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Perfect Tracking of Repetitive Signals for a Class of Nonlinear Systems

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Abstract: Perfect tracking of the output of a class of nonlinear systems that has a unique response for a given input and is subject to repetitive reference input is considered in this paper. A conditional learning scheme guaranteeing sufficient knowledge can be learned iteratively to improve the input to achieve perfect tracking is proposed. The sufficient condition for monotonic convergence of the input sequence and the choice of the learning gains are given. The tracking performance of the proposed scheme is illustrated by a simulated example.

1. INTRODUCTION

Tracking repetitive signals or completing repetitive tasks is an important practice task, in the fields of electronic circuits, robotics, servo mechanisms and electric motors (Alleyne and Pomykalski, 2000; Hu and Tomizuka, 1993). As iterative learning control (ILC) is able to control nonlinear dynamic systems with imperfect knowledge for these repetitive tasks (Arimoto, Kawamura and Miyazaki, 1984; Xu and Tan, 2003), it is considered in this paper to solve this tracking problem for global Lipschitz continuous nonlinear systems with unknown dynamics.

Generally, ILC based on contraction mapping takes the assumption of identical initial conditions (i.i.c.) in each of the iterations (Xu and Tan, 2003), namely, the initial state must be reposed to the same position at the beginning of every iteration. However, since the tracking process is continuous in time rather than with separate iterations that have distinct starting and ending points, i.i.c. generally is not available. In fact, the tracking problem belongs to the most general case, in which the end state of the previous iteration becomes the initial state of the current iteration, referred as the alignment initial condition. Although the relaxation of i.i.c. is widely studied (Park and Bien, 2000; Sun and Wang, 2002; Xu and Yan, 2005), the alignment initial condition is unfortunately not available in the contraction mapping ILC (Xu and Yan, 2005). Here, this problem is well handled by the proposed conditional learning scheme for the contraction mapping ILC. Although besides our method, another ILC approach based on composite energy function (CEF) (Sun, Ge and Mareels, 2006; Xu and Tan, 2003; Xu and Yan, 2005) can deal with the alignment initial condition, it requires the measurement of the system states, which may not always be available in practice (Xu and Tan, 2003). In contrast, ILC based on contraction mapping requires only the measurements of the output, which are more readily available. Further, discrete CEF based ILC is still under development (Xu and Tan, 2003).

The objective of the paper is to relax the i.i.c. requirement, such that the alignment initial condition can be applied to ILC derived from contraction mapping. It is achieved by introducing conditional learning for a class of nonlinear systems that has a unique steady-state response for a given input. In the conditional learning scheme, the input is updated only if the system response is dominated by the steady-state component. If this condition is not satisfied, the same input as that in the previous iteration will be used. In this approach, "learning" from the output error can still be performed despite the initial condition is unknown, and sufficient "knowledge" can be learned to improve iteratively the input to achieve perfect tracking of the output. The input updating scheme proposed here is a PD-type consisting of two learning gains (Xu and Tan, 2003). The learning gains are given, such that the input sequences are monotonically convergent in the sense of L^2 norm, thus avoiding the inconvenient λ-norm proposed in (Xu and Tan, 2003). The sufficient condition for the existence of monotonically convergent input sequence for the proposed scheme is also presented.

The paper is organized in follows. Section 2 states the problem formally with other preliminaries. Section 3 studies the response of nonlinear systems that have a unique steady-state for a given input. Section 4 proposes the ILC with conditional learning, and is illustrated by an example presented in Section 5.

2. PRELIMINARIES

Consider the following SISO dynamic system,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}\left(\mathbf{x}(t), u(t)\right) \\ y(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{1}$$

where $u(t)$, $\mathbf{x}(t)$ and $y(t)$ are respectively the system input, state and output. All variables and functions have appropriate dimensions, **f** and C are unknown global Lipschitz continuous functions that satisfy the following assumption.

Assumption 1: $\mathbf{f}_\mathbf{x}$ is a symmetrical matrix and there exist constants α_1, α_2, β, γ_1, γ_2 and ε such that

$$\alpha_1 \le \lambda(\mathbf{f}_\mathbf{x}) \le \alpha_2 \tag{2}$$

$$\left\|\mathbf{f}_u\right\|_2 \le \beta \tag{3}$$

Without loss of generality, the case of negative values are similarly defined, and

$$0 < \gamma_1 \le \mathbf{C}\mathbf{f}_u \le \gamma_2 \tag{4}$$

$$\|\mathbf{C}\|_2 \leq \varepsilon \tag{5}$$

where

$$\mathbf{f}_{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} f_{1,\mathbf{x}}\left(\mathbf{x}+\xi_1(\mathbf{x}^*-\mathbf{x}),u\right) \\ \vdots \\ f_{n,\mathbf{x}}\left(\mathbf{x}+\xi_n(\mathbf{x}^*-\mathbf{x}),u\right) \end{pmatrix}$$
$$\mathbf{f}_u = \frac{\partial \mathbf{f}}{\partial u} = \begin{pmatrix} f_{1,u}\left(\mathbf{x},u+\zeta_1(u^*-u)\right) \\ \vdots \\ f_{n,u}\left(\mathbf{x},u+\zeta_n(u^*-u)\right) \end{pmatrix} \tag{6}$$

n is the dimension of $\mathbf{x}$, $f_{n,\mathbf{x}}$ is the partial derivative of the n^{th} element of $\mathbf{f}$ to $\mathbf{x}$ and $f_{n,u}$ is to u; $\xi_1,\ldots,\xi_n$, $\zeta_1,\ldots$, and ζ_n belong to (0,1), $\lambda(\mathbf{f}_{\mathbf{x}})$ is the eigenvalue of $\mathbf{f}_{\mathbf{x}}$, $\|.\|_2$ is the 2-norm. Note that both $\lambda(\mathbf{f}_{\mathbf{x}})$ and $\|\mathbf{f}_u\|_2$ are functions of time. Since the mean value theorem will be used extensively here, the following notations are used to avoid the use of $\xi_1,\ldots,\xi_n$, $\zeta_1,\ldots$, and ζ_n,

$$\begin{aligned}\mathbf{f}(\mathbf{x},u)-\mathbf{f}(\mathbf{x}^*,u) &= \mathbf{f}_{\mathbf{x}}(\mathbf{x}-\mathbf{x}^*) \\ \mathbf{f}(\mathbf{x},u)-\mathbf{f}(\mathbf{x},u^*) &= \mathbf{f}_u(u-u^*)\end{aligned} \tag{7}$$

The second assumption that is related to the unique existence of the input for the desired output (Xu and Tan, 2003) is now presented.

Assumption 2: For a given desired output $y_d(t)$ that repeats at a period of T, there exists a unique desired input $u_d(t)$ such that

$$\begin{aligned}\dot{\mathbf{x}}_d(t) &= \mathbf{f}\left(\mathbf{x}_d(t),u_d(t)\right) \\ y_d(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned} \tag{8}$$

From Assumption 2, $u_d(t)$ exists uniquely. It follows that if the input $u(t)$ converges to $u_d(t)$, the state and output tracking errors will vanish. An input updating law must find $u_d(t)$ from an arbitrary input. Note that the dynamic process is manipulated continuously and the state $\mathbf{x}(t)$ is aligned, and hence it is also referred to as the alignment iterations, such that contraction mapping ILC can be applied to achieve perfect tracking. This is presented in more details below (Sun, *et al.*, 2006):

1) each iteration begins at zero and ends in a finite time T, i.e., $t\in[0, T]$;
2) the initial condition of the system at the beginning of each iteration is aligned with the final position of the previous one, such that $\mathbf{x}_i(0) = \mathbf{x}_{i-1}(T)$, where i is the iteration index and $\mathbf{x}_i(t)$, $t\in[0, T]$, is the system state at the i^{th} iteration, i.e., $\mathbf{x}_d(T) = \mathbf{x}_d(0)$;
3) the desired output $y_d(t)$ is given *a priori* over [0, T] and from 2), $y_d(T)= y_d(0)$.

In the following analysis, L^2 norm for vectors and square matrixes defined below is used.

$$\begin{aligned}\|\mathbf{x}(t)\|_T &= \left(\frac{1}{T}\int_0^T \mathbf{x}(t)'\mathbf{x}(t)dt\right)^{1/2} \\ \|\mathbf{A}(t)\|_T &= \left(\frac{1}{T}\int_0^T \lambda_{\max}\left(\mathbf{A}(t)'\mathbf{A}(t)\right)dt\right)^{1/2}\end{aligned} \tag{9}$$

The norm is well defined having the following properties:

$$\begin{aligned}&\|.\|_T \leq \max_{t\in[0,T]}\|.\|_2 \\ &\|\mathbf{A}(t)\mathbf{x}(t)\|_T \leq \|\mathbf{A}(t)\|_T\,\|\mathbf{x}(t)\|_T \\ &\|\mathbf{P}(t)\mathbf{x}(t)\|_T = \|\mathbf{x}(t)\|_T \\ &\|\mathbf{P}(t)\mathbf{A}(t)\mathbf{Q}(t)\|_T = \|\mathbf{A}(t)\|_T\end{aligned} \tag{10}$$

where $\mathbf{P}(t)$ and $\mathbf{Q}(t)$ are time-varying orthogonal matrixes. Without causing confusion, the time t, where $t\in[0, T]$, is omitted hereafter.

3. UNIQUE STEADY-STATE

The i.i.c. does not hold here, as the initial states are assumed to be aligned. The following assumption is made to overcome the problem of non-identical initial condition.

Assumption 3: The system (1) has a unique steady-state solution (Chua and Green, 1976), such that all the solutions of (1) are bounded and independent of the initial condition, and any pair of solutions $\mathbf{x}$ and $\mathbf{x}^*$ of (1) tend to each other asymptotically, i.e.,

$$\lim_{t\to+\infty}\|\mathbf{x}-\mathbf{x}^*\| = 0 \tag{11}$$

where $\|.\|$ is an arbitrary norm. That is, $\mathbf{x}(t)$ can be decomposed into the transient response $\mathbf{x}_{tr}$ and the steady-state response $\mathbf{x}_{ss}$, as follows,

$$\mathbf{x} = \mathbf{x}_{tr} + \mathbf{x}_{ss} \tag{12}$$

where $\mathbf{x}_{tr} \to 0$ as $t \to \infty$.

As the steady-state response is eventually not affected by the initial condition and is determined by the input, ILC is also applicable to systems with non-identical initial condition. It is shown in Theorem A.2 of Chua and Green (1976) that the steady-state of system (1) exists, if an incremental Lyapunov function V_Δ exists, such that

$$V_\Delta(\mathbf{x},\mathbf{x}^*)\begin{cases} = 0 & \text{if } \mathbf{x} = \mathbf{x}^* \\ > 0 & \text{if } \mathbf{x} \neq \mathbf{x}^* \end{cases} \tag{13}$$

and

$$\frac{dV_\Delta(\mathbf{x},\mathbf{x}^*)}{dt}\begin{cases} = 0 & \text{if } \mathbf{x} = \mathbf{x}^* \\ < 0 & \text{if } \mathbf{x} \neq \mathbf{x}^* \end{cases} \tag{14}$$

where both $\mathbf{x}$ and $\mathbf{x}^*$ are the response of (1) for the same input under different initial conditions. If $\mathbf{x} \neq \mathbf{x}^*$, then from (11) and (14), the non-negative V_Δ approaches to 0. Consequently, $\mathbf{x}$ and $\mathbf{x}^*$ approach to each other for sufficiently large time, as given below.

Theorem 1: If $\mathbf{f}_{\mathbf{x}}$ is negative definite for all $\mathbf{x}$ and u, then system (1) has a unique steady-state response.

Proof: Define

$$V_\Delta(\mathbf{x},\mathbf{x}^*) = \frac{1}{2}(\mathbf{x}-\mathbf{x}^*)'(\mathbf{x}-\mathbf{x}^*)$$

and

$$\frac{dV_\Delta(\mathbf{x},\mathbf{x}^*)}{dt} = (\mathbf{x}-\mathbf{x}^*)'\left(\mathbf{f}(\mathbf{x},u)-\mathbf{f}(\mathbf{x}^*,u)\right) \tag{15}$$

Applying the mean value theorem (Ortega and Rheinboldt, 1970), gives

$$\mathbf{f}(\mathbf{x},u)-\mathbf{f}(\mathbf{x}^*,u) = \mathbf{f}_{\mathbf{x}}(\mathbf{x}-\mathbf{x}^*) \tag{16}$$

Substituting (16) into (15), and since $\mathbf{f_x}$ is negative definite, (15) also satisfies (13), hence system (1) has a unique steady-state. This completes the proof. □

Remark 1: Under Assumption 3, if an arbitrary input is applied repetitively at a fixed period, all the response will eventually approach to a repetitive steady-state response at the same period. Clearly, if $\mathbf{f_x}$ is negative definite, $\alpha_2<0$.

The properties of the steady-state response $\mathbf{x}_{ss}$ and the transient response $\mathbf{x}_{tr}$ are given in the following theorem.

Theorem 2: If system (1) satisfies Assumption 2, and its steady-state response is bounded and satisfies,

$$\|\mathbf{x}_{ss} - \mathbf{x}^*_{ss}\|_T \le \frac{\beta}{|\alpha_2|}\|u - u^*\|_T = G\|u - u^*\|_T \tag{17}$$

then its transient response converges to zero exponentially,

$$\|\mathbf{x}_{tr}(0)\|_2 \exp(\alpha_1 t) \le \|\mathbf{x}_{tr}(t)\|_2 \le \|\mathbf{x}_{tr}(0)\|_2 \exp(\alpha_2 t) \tag{18}$$

Details of the proof are given in Appendix A. In the next section, a novel ILC with conditional learning is proposed, such that the contraction mapping ILC can be applied to non-identical initial condition.

4. ILC WITH CONDITIONAL LEARNING

4.1 Contraction Mapping ILC and Conditional Learning

For convenience, the following notations for the difference between the desired and the actual values are introduced first (Xu and Tan, 2003),

$$\begin{aligned}\Delta u_i &\triangleq u_d - u_i\\ \Delta \mathbf{x}_i &\triangleq \mathbf{x}_d - \mathbf{x}_i = \Delta \mathbf{x}_{i,tr} + \Delta \mathbf{x}_{i,ss}\\ \Delta y_i &\triangleq y_d - y_i = \Delta y_{i,tr} + \Delta y_{i,ss}\end{aligned} \tag{19}$$

where u_d, $\mathbf{x}_d$ and y_d are the desired input, desired state vector and desired output, u_i, $\mathbf{x}_i$ and y_i are the input, state vector and output at the i^{th} iteration. In the contraction mapping ILC, the input is updated iteratively after each operation based on the output errors in the previous iteration. Since the relative degree of the nonlinear system (1) is 1, PD-type ILC, as shown below, is popular (Xu and Tan, 2003) and will be used here.

$$u_{i+1} = u_i + q_1\Delta y_i + q_2\Delta\dot{y}_i \tag{20}$$

where q_1 and q_2 are the learning gains.

In general, perfect tracking can only be achieved if $u_{i+1} = u_d$ and $\mathbf{x}_{i+1}(0) = \mathbf{x}_d(0)$. Although the i.i.c. does not necessarily hold here, Assumption 3 implies that the effect of the initial condition will eventually vanish. Therefore, in order to achieve the same result as using the assumption of i.i.c., it is only necessary to ebsure $u_{i+1} = u_d$ through learning from previous output error Δy_i (Xu and Tan, 2003). However, the learning processes are different if the i.i.c. does not hold. For the system (1), the i.i.c. assumption implies that Δy_i depends only on Δu_i, which in fact guarantees that in the "learning", all the learned information is used to improve Δu_i. In contrast, without the i.i.c. assumption, Δy_i is determined by both $\Delta \mathbf{x}_i$ and Δu_i, and hence the information in Δy_i for learning includes also the transient componenet $\Delta y_{i,tr}$ and the steady-state component $\Delta y_{i,ss}$. In order to learn sufficient information about Δu_i to improve the input, the conditional learning scheme is proposed here. The input is updated only if the information about Δu_i plays a major part, i.e., $\Delta y_{i,ss}$ is the major part of Δy_i with $\Delta y_{i,tr}$ being a minor one. Let the condition for learning be given by,

$$\|\Delta \mathbf{x}_{i,tr}\|_T \le m\|\Delta \mathbf{x}_{i,ss}\|_T \tag{21}$$

where m is a positive constant less than 1. However, as (21) may not be satisfied in each iteration, additional conditions have to be devised to ensure that (21) is satisfied.

Theorem 3: The transient response of system (1) under Assumptions 1 and 3 has the following properties,

$$\|\Delta \mathbf{x}_{i,tr}\|_T \le p\|\Delta \mathbf{x}_{i-1,tr}\|_T + (K+G)\|u_i - u_{i-1}\|_T \tag{22}$$

where G is given by (17) and

$$p = \exp(\alpha_2 T) < 1 \tag{23}$$

$$K = \beta T \exp(-\alpha_1 T) \tag{24}$$

Details of the proof of Theorem 3 are given in Appendix B. Since from (22), $\Delta \boldsymbol{x}_{0,\,tr} = 0$, then

$$\|\Delta \mathbf{x}_{i,tr}\|_T \le \sum_{j=1}^{i}(K+G)\|u_j - u_{j-1}\|_T\, p^{j-1} \tag{25}$$

Theorem 4: The inequality (21) is satisfied if,

$$\sum_{j=1}^{i}(K+G)\|u_j - u_{j-1}\|_T\, p^{j-1} \le \frac{m}{(1+m)\varepsilon}\|\Delta y_i\|_T \tag{26}$$

Details of the proof are given in Appendix C. It is clear that (26) can be easily verified, as the input sequence is known and Δy_i can be readily computed. Therefore, the input updating law given by (20) is modified as follows,

$$u_1 = q_1 y_d + q_2\dot{y}_d$$

$$u_{i+1} = \begin{cases} u_i + q_1\Delta y_i + q_2\Delta\dot{y}_i & \text{if (21) is satisfied}\\ u_i & \text{otherwise}\end{cases} \tag{27}$$

From (27), the procedure to update the input consists of either: (i) *update*, i.e., $u_{i+1} = u_i + q_1\Delta y_i + q_2\Delta\dot{y}_i$, or (ii) *idle*, i.e., $u_{i+1} = u_i$. Clearly, if the learning gains q_1 and q_2 are suitably chosen, such that $\|\Delta u_{i+1}\|_T < \|\Delta u_i\|_T$ is satisfied in each *update*, the updated input will monotonically converge to the desired input u_d. In this case, perfect output tracking can be achieved.

4.2 Monotonically Convergent Control Sequence

The result discussed previously that if the learning gains q_1 and q_2 are chosen, such that $\|\Delta u_{i+1}\|_T < \|\Delta u_i\|_T$ is satisfied, the updated input converges monotonically to $u_d(t)$ is presented in the following theorem.

Theorem 5: Under Assumptions 1 to 3, if $\varepsilon G(\alpha_2-\alpha_1)/2<\gamma_1$, there exist m and learning gains q_1 and q_2, such that $\|\Delta u_{i+1}\|_T < \|\Delta u_i\|_T$ is strictly guaranteed in each update of the input.

Proof: From Assumptions 1 to 3, and applying Taylor's theorem,

$$
\begin{aligned}
&\Delta u_{i+1} \\
&= u_d - u_{i+1} \\
&= u_d - u_i + u_i - u_{i+1} \\
&= \Delta u_i - q_1 \Delta y_i - q_2 \Delta \dot{y}_i \\
&= \Delta u_i - q_1 \Delta y_i - q_2 \mathbf{C}(\mathbf{f}(x_d, u_d) - \mathbf{f}(x_i, u_i)) \\
&= \Delta u_i - q_1 \Delta y_i - q_2 \mathbf{C}(\mathbf{f}_u \Delta u_i + \mathbf{f}_\mathbf{x} \Delta \mathbf{x}_i) \\
&= (1 - q_2 \mathbf{C}\mathbf{f}_u) \Delta u_i - (q_1 \Delta y_i + q_2 \mathbf{C}\mathbf{f}_\mathbf{x} \Delta \mathbf{x}_i)
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\left\|\Delta u_{i+1}\right\|_T \\
&= \left\|(1 - q_2 \mathbf{C}\mathbf{f}_u)\Delta u_i - q_1 \Delta y_i + q_2 \mathbf{C}\mathbf{f}_\mathbf{x} \Delta \mathbf{x}_i\right\|_T \\
&\le \left\|1 - q_2 \mathbf{C}\mathbf{f}_u\right\|_T \left\|\Delta u_i\right\|_T + \varepsilon \left\|q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right\|_T \left\|\Delta \mathbf{x}_i\right\|_T \\
&\le \left\|1 - q_2 \mathbf{C}\mathbf{f}_u\right\|_T \left\|\Delta u_i\right\|_T + \varepsilon \left\|q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right\|_T (1+m) \left\|\Delta \mathbf{x}_{i,ss}\right\|_T \\
&\le \left\|1 - q_2 \mathbf{C}\mathbf{f}_u\right\|_T \left\|\Delta u_i\right\|_T + (1+m)\varepsilon G \left\|q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right\|_T \left\|\Delta u_i\right\|_T \\
&\le \max\left(\left\|1 - q_2 \mathbf{C}\mathbf{f}_u\right\|_T + (1+m)\varepsilon G \left\|q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right\|_T\right) \left\|\Delta u_i\right\|_T
\end{aligned}
\tag{28}
$$

To ensure that $\|\Delta u_{i+1}\|_T < \|\Delta u_i\|_T$, it is necessary that

$$
\max\left(\left\|1 - q_2 \mathbf{C}\mathbf{f}_u\right\|_T + (1+m)\varepsilon G \left\|q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right\|_T\right) < 1 \tag{29}
$$

Therefore, if q_1 and q_2 are chosen satisfying the inequality (29), then there exists at least a pair of q_1 and q_2, such that $\|\Delta u_{i+1}\|_T < \|\Delta u_i\|_T$ is satisfied. To complete the proof, it is only necessary to show that the optimal choice of q_1 and q_2 given below satisfies (29),

$$
(q_1, q_2) = \arg\min_{q_1, q_2}\left(\max\left(\left\|1 - q_2 \mathbf{C}\mathbf{f}_u\right\|_T + (1+m)\varepsilon G \left\|q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right\|_T\right)\right) \tag{30}
$$

Since $\mathbf{f}_\mathbf{x}$ is symmetrical and from (2) $\alpha_1 \le \lambda(\mathbf{f}_\mathbf{x}) \le \alpha_2$ and (4) $0 < \gamma_1 \le \mathbf{C}\mathbf{f}_u \le \gamma_2$, hence $q_2 > 0$. For a given q_2, there exists a time-varying orthogonal matrix $\mathbf{P}(t)$ such that

$$
q_1 = \arg\min_{q_1}\left(\max\left(\left\|q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right\|_T\right)\right) = -q_2 \frac{\alpha_1 + \alpha_2}{2} \tag{31}
$$

and

$$
\begin{aligned}
&\max\left((1+m)\varepsilon G \left\|q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right\|_T\right) \\
&= (1+m)\varepsilon G \max\left(\left\|\mathbf{P}(t)\left(q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right)\mathbf{P}'(t)\right\|_T\right) \\
&= (1+m)\varepsilon G \max\left(\left\|q_1 \mathbf{I} + q_2 diag\left[\lambda\left(\mathbf{f}_\mathbf{x}\right)\right]\right\|_T\right) \\
&= q_2 (1+m)\varepsilon G \frac{\alpha_2 - \alpha_1}{2}
\end{aligned}
\tag{32}
$$

Substituting (31) into (29) gives

$$
\begin{aligned}
&\max\left(\left\|1 - q_2 \mathbf{C}\mathbf{f}_u\right\|_T + (1+m)\varepsilon G \left\|q_1 \mathbf{I} + q_2 \mathbf{f}_\mathbf{x}\right\|_T\right) \\
&= \begin{cases}
1 - q_2 \gamma_1 + q_2 (1+m)\varepsilon G \dfrac{\alpha_2 - \alpha_1}{2} & 0 < q_2 < \dfrac{2}{\gamma_1 + \gamma_2} \\
q_2 \gamma_2 - 1 + q_2 (1+m)\varepsilon G \dfrac{\alpha_2 - \alpha_1}{2} & q_2 \ge \dfrac{2}{\gamma_1 + \gamma_2}
\end{cases}
\end{aligned}
\tag{33}
$$

Since (33) must be smaller than 1, it can be readily shown that the following conditions are satisfied, and (29) holds,

$$
\varepsilon G \frac{\alpha_2 - \alpha_1}{2} < \gamma_1 \tag{34}
$$

$$
0 < q_2 < \frac{2}{\gamma_2 + (1+m)\varepsilon G \dfrac{\alpha_2 - \alpha_1}{2}} \tag{35}
$$

Therefore, if the system satisfies (34), the learning gains q_1 and q_2 exist. The optimal q_1 and q_2 given by (30) can be readily obtained as follows,

$$
\begin{aligned}
q_1 &= -q_2 \frac{\alpha_1 + \alpha_2}{2} \\
q_2 &= \frac{2}{\gamma_1 + \gamma_2}
\end{aligned}
\tag{36}
$$

This completes the proof. □

Remark 2: The proof also provides a method of choosing the learning gains q_1 and q_2.

Remark 3: The existence condition (34) indicates that the effect of the state uncertainty must be smaller than that of the input to some degree. From (17), (34) is equivalent to

$$
\frac{\alpha_1}{\alpha_2} - 2\frac{\gamma_1}{\varepsilon\beta} < 1 \tag{37}
$$

and from (2) to (4), let

$$
\begin{aligned}
\mu &= \frac{\alpha_1}{\alpha_2} = \frac{\left\|\mathbf{f}_\mathbf{x}\right\|_{T,\max}}{\left\|\mathbf{f}_\mathbf{x}\right\|_{T,\min}} \\
\nu &= \frac{\varepsilon\beta}{\gamma_1} = \frac{\left\|\mathbf{C}\mathbf{f}_u\right\|_{T,\max}}{\left\|\mathbf{C}\mathbf{f}_u\right\|_{T,\min}}
\end{aligned}
\tag{38}
$$

where μ and ν are regarded as evaluations of the uncertainty of $\mathbf{f}_\mathbf{x}$ and $\mathbf{C}\mathbf{f}_u$, then (34) can be rewritten as,

$$
\nu(\mu - 1) < 2 \tag{39}
$$

and the existence region of monotonically convergent input sequence is shown by Fig. 1. In general, if the system can be modelled accurately, such that the system uncertainty is small and can be located in the existence region, then a monotonically convergent input sequence can be found.

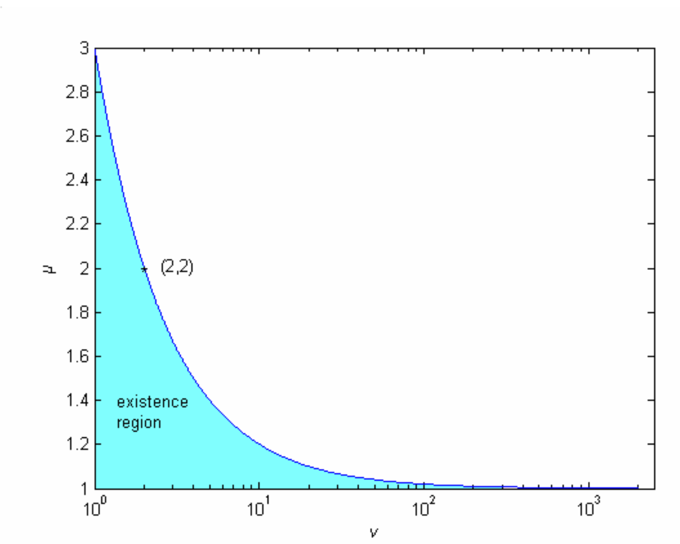

Fig. 1. Existence region of monotonically convergent input

5. SIMULATED EXAMPLE

Let us consider an unknown nonlinear dynamic system,

$$\dot{x}_1 = -0.1(\cos^2 x_1 + 5x_1) + u$$
$$\dot{x}_2 = -0.1(\sin^2 x_2 + 5x_2) + \frac{u}{3+t^2} \quad (40)$$
$$x_1(0) = x_2(0) = 0$$
$$y = 0.2x_1 - 0.1x_2$$

Let the desired output be

$$y_d = \left|(2t-1)^3\right| \quad t \in [0,1] \quad (41)$$

Since the system is unknown, the following parameters are selected: α_1 = -0.65, α_2 = -0.35, β_1 = 1, β_2 = 1.06, γ_1 = 0.15, γ_2 = 0.2, ε = 0.25, and G = 3.0286, p = 0.7047 and K = 2.0305. Clearly, (34) is satisfied, and hence there exists a set of q_1, q_2 and m that ensures perfect tracking of the output. Since the system is unknown, m is set to 0.3208, then q_1 and q_2 to 2.8571 and 5.7143 from (36). It is shown in Fig. 2 that the input sequence converges monotonically to the desired one and perfect output tracking can eventually be achieved.

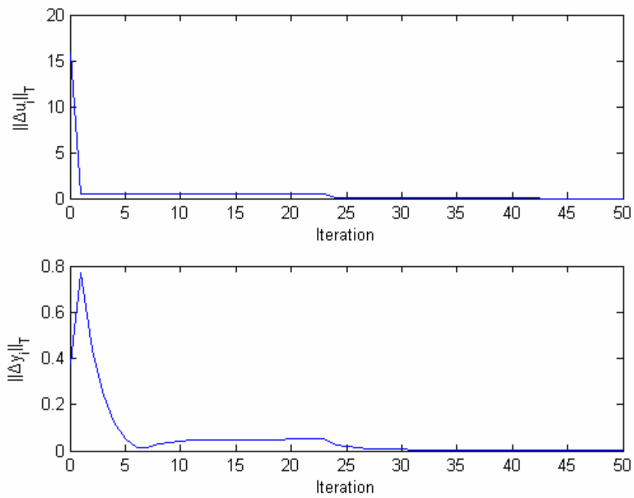

Fig. 2. $||\Delta u_i||_T$ and $||\Delta y_i||_T$ with q_1 = 2.8571 and q_2 = 5.7143.

6. CONCLUSION

Although ILC is effective to track repetitive trajectories for nonlinear systems, it requires i.i.c. for perfect tracking. However, for many practical systems i.i.c. is not achievable, as these systems are operating under the alignment initial condition without distinct starting and ending points. In this paper, ILC derived from contraction mapping with conditional learning is proposed for a class of nonlinear systems that has a unique steady-state response for an input. In this method, the updated input converges monotonically to the desired input, thus achieving perfect tracking of the desired output. The proposed ILC also avoids the practical difficulties of ILC derived based on the Lyapunov function. The sufficient condition for the monotonic convergence of the updated input is presented, and the corresponding settings of the input updating law are also derived. The performance of the proposed conditional learning ILC is illustrated by an example.

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Appendix A. PROOF OF THEOREM 2

First, the boundary of steady-state response is proven. From (17),

$$\mathbf{z} = \Delta\mathbf{x}_{ss} = \mathbf{x}_{ss} - \mathbf{x}^*_{ss}$$
$$w = \Delta u = u - u^* \quad (A1)$$

And from (6) and (7),

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{x}_{ss}, u) - \mathbf{f}(\mathbf{x}^*_{ss}, u^*) = \mathbf{f}_\mathbf{x}\mathbf{z} + \mathbf{f}_u w \quad (A2)$$

Then we have

$$\frac{1}{2}\frac{d\left(\mathbf{z}'\mathbf{z}\right)}{dt} = \mathbf{z}'\dot{\mathbf{z}} = \mathbf{z}'\mathbf{f}_\mathbf{x}\mathbf{z} + \mathbf{z}'\mathbf{f}_u w \quad (A3)$$

Since **z'z** is a sinusoidal function, its derivative is also a sinusoidal function without constant term; consequently, the integral in (9) of the left hand side term in (A3) over a period T is zero, and then

$$\frac{1}{T}\int_0^T -\mathbf{z}'\mathbf{f}_\mathbf{x}\mathbf{z}dt = \frac{1}{T}\int_0^T \mathbf{z}'\mathbf{f}_u w dt \quad (A4)$$

Since following square inequality always holds for any positive constant a,

$$\mathbf{z}'\mathbf{f}_u w \le \frac{1}{2}\left(a\mathbf{z}'\mathbf{z} + \frac{w^2}{a}\mathbf{f}_u'\mathbf{f}_u\right) \quad (A5)$$

we have

$$\frac{1}{T}\int_0^T -\mathbf{z}'\mathbf{f}_\mathbf{x}\mathbf{z} - \frac{a}{2}\mathbf{z}'\mathbf{z}dt \le \frac{1}{T}\int_0^T \frac{w^2}{2a}\mathbf{f}_u'\mathbf{f}_u dt \tag{A6}$$

then

$$G = \frac{\frac{1}{T}\int_0^T \mathbf{z}'\mathbf{z}dt}{\frac{1}{T}\int_0^T w^2 dt} \le \frac{\|\mathbf{f}_u\|_T^2}{-a(2\alpha_2 + a)} \le \frac{\beta^2}{-a(2\alpha_2 + a)} \tag{A7}$$

It is clear that the right-hand-side term is minimized, i.e., reaching the infimum of G, when

$$a = -\alpha_2 \tag{A8}$$

and then

$$G \le \frac{\|\mathbf{f}_u\|_T}{-\alpha_2} \le \frac{\beta}{|\alpha_2|} \tag{A9}$$

Equation (18) is proved as follows:

$$\dot{\mathbf{x}}_{tr} = \mathbf{f}(\mathbf{x},u) - \mathbf{f}(\mathbf{x}_{ss},u) = \mathbf{f}_\mathbf{x}\mathbf{x}_{tr} \tag{A10}$$

and

$$\frac{d\left(\mathbf{x}_{tr}'\mathbf{x}_{tr}\right)}{dt} = 2\mathbf{x}_{tr}'\dot{\mathbf{x}}_{tr} = 2\mathbf{x}_{tr}'\mathbf{f}_\mathbf{x}\mathbf{x}_{tr} \tag{A11}$$

Since

$$\lambda_{\min}(\mathbf{f}_\mathbf{x})\mathbf{x}_{tr}'\mathbf{x}_{tr} \le \mathbf{x}_{tr}'\mathbf{f}_\mathbf{x}\mathbf{x}_{tr} \le \lambda_{\max}(\mathbf{f}_\mathbf{x})\mathbf{x}_{tr}'\mathbf{x}_{tr} \tag{A12}$$

then

$$2\lambda_{\min}(\mathbf{f}_\mathbf{x})\mathbf{x}_{tr}'\mathbf{x}_{tr} \le \frac{d\left(\mathbf{x}_{tr}'\mathbf{x}_{tr}\right)}{dt} \le 2\lambda_{\max}(\mathbf{f}_\mathbf{x})\mathbf{x}_{tr}'\mathbf{x}_{tr} \tag{A13}$$

Therefore

$$\|\mathbf{x}_{tr}(0)\|_2^2 \exp(2\alpha_1) \le \|\mathbf{x}_{tr}\|_2^2 \le \|\mathbf{x}_{tr}(0)\|_2^2 \exp(2\alpha_2) \tag{A14}$$

and then

$$\|\mathbf{x}_{tr}(0)\|_2 \exp(\alpha_1) \le \|\mathbf{x}_{tr}\|_2 \le \|\mathbf{x}_{tr}(0)\|_2 \exp(\alpha_2) \tag{A15}$$

Equation (18) can now be obtained from (A15), and hence the proof is completed. □

Appendix B. PROOF OF THEOREM 3

For the i^{th} iteration, consider u_i and u^*_I with $u^*_i = u_{i-1}$, then

$$\begin{aligned}\Delta\mathbf{x}^*_{i,tr} &= \Delta\mathbf{x}^*_i - \Delta\mathbf{x}^*_{i,ss}\\ \Delta\mathbf{x}_{i,tr} &= \Delta\mathbf{x}_i - \Delta\mathbf{x}_{i,ss}\end{aligned} \tag{A16}$$

Hence

$$\|\Delta\mathbf{x}_{i,tr}\|_T \le \|\Delta\mathbf{x}^*_{i,tr}\|_T + \|\Delta\mathbf{x}_i - \Delta\mathbf{x}^*_i\|_T + \|\Delta\mathbf{x}_{i,ss} - \Delta\mathbf{x}^*_{i,ss}\|_T \tag{A17}$$

From (18),

$$\|\Delta\mathbf{x}^*_{i,tr}\|_T \le \exp(\alpha_2 T)\|\Delta\mathbf{x}_{i-1,tr}\|_T = p\|\Delta\mathbf{x}_{i-1,tr}\|_T \tag{A18}$$

Now consider,

$$\Delta\mathbf{x}_i - \Delta\mathbf{x}^*_i = \int_0^t \mathbf{f}_\mathbf{x}(\Delta\mathbf{x}_i - \Delta\mathbf{x}^*_i) + \mathbf{f}_u(u_i - u_{i-1})d\tau \tag{A19}$$

Applying the Gronwall-Bellman lemma (Apartsyn, 2003), gives,

$$\begin{aligned}\|\Delta\mathbf{x}_i - \Delta\mathbf{x}^*_i\|_T &= \left\|\int_0^t \mathbf{f}_\mathbf{x}(\Delta\mathbf{x}_i - \Delta\mathbf{x}^*_i) + \mathbf{f}_u(u_i - u_{i-1})d\tau\right\|_T\\ &\le -\alpha_1\int_0^t \|\Delta\mathbf{x}_i - \Delta\mathbf{x}^*_i\|_T\, d\tau + \beta\int_0^t \|u_i - u_{i-1}\|_T\, d\tau\\ &\le -\alpha_1\int_0^t \|\Delta\mathbf{x}_i - \Delta\mathbf{x}^*_i\|_T\, d\tau + \beta t\|u_i - u_{i-1}\|_T\\ &\le \beta t\|u_i - u_{i-1}\|_T \exp(-\alpha_1 t)\\ &\le \beta T\exp(-\alpha_1 T)\|u_i - u_{i-1}\|_T\\ &= K\|u_i - u_{i-1}\|_T\end{aligned} \tag{A20}$$

From the result on the boundedness of the steady-state response in (17), it is obtained that

$$\|\Delta\mathbf{x}_{i,tr}\|_T \le p\|\Delta\mathbf{x}_{i-1,tr}\|_T + (K+G)\|u_i - u_{i-1}\|_T \tag{A21}$$

This completes the proof. □

Appendix C. PROOF OF THEOREM 4

From (25) and (5), it follows from (26) that

$$\|\Delta\mathbf{x}_{i,tr}\|_T \le \frac{m}{(1+m)\|\mathbf{C}\|_T}\|\Delta y_i\|_T \tag{A22}$$

This is equivalent to

$$\|\Delta\mathbf{x}_{i,tr}\|_T \le \frac{m\left(\|\Delta y_i\|_T - \|g_\mathbf{x}\|_T\|\Delta\mathbf{x}_{i,tr}\|_T\right)}{\|\mathbf{C}\|_T} \le \frac{m\left(\|\Delta y_i\|_T - \|\Delta y_{i,tr}\|_T\right)}{\|\mathbf{C}\|_T} \tag{A23}$$

Since

$$\frac{\|\Delta y_i\|_T - \|\Delta y_{i,tr}\|_T}{\|\mathbf{C}\|_T} \le \frac{\|\Delta y_{i,ss}\|_T}{\|\mathbf{C}\|_T} \le \|\Delta\mathbf{x}_{i,ss}\|_T \tag{A24}$$

The result is obtained by substituting (A24) into (A23). This completes the proof. □