 is a positive constant to be determined. To perform backstep, apply the variable change

$$y_2(t) = x_2(t) - \hat{x}_2(t) = x_2(t) + \alpha_1 x_1(t) - \beta_1 r(t) - f_1 w_1(t) \tag{7}$$

The robust control input $w_1(t)$ is given by

$$w_1(t) = -F_1(s)\hat{\phi}_1(t) \tag{8}$$

where $F_1(s)$ is a robust filter of the form

$$F_1(s) = \frac{1}{s + f_1}$$

and

$$\hat{\phi}_1 = \phi_1 + y_2(t)$$
$$= \phi_1 + x_2(t) + \alpha_1 y_1(t) + \alpha_1 y_m(t) - \beta_1 r(t) - f_1 w_1(t)$$

It should be pointed out that, to get the robust control input $w_1(t)$, only the states $x_1(t)$ and $r(t)$ are needed. If fact,

$$w_1(t) = -F_1(s)\hat{\phi}_1(t)$$
$$= -\left(1 + \frac{\alpha_1}{s}\right)x_1(t) + \frac{\beta_1}{s} r(t) \tag{9}$$

Define

$$y_1(t) = y_p(t) - y_m(t) = x_1(t) - y_m(t) \tag{10}$$

then one has

$$w_1(t) = -\left(1 + \frac{\alpha_1}{s}\right)y_1(t) \tag{11}$$

and

$$\dot{y}_1(t) = -\alpha_1 y_1(t) + \hat{\phi}_1(t) + f_1 w_1(t) \tag{12}$$

From (1), (7), (8) and (10), one has

$$\dot{y}_2(t) = x_3(t) + \tilde{\phi}_2(t) \tag{13}$$

where

$$\tilde{\phi}_2(t) = \phi_2(t) - \alpha_1^2 y_1(t) + (f_1 + \alpha_1)\left[\hat{\phi}_1(t) + f_1 w_1(t)\right] + \rho_m(t)$$
$$\rho_m(t) = \alpha_1^2 y_m(t) + \alpha_1\beta_1 r(t) - \beta_1 \dot{r}(t) \tag{14}$$

As the second step, consider the subsystem (13) with $\tilde{\phi}_2(t)$ as a disturbance and $x_3(t)$ as a virtual control input and continue the design procedure. At the k th-step, consider the subsystem

$$\dot{y}_k(t) = x_{k+1}(t) + \tilde{\phi}_k(t)$$

and regard $x_{k+1}(t)$ as a virtual control input with the form

$$\hat{x}_{k+1}(t) = u_k(t) + f_k w_k(t)$$

where $u_k(t)$ is a nominal control input given by

$$u_k(t) = -\alpha_k y_k(t)$$

where α_k is a positive constant. Let

$$y_{k+1}(t) = x_{k+1}(t) - \hat{x}_{k+1}(t)$$
$$= x_{k+1}(t) + \alpha_k y_k(t) - f_k w_k(t)$$

Then

$$\dot{y}_k(t) = -\alpha_k y_k(t) + \hat{\phi}_k(t) + f_k w_k(t)$$

where

$$\hat{\phi}_k(t) = \tilde{\phi}_k(t) + y_{k+1}(t)$$

The robust control input $w_k(t)$ is constructed as

$$w_k(t) = -F_k(s)\hat{\phi}_k(t)$$
$$F_k(s) = \frac{1}{s+f_k}$$

where f_k is a positive constant to be determined.
Note that

$$\hat{\phi}_k(t) = (s+\alpha_k)y_k(t) - f_k w_k(t)$$

So the robust control input $w_k(t)$ can also be given by

$$w_k(t) = -F_k(s)\hat{\phi}_k(t) = -\left(1+\frac{\alpha_k}{s}\right)y_k(t)$$

Differentiating $y_{k+1}(t)$, one has

$$\dot{y}_{k+1}(t) = x_{k+2}(t) + \tilde{\phi}_{k+1}(t)$$

where

$$\tilde{\phi}_{k+1}(t) = \phi_{k+1}(t) - \alpha_k^2 y_k(t) + (f_k+\alpha_k)\left[\hat{\phi}_k(t) + f_k w_k(t)\right]$$

Finally, one has

$$\dot{y}_n(t) = u(t) + \tilde{\phi}_n(t) \tag{15}$$

where

$$y_n(t) = x_n(t) + \alpha_{n-1}y_{n-1}(t) - f_{n-1}w_{n-1}(t)$$
$$\tilde{\phi}_n(t) = \phi_n(t) - \alpha_{n-1}^2 y_{n-1}(t) + (f_{n-1}+\alpha_{n-1})\left[\hat{\phi}_{n-1}(t) + f_{n-1}w_{n-1}(t)\right]$$

The control input $u(t)$ is constructed as

$$u(t) = u_n(t) + f_n w_n(t)$$

with the nominal control input $u_n(t)$ is given by

$$u_n(t) = -\alpha_n y_n(t) \tag{16}$$

and the robust control input $w_n(t)$ by

$$w_n(t) = -F_n(s)\hat{\phi}_n(t)$$
$$F_n(s) = \frac{1}{s+f_n}$$

where $\hat{\phi}_n(t) = \tilde{\phi}_n(t)$, α_n and f_n are positive constants .
Then the following differential equation holds

$$\dot{w}_n(t) = -f_n w_n(t) - \hat{\phi}_n(t)$$

and $w_n(t)$ can be implemented as

$$w_n(t) = -F_n(s)\hat{\phi}_n(t) = -\left(1+\frac{\alpha_n}{s}\right)y_n(t)$$

From (15) and (16) it follows that

$$\dot{y}_n(t) = -\alpha_n y_n(t) + \hat{\phi}_n(t) + f_n w_n(t)$$

Summarizing the design results, one has

$$\begin{bmatrix}\dot{y}_k(t)\\ \dot{w}_k(t)\end{bmatrix} = \begin{bmatrix}-\alpha_k & f_k\\ 0 & -f_k\end{bmatrix}\begin{bmatrix}y_k(t)\\ w_k(t)\end{bmatrix} + \begin{bmatrix}1\\ -1\end{bmatrix}\hat{\phi}_k(t),$$
$$k = 1,2,\cdots,n$$

and the whole controller description

$$\begin{aligned}
u(t) &= u_n(t) + f_n w_n(t)\\
y_1(t) &= x_1(t) - y_m(t)\\
y_2(t) &= x_2(t) + \alpha_1 y_1(t) - f_1 w_1(t) + \alpha_1 y_m(t) - \beta_1 r(t)\\
y_k(t) &= x_k(t) + \alpha_{k-1}y_{k-1}(t) - f_{k-1}w_{k-1}(t), k = 3,4,\cdots,n\\
u_1(t) &= -\alpha_1 y_1(t) - \alpha_1 y_m(t) + \beta_1 r(t)\\
u_k(t) &= -\alpha_k y_k(t), \quad k = 2,3,\cdots,n\\
w_k(t) &= -\left(1+\frac{\alpha_k}{s}\right)y_k(t), \quad k = 1,2,\cdots,n
\end{aligned} \tag{17}$$

One sees that the designed controller is a linear time-invariant one.

4. CLOSED-LOOP CONTROL PROPERTIES

Let

$$x(t) = \begin{bmatrix}x_1(t) & x_2(t) & \cdots & x_n(t)\end{bmatrix}^T$$
$$w(t) = \begin{bmatrix}w_1(t) & w_2(t) & \cdots & w_n(t)\end{bmatrix}^T$$
$$\tau(t) = \begin{bmatrix}\tau_1(t) & \tau_2(t) & \cdots & \tau_n(t)\end{bmatrix}^T = \begin{bmatrix}y_m(t) & -\alpha_1 y_m(t) + \beta_1 r(t) & 0 & \cdots & 0\end{bmatrix}^T$$

Assumption C

The parameters $f_i\,(i = 1,2,\cdots,n)$ are sufficiently large, f_{i+1} is much larger than $f_i\,(i = 1,2,\cdots,n-1)$ and

$$f_1 \gg \max\{\beta_1, \alpha_i, \xi_{ij}, i, j = 1,2,\cdots,n, i \ge j\}$$
$$f_1 \gg \max\left\{\|\tau\|_\infty, \|\rho_m\|_\infty, \varsigma_j\left(\|d\|_\infty, \|\eta(t_0)\|\right), j = 1,2,\cdots,n\right\}$$

It will be shown that the conclusions stated in the following theorem hold.

Theorem 1. For any given and bounded initial states $x(t_0)$ and $w(t_0)$ and for any given positive constant ε, one can find sufficiently large parameters which satisfy Assumption C, such that the states $x(t)$, $\eta(t)$ and $w(t)$ are bounded and the following statements hold.

(1) If the initial states $x(t_0)$ and $w(t_0)$ are nonzero, then there is a $T \ge t_0$ such that

$$|y_k(t)| \le \varepsilon, \quad |w_k(t)| \le \varepsilon, \quad t \ge T, \quad k = 1,2,\cdots,n$$

(2) If the initial states $x(t_0)$ and $w(t_0)$ are zero, then

$$\left|y_k(t)\right| \le \varepsilon, \quad \left|w_k(t)\right| \le \varepsilon, \quad t \ge t_0, \quad k = 1,2,\cdots,n$$

Before showing the proof of the main results stated in Theorem 1, several preliminary lemmas are stated and proven.

Lemma 1. Let

$$\upsilon(f) = \left[\upsilon_{ij}(f)\right] \triangleq \begin{bmatrix} 1 & & & & 0 \\ -f_1-\alpha_1 & 1 & & & \\ & -f_2-\alpha_2 & 1 & & \\ 0 & & \ddots & 1 & \\ & & & -f_{n-1}-\alpha_{n-1} & 1 \end{bmatrix}^{-1}$$

then

$$\upsilon_{ij}(f) = 1, i = j$$

$$\upsilon_{ij}(f) = \prod_{k=j}^{i-1}(f_k + \alpha_k), i > j$$

$$\upsilon_{ij}(f) = 0, i < j$$

Proof. The proof is omitted since it is straightforward.

Let

$$\mu_{y11} = \xi_{11}, \mu_{yk[k]} = \begin{bmatrix} \xi_{k1} + \alpha_1\xi_{k2} \\ \xi_{k2} + \alpha_2\xi_{k3} \\ \vdots \\ \xi_{k(k-1)} + \alpha_{k-1}\xi_{kk} \\ \xi_{kk} \end{bmatrix}^T,$$

$$\mu_{wk[k-1]}(f) = \begin{bmatrix} f_1\xi_{k2} \\ f_2\xi_{k3} \\ \vdots \\ f_{k-1}\xi_{kk} \end{bmatrix}^T, \quad k = 2,3,\cdots,n$$

Lemma 2. $\phi_k\ (k = 1,2,\cdots,n)$ satisfy that

$$\left|\phi_1(x,\eta,d,t)\right| \le \xi_{11}\left|y_1(t)\right|_\infty + \xi_{11}\left|\tau_1(t)\right|_\infty + \varsigma_1\left(\left\|d(t)\right\|_\infty, \left\|\eta(t_0)\right\|\right)$$

$$\begin{aligned}\left|\phi_k(x,\eta,d,t)\right| \le\ & \mu_{yk[k]}\left|y_{[k]}(t)\right|_\infty \\ & + \mu_{wk[k-1]}(f)\left|w_{[k-1]}(t)\right|_\infty \\ & + \xi_{k[k]}\left|\tau_{[k]}(t)\right|_\infty \\ & + \varsigma_k\left(\left\|d(t)\right\|_\infty, \left\|\eta(t_0)\right\|\right)\end{aligned}$$

$$k = 2,3,\cdots,n$$

Proof. From the definition of $y_k\ (k = 1,2,\cdots,n)$ it follows that

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} - \begin{bmatrix} 0 \\ \alpha_1 y_1(t) \\ \vdots \\ \alpha_{n-1}y_{n-1}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f_1 w_1(t) \\ \vdots \\ f_{n-1}w_{n-1}(t) \end{bmatrix} + \tau(t) \tag{18}$$

So one has

$$\begin{aligned} &\xi_{1[1]}\left|x_{[1]}(t)\right| \le \xi_{11}\left|y_1(t)\right| + \xi_{11}\left|\tau_1(t)\right| \\ &\xi_{k[k]}\left|x_{[k]}(t)\right| \le \mu_{yk[k]}\left|y_{[k]}(t)\right| + \xi_{k[k]}\left|\tau_{[k]}(t)\right| \\ &\qquad + \mu_{wk[k-1]}(f)\left|w_{[k-1]}(t)\right| \\ &k = 2,3,\cdots,n \end{aligned} \tag{19}$$

From Assumption A and inequalities (19), one sees that the conclusions of Lemma 2 hold.

Let $\upsilon_{k0}(f) = 0, \quad k = 2,3,\cdots,n$.

Lemma 3. $\hat{\phi}_k(t)\left(k = 1,2,\cdots,n\right)$ satisfy the following inequalities.

$$\begin{aligned} &\left|\hat{\phi}_1(t)\right| \le \left|y_2(t)\right|_\infty + \hat{\mu}_{y11}\left|y_1(t)\right|_\infty + \hat{\varsigma}_1(t) \\ &\left|\hat{\phi}_k(t)\right| \le \left|y_{k+1}(t)\right|_\infty + \hat{\mu}_{yk[k]}(f)\left|y_{[k]}(t)\right|_\infty \\ &\qquad + \hat{\mu}_{wk[k-1]}(f)\left|w_{[k-1]}(t)\right|_\infty + \hat{\varsigma}_k(f,t) \\ &\quad k = 2,3,\cdots,n-1 \\ &\left|\hat{\phi}_n(t)\right| \le \hat{\mu}_{yn[n]}(f)\left|y_{[n]}(t)\right|_\infty + \hat{\varsigma}_n(f,t) \\ &\qquad + \hat{\mu}_{wn[n-1]}(f)\left|w_{[n-1]}(t)\right|_\infty \end{aligned} \tag{20}$$

where

$$\begin{aligned} &\hat{\mu}_{y11} = \xi_{11}, \\ &\hat{\varsigma}_1(t) = \xi_{11}\left|\tau_1(t)\right|_\infty + \varsigma_1\left(\left\|d(t)\right\|_\infty, \left\|\eta(t_0)\right\|\right) \\ &\hat{\mu}_{ykj}(f) = \left|\upsilon_{k(j-1)}(f) - \upsilon_{k(j+1)}(f)\alpha_j^2\right| + \sum_{i=j}^{k}\upsilon_{ki}(f)\mu_{yij} \\ &\qquad j = 1,2,\cdots,k \\ &\hat{\mu}_{wkj}(f) = \upsilon_{kj}(f)f_j + \sum_{i=j}^{k-1}\upsilon_{k(i+1)}(f)\mu_{w(i+1)j}(f) \\ &\quad j = 1,2,\cdots,k-1 \\ &\hat{\varsigma}_k(f,t) = \sum_{j=1}^{k}\upsilon_{kj}(f)\left[\xi_{j[j]}\left|\tau_{[j]}(t)\right|_\infty + \varsigma_k\left(\left\|d(t)\right\|_\infty, \left\|\eta(t_0)\right\|\right)\right] \\ &\qquad + \upsilon_{k2}(f)\left|\rho_m(t)\right|, \quad k = 2,3,\cdots,n \end{aligned} \tag{21}$$

Proof. From the definition of $\hat{\phi}_k(t)\left(k = 1,2,\cdots,n\right)$ and from Lemmas 1 and 2, one has that

$$\begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \\ \hat{\phi}_3(t) \\ \vdots \\ \hat{\phi}_{n-1}(t) \\ \hat{\phi}_n(t) \end{bmatrix} = \upsilon(f) \begin{bmatrix} y_2(t) + \phi_1(t) \\ y_3(t) - \alpha_1^2 y_1(t) + \phi_2(t) + (f_1 + \alpha_1) f_1 w_1(t) + \rho_m(t) \\ y_4(t) - \alpha_2^2 y_2(t) + \phi_3(t) + (f_2 + \alpha_2) f_2 w_2(t) \\ \vdots \\ y_n(t) - \alpha_{n-2}^2 y_{n-2}(t) + \phi_{n-1}(t) + (f_{n-2} + \alpha_{n-2}) f_{n-2} w_{n-2}(t) \\ -\alpha_{n-1}^2 y_{n-1}(t) + \phi_n(t) + (f_{n-1} + \alpha_{n-1}) f_{n-1} w_{n-1}(t) \end{bmatrix}$$

Together with Assumption A and Lemmas 1 and 2, the conclusions of Lemma 3 hold.

Lemma 4. For any given positive constant ε_ϕ, if Assumption C is satisfied, then

$$\frac{|\hat{\phi}_1(t)|}{\sqrt{f_1}} \le \varepsilon_\phi \left[\|y_{[2]}(t)\|_\infty + 1 \right] \tag{22}$$

$$\frac{|\hat{\phi}_k(t)|}{\sqrt{f_k}} \le \varepsilon_\phi \left[\|y_{[k+1]}(t)\|_\infty + \|w_{[k-1]}(t)\|_\infty + 1 \right] \tag{23}$$

$$k = 2, 3, \cdots, n-1$$

$$\frac{|\hat{\phi}_n(t)|}{\sqrt{f_n}} \le \varepsilon_\phi \left[\|y_{[n]}(t)\|_\infty + \|w_{[n-1]}(t)\|_\infty + 1 \right] \tag{24}$$

Proof. From the definition of $\hat{\mu}_{ykj}(f)$ and $\hat{\mu}_{wkj}(f)$, one sees that for any given positive constant ε_k, if $f_j \ (j = 1, 2, \cdots, n)$ are sufficiently large and satisfy Assumption C, then

$$\frac{1}{\sqrt{f_k}} \le \varepsilon_k, \quad k = 1, 2, \cdots, n$$

$$\frac{|\hat{\mu}_{ykj}(f)|}{\sqrt{f_k}} \le \varepsilon_k, \quad k = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, k$$

$$\frac{|\hat{\mu}_{wkj}(f)|}{\sqrt{f_k}} \le \varepsilon_k, \quad k = 2, 3, \cdots, n; \quad j = 1, 2, \cdots, k-1$$

$$\frac{|\hat{\varsigma}_k(f,t)|_\infty}{\sqrt{f_k}} \le \varepsilon_k, \quad k = 1, 2, \cdots, n \tag{25}$$

We only show the proof of (23). The proof of (22) and (24) can be performed in a similar way. For $k = 2, 3, \cdots, n-1$, from (20) and (25), one has

$$\frac{|\hat{\phi}_k(t)|}{\sqrt{f_k}} \le \frac{1}{\sqrt{f_k}} |y_{k+1}(t)|_\infty + \frac{\hat{\mu}_{yk[k]}(f)}{\sqrt{f_k}} |y_{[k]}(t)|_\infty + \frac{\hat{\mu}_{wk[k-1]}(f)}{\sqrt{f_k}} |w_{[k-1]}(t)|_\infty + \frac{1}{\sqrt{f_k}} \hat{\varsigma}_k(f,t)$$

$$\le \varepsilon_k \left[\sum_{i=1}^{k+1} |y_i(t)|_\infty + \sum_{i=1}^{k-1} |w_i(t)|_\infty + 1 \right]$$

$$\le \varepsilon_k \left[\sqrt{k+1} \|y_{[k+1]}(t)\|_\infty + 1 + \sqrt{k-1} \|w_{[k-1]}(t)\|_\infty \right]$$

If choose ε_k such that $\varepsilon_k \sqrt{k+1} \le \varepsilon_\phi$, then (23) holds.

Now we are ready to prove Theorem 1. For $k = 1, 2, \cdots, n$, let

$$P_k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Consider the following positive function

$$V = \sum_{k=1}^{n} v_k$$

where

$$v_k = \begin{bmatrix} y_k(t) & w_k(t) \end{bmatrix} P_k \begin{bmatrix} y_k(t) \\ w_k(t) \end{bmatrix}$$

The derivative of V along the trajectories of the closed-loop system is given by

$$\begin{aligned} \dot{V} &= \sum_{k=1}^{n} \dot{v}_k \\ &= -2 \sum_{k=1}^{n} \left[\alpha_k y_k^2(t) + \alpha_k y_k(t) w_k(t) + f_k w_k^2(t) + w_k(t) \hat{\phi}_k(t) \right] \\ &= -\sum_{k=1}^{n} \left[\alpha_k v_k + \alpha_k y_k^2(t) + 2(f_k - \alpha_k) w_k^2(t) + 2 w_k(t) \hat{\phi}_k(t) \right] \\ &\le -\sum_{k=1}^{n} \left\{ \alpha_k v_k + \alpha_k y_k^2(t) + (f_k - 2\alpha_k) w_k^2(t) - \left[\frac{|\hat{\phi}_k(t)|}{\sqrt{f_k}} \right]^2 \right\} \\ &\le -\alpha_0 V + \sum_{k=1}^{n} \left[\frac{|\hat{\phi}_k(t)|}{\sqrt{f_k}} \right]^2 \end{aligned}$$

where $\alpha_0 = \min\{\alpha_k, k = 1, 2, \cdots, n\}$. So one has

$$V(t) \le e^{-\alpha_0(t-t_0)} V(t_0) + \frac{1}{\alpha_0} \sum_{k=1}^{n} \left[\frac{|\hat{\phi}_k(t)|_\infty}{\sqrt{f_k}} \right]^2 \tag{26}$$

From Lemma 4, it follows that for any given positive constant ε_ϕ, if Assumption C is satisfied, one has

$$\begin{aligned} \sum_{k=1}^{n} \left[\frac{|\hat{\phi}_k(t)|_\infty}{\sqrt{f_k}} \right]^2 &\le 2\varepsilon_\phi^2 \left[\|y_{[2]}(t)\|_\infty^2 + 1 \right] \\ &+ \sum_{k=2}^{n-1} 3\varepsilon_\phi^2 \left[\|y_{[k+1]}(t)\|_\infty^2 + \|w_{[k-1]}(t)\|_\infty^2 + 1 \right] \\ &+ 3\varepsilon_\phi^2 \left[\|y_{[n]}(t)\|_\infty^2 + \|w_{[n-1]}(t)\|_\infty^2 + 1 \right] \end{aligned}$$

$$\le 3\varepsilon_\phi^2\left\{(n+1)\left[\|y(t)\|_\infty^2+\|w(t)\|_\infty^2\right]+n\right\}$$
$$\le \frac{3\varepsilon_\phi^2(n+1)}{\lambda_0}\|V(t)\|_\infty+3\varepsilon_\phi^2 n \tag{27}$$

where $\lambda_0=\lambda_{\min}(P_k)=\left(3-\sqrt{5}\right)/2$. From (26) and (27) it follows that

$$\|V(t)\|_\infty \le V(t_0)+\frac{3\varepsilon_\phi^2(n+1)}{\lambda_0\alpha_0}\|V(t)\|_\infty+\frac{3\varepsilon_\phi^2 n}{\alpha_0}$$

If

$$\varepsilon_\phi<\sqrt{\frac{\lambda_0\alpha_0}{3(n+1)}} \tag{28}$$

then

$$\|V(t)\|_\infty \le \frac{1}{\pi}\left[V(t_0)+\frac{3\varepsilon_\phi^2 n}{\alpha_0}\right] \tag{29}$$

where $\pi=1-\dfrac{3\varepsilon_\phi^2(n+1)}{\lambda_0\alpha_0}$. From (26), (27) and (29) one has

$$\sum_{k=1}^{n}\left[y_k^2(t)+w_k^2(t)\right]\le\frac{e^{-\alpha_0(t-t_0)}}{\lambda_0}V(t_0)$$
$$+\frac{3\varepsilon_\phi^2}{\lambda_0\alpha_0}\left\{\frac{(n+1)}{\lambda_0\pi}\left[V(t_0)+\frac{3\varepsilon_\phi^2 n}{\alpha_0}\right]+n\right\}$$

It can be shown (see Lemma A in Appendix A) that if ε_ϕ is sufficiently small so that

$$\varepsilon_\phi<\sqrt{\frac{\lambda_0^2\alpha_0\varepsilon}{3\left[(n+1)V(t_0)+n\lambda_0+(n+1)\lambda_0\varepsilon\right]}} \tag{30}$$

then

$$\sum_{k=1}^{n}\left[y_k^2(t)+w_k^2(t)\right]\le\frac{e^{-\alpha_0(t-t_0)}}{\lambda_0}V(t_0)+\varepsilon$$

The inequality above implies the conclusions of Theorem 1.

5. NUMERICAL EXAMPLE

Consider the following plant as a numerical example (Jiang and Hill, 1999).

$$\dot{x}(t)=u(t)+\phi(t)$$
$$\dot{\eta}(t)=-\frac{k}{m}\eta(t)+x(t)$$
$$y_p(t)=x(t)$$

where u is the torque input, η is the unmeasured unmodeled dynamics, x is the state, y_p is the output. The reference model is

$$\dot{y}_m(t)=-\alpha y_m(t)+\beta r(t),\quad r(t)=1,\quad t\ge 0$$

where $\alpha=10,\beta=10$. It is required that $\varepsilon=0.03$.
For the simulation, we take

$$g=9.8,\ m=k/10=g^{-2},$$
$$d(t)=\sin(3t),\ L(t)=g+3\sin(2t)$$
$$\phi(t)=L(t)mg\sin\left(\frac{1}{L(t)m}\eta(t)\right)+d(t)$$

The initial values are given by $x(0)=0,\eta(0)=0$. The robust controller designed by the method shown in the last section is as follows:

$$u(t)=-\alpha x(t)+\beta r(t)+f_1 w_1(t)$$

where

$$w_1(t)=-\left(1+\frac{\alpha}{s}\right)y_1(t),\ w_1(0)=0$$

Choose ε_ϕ so that

$$\varepsilon_\phi<\sqrt{\frac{\lambda_0\alpha_0}{3(n+1)}}\approx 0.2523$$
$$\varepsilon_\phi<\sqrt{\frac{\lambda_0^2\alpha_0\varepsilon}{3\left[(n+1)V(t_0)+n\lambda_0+(n+1)\lambda_0\varepsilon\right]}}\approx 0.0779$$

and choose f_1 to satisfy

$$f_1\ge\left(\frac{gm}{\varepsilon_\phi k}\right)^2\approx 158.2622$$
$$f_1\ge\left(\frac{gm}{\varepsilon_\phi k}|y_m(t)|_\infty+\frac{g}{\varepsilon_\phi}|\eta(0)|+\frac{1}{\varepsilon_\phi}|d(t)|\right)^2\approx 646.0341$$

So we set $f_1=700$. Simulation results are shown in Figs 1 through 3. We can see that the state x, unmeasured unmodeled dynamics η are globally bounded. The tracking error is driven into the desired small neighborhood of the origin with desired transient performance.

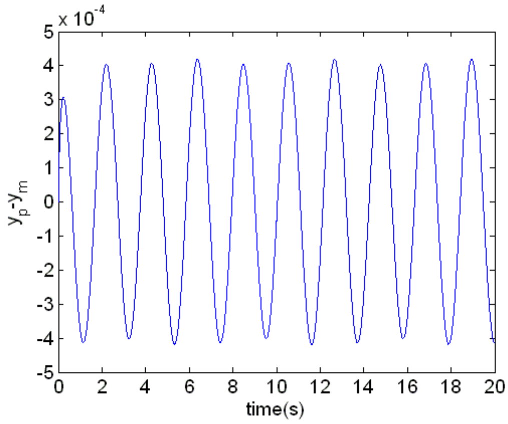

Fig. 1. Plot of tracking error

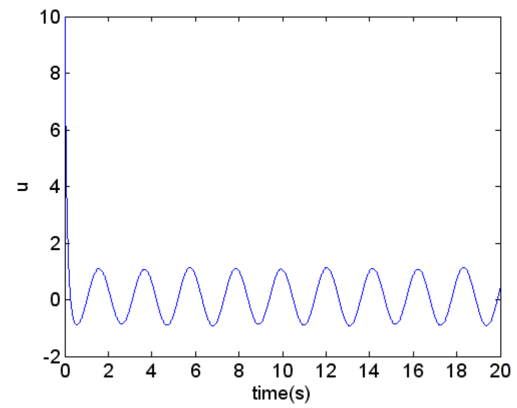

Fig. 2. Plot of robust control input

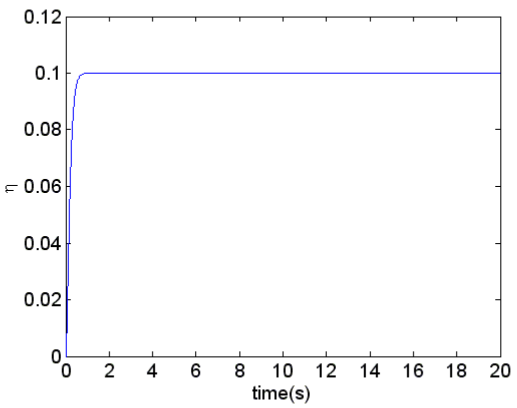

Fig. 2. Plot of unmodeled dynamics

6. CONCLUSIONS

For strict-feedback systems with time-varying nonlinear uncertainties and unmodeled dynamics, a new method is proposed of designing robust controller. A nominal controller is designed to get exact output tracking for the nominal plant; a robust compensator is added to achieve robust properties. Under Assumptions A, B and C, robust tracking property and robust tracking transient performance are achieved. The robust controller is a linear and time-invariant one, so it can be realized easily.

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Appendix A.

Lemma A If ε_ϕ is sufficiently small so that (28) holds and that

$$\varepsilon_\phi < \sqrt{\frac{\lambda_0^2\alpha_0\varepsilon}{3\left[(n+1)V(t_0)+n\lambda_0+(n+1)\lambda_0\varepsilon\right]}} \tag{A.1}$$

then

$$\frac{3\varepsilon_\phi^2}{\lambda_0\alpha_0}\left\{\frac{(n+1)}{\lambda_0\pi}\left[V(t_0)+\frac{3\varepsilon_\phi^2 n}{\alpha_0}\right]+n\right\}<\varepsilon \tag{A.2}$$

Proof. If (28) holds, then $\lambda_0^2\alpha_0-3\varepsilon_\phi^2(n+1)\lambda_0>0$. In this case, from (A.1) it follows that

$$\frac{3\varepsilon_\phi^2\left[(n+1)V(t_0)+(n\lambda_0)\right]}{\lambda_0^2\alpha_0-3\varepsilon_\phi^2(n+1)\lambda_0}<\varepsilon$$

which implies that

$$\frac{3\varepsilon_\phi^2\left[(n+1)\alpha_0V(t_0)+3\varepsilon_\phi^2(n+1)n+\lambda_0\pi n\alpha_0\right]}{\lambda_0^2\pi\alpha_0^2}<\varepsilon$$

So (A.2) holds.