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A design method of robust stabilizing simple multi-period repetitive controllers

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Abstract: A multi-period repetitive control system is a type of servomechanism for a periodic reference input. Even if a plant does not include time-delay, the transfer function from the periodic reference input to the output and that from the disturbance to the output of the multi-period repetitive control system generally have an infinite number of poles. In order to specify the input-output characteristic and the disturbance attenuation characteristic easily, Yamada et al. propose the concept of simple multi-period repetitive control systems such that the controller works as a stabilizing multi-period repetitive controller and the transfer function from the periodic reference input to the output and that from the disturbance to the output have a finite number of poles. However, the method by Yamada et al. cannot apply for the plant with uncertainty. The purpose of this paper is to propose the concept of robust stabilizing simple multi-period repetitive controllers for the plant with uncertainty and to clarify the parametrization of all robust stabilizing simple multi-period repetitive controllers.

Keywords: periodic signal, multi-period repetitive controller, finite number of poles, parametrization

1. INTRODUCTION

In this paper, we investigate a problem to seek all robust stabilizing simple multi-period repetitive controllers for a plant with uncertainty. A repetitive control system is a type of servomechanism for a periodic reference input for the periodic signal without steady state error (Inoue *et al.*, 1981; Omata *et al.*, 1987; Watanabe and Yamatari, 1986; Hara *et al.*, 1988; Yamamoto and Hara, 1988; Nakano *et al.*, 1989; Ikeda and Takano, 1990; Yamamoto and Hara, 1992; Yamamoto, 1993; Katoh and Funahashi, 1996; Weiss, 1997). That is, the repetitive control system follows the periodic reference input without steady state error, even if there exists a periodic disturbance or the plant has uncertainty (Inoue *et al.*, 1981; Omata *et al.*, 1987; Watanabe and Yamatari, 1986; Hara *et al.*, 1988; Yamamoto and Hara, 1988; Nakano *et al.*, 1989; Ikeda and Takano, 1990; Yamamoto and Hara, 1992; Yamamoto, 1993; Katoh and Funahashi, 1996; Weiss, 1997). The repetitive control system was proposed for 'high accuracy control for magnet power supply of proton synchrotron in recurrent operation' (Inoue *et al.*, 1981). Various papers on the repetitive control have been studied (Inoue *et al.*, 1981; Omata *et al.*, 1987; Watanabe and Yamatari, 1986; Hara *et al.*, 1988; Yamamoto and Hara, 1988; Nakano *et al.*, 1989; Ikeda and Takano, 1990; Yamamoto and Hara, 1992; Yamamoto, 1993; Katoh and Funahashi, 1996; Weiss, 1997).

On the other hand, there exists important control problem to find all stabilizing controllers named the parameterization problem (Youla *et al.*, 1976; Kucera, 1979; Glaria and Goodwin, 1994; Vidyasagar, 1985). First, the parametrization of all stabilizing modified repetitive controllers that follow the periodic reference input with small steady state error even if there exists a periodic disturbance or the uncertainty of the plant was studied by (Hara and Yamamoto, 1986). In (Hara and Yamamoto, 1986), since the stability sufficient condition of repetitive control system is decided as H_∞ norm problem, the parametrization for repetitive control system is given by resolving into the interpolation problem of Nevanlinna-Pick. Katoh and Funahashi gave the parametrization of all stabilizing repetitive controllers for minimum phase systems by solving exactly Bezout equation (Katoh and Funahashi, 1996). However, Katoh and Funahashi (Katoh and Funahashi, 1996) assumed the plant is asymptotically stable. This implies that they gave the parametrization of all causal repetitive controllers for an asymptotically stable and minimum phase plant. That is, they do not give the explicit parametrization for minimum phase systems (Katoh and Funahashi, 1996). In addition, in (Katoh and Funahashi, 1996) it is assumed that the relative degree of low-pass filter in the repetitive compensator is equal to that of the plant. Extending the results in (Katoh and Funahashi, 1996), Yamada and Okuyama gave the parametrization of all stabilizing repetitive controllers for minimum phase systems (Yamada and Okuyama, 2000). Yamada et al. gave the parametrization of all stabilizing repetitive controllers for the certain class of non-minimum phase systems (Yamada *et al.*, 2002*a*). They obtained the parametrization of all repetitive controllers using fusion of the parallel compensation technique and the solution of Bezout equation. However, they gave the parametrization of all repetitive controllers for limited class of non-minimum phase systems. Yamada et al. gave the complete parametrization of all stabilizing modified repetitive controllers for non-minimum phase single-input/single-output systems (Yamada *et al.*, 2002*b*). They obtained the

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parametrization of all repetitive controllers using fusion of the parallel compensation technique and the solution of Bezout equation. However, they gave the parametrization of all repetitive controllers for limited class of non-minimum phase systems. Yamada et al. gave the complete parametrization of all stabilizing modified repetitive controllers for non-minimum phase single-input/single-output systems (Yamada *et al.*, 2002*b*). In this way, the parametrization of all stabilizing modified repetitive controllers has been considered.

However the modified repetitive controllers has bad disturbance attenuation characteristic (Gotou *et al.*, 1987). In order to improve the disturbance attenuation characteristic of the modified repetitive controllers, the multi-period repetitive controllers was proposed by (Gotou *et al.*, 1987). The parametrization of all stabilizing multi-period repetitive controllers for non-minimum phase systems, which is used to improve the disturbance attenuation characteristics of the modified repetitive controller, was solved in (Yamada *et al.*, 2004). Since the method in (Yamada *et al.*, 2004) cannot apply for the plant with uncertainty, Satoh, Yamada and Mei proposed the parametrization of all robust stabilizing multi-period repetitive controllers for the plant with uncertainty (Satoh *et al.*, 2006). However, using the method in (Satoh *et al.*, 2006), the transfer function from the periodic reference input to the output and that from the disturbance to the output have an infinite number of poles, even if the uncertainty does not exist. When the transfer function from the periodic reference input to the output and that from the disturbance to the output have an infinite number of poles, it is difficult to specify the input-output characteristic and the disturbance attenuation characteristic. From the practical point of view, it is desirable that the input-output characteristic and the disturbance attenuation characteristic are easily specified. In order to specify the input-output characteristic and the disturbance attenuation characteristic easily, the transfer function from the periodic reference input to the output and that from the disturbance to the output are desirable to have a finite number of poles.

In this paper, we propose the concept of robust stabilizing simple multi-period repetitive controllers for the plant with uncertainty and clarify the parametrization of all robust stabilizing simple multi-period repetitive control systems such that the controller works as a robust stabilizing multi-period repetitive controller and the transfer function from the periodic reference input to the output and that from the disturbance to the output have a finite number of poles.

Notation

R	the set of real numbers.
R_+	$R \cup \{\infty\}$.
$R(s)$	the set of real rational function with s.
RH_∞	the set of stable proper real rational functions.
H_∞	the set of stable causal functions.
$D^\perp$	orthogonal complement of D, i.e., $\left[\, D \;\; D^\perp \,\right]$ or $\begin{bmatrix} D \\ D^\perp \end{bmatrix}$ is unitary.
A^T	transpose of A.
$A^\dagger$	pseudo inverse of A.
$\rho(\{\cdot\})$	spectral radius of $\{\cdot\}$.
$\\|\{\cdot\}\\|_\infty$	H_∞ norm of $\{\cdot\}$.
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	represents the state space description $C(sI-A)^{-1}B+D$.

2. PROBLEM FORMULATION

Consider the unity feedback system in

$$\begin{cases} y = G(s)u + d \\ u = C(s)(r-y) \end{cases}, \tag{1}$$

where $G(s) \in R(s)$ is the plant, $C(s)$ is the controller, $u \in R$ is the control input, $d \in R$ is the disturbance, $y \in R$ is the output and $r \in R$ is the periodic reference input with period T satisfying

$$r(t+T) = r(t) \quad (\forall t \geq 0). \tag{2}$$

The nominal plant of $G(s)$ is denoted by $G_m(s) \in R(s)$. Both $G(s)$ and $G_m(s)$ are assumed to have no zero or pole on the imaginary axis. In addition, it is assumed that the number of poles of $G(s)$ in the closed right half plane is equal to that of $G_m(s)$. The relation between the plant $G(s)$ and the nominal plant $G_m(s)$ is written as

$$G(s) = G_m(s)(1+\Delta(s)). \tag{3}$$

The set of $\Delta(s)$ is all rational functions satisfying

$$|\Delta(j\omega)| < |W_T(j\omega)| \quad (\forall\omega \in R_+), \tag{4}$$

where $W_T(s)$ is an asymptotically stable rational function.

The robust stability condition for the plant $G(s)$ with uncertainty $\Delta(s)$ satisfying (4) is given by

$$\|T(s)W_T(s)\|_\infty < 1, \tag{5}$$

where $T(s)$ is the complementary sensitivity function given by

$$T(s) = \frac{G_m(s)C(s)}{1+G_m(s)C(s)}. \tag{6}$$

According to (Gotou *et al.*, 1987), the multi-period repetitive controller $C(s)$ in (1) is written by the form in

$$C(s) = C_0(s) + \frac{\displaystyle\sum_{i=1}^{N} C_i(s)e^{-sT_i}}{1-\displaystyle\sum_{i=1}^{N} q_i(s)e^{-sT_i}}, \tag{7}$$

where N is arbitrary positive integer, $C_0(s) \in R(s)$, $C_i(s) \neq 0 \in R(s)(i = 1,\ldots,N)$, $q_i(s) \in R(s)(i = 1,\ldots,N)$ are low-pass filter satisfying $\sum_{i=1}^{N} q_i(0) = 1$ and $T_i \in R(i = 1,\ldots,N)$.

Using the modified repetitive controller $C(s)$ in (7), the transfer function from the periodic reference input r to the output y and that from the disturbance d to the output y in (1) are written as

$$\begin{aligned}\frac{y}{r} &= \frac{C(s)G(s)}{1+C(s)G(s)} \\ &= \left\{ C_0(s)G_m(s)\left(1+\Delta(s)\right) - \sum_{i=1}^{N} \left(C_0(s)q_i(s)\right.\right. \\ &\quad \left.\left. -C_i(s)\right) e^{-sT_i} G_m(s)\left(1+\Delta(s)\right)\right\} [1+ \\ &\quad C_0(s)G_m(s)\left(1+\Delta(s)\right) - \sum_{i=1}^{N} [q_i(s)\{1+ \\ &\quad C_0(s)G_m(s)\left(1+\Delta(s)\right)\} - C_i(s)G_m(s) \\ &\quad \left(1+\Delta(s)\right)] e^{-sT_i}]^{-1} \end{aligned} \tag{8}$$

and

$$\begin{aligned}\frac{y}{d} &= \frac{1}{1+C(s)G(s)} \\ &= \left\{1 - \sum_{i=1}^{N} q_i(s)e^{-sT_i}\right\} [1 + C_0(s)G_m(s) \\ &\quad \left(1+\Delta(s)\right) - \sum_{i=1}^{N} [q_i(s)\{1 + C_0(s)G_m(s) \\ &\quad \left(1+\Delta(s)\right)\} - C_i(s)G_m(s) \\ &\quad \left(1+\Delta(s)\right)] e^{-sT_i}]^{-1}, \end{aligned} \tag{9}$$

respectively. Generally, the transfer function from the periodic reference input r to the output y in (8) and that from the disturbance d to the output y in (9) have an infinite number of poles, even if $\Delta(s) = 0$. When the transfer function from the periodic reference input r to the output y and that from the disturbance d to the output y have an infinite number of poles, it is difficult to specify the input-output characteristic and the disturbance attenuation characteristic. From the practical point of view, it is desirable that the input-output characteristic and the disturbance attenuation characteristic are easily specified. In order to specify the input-output characteristic and the disturbance attenuation characteristic easily, the transfer function from the periodic reference input r to the output y and that from the disturbance d to the output y are desirable to have a finite number of poles.

From above practical requirement, we clarify the parametrization of all robust stabilizing simple multi-period repetitive controllers defined as follows:

Definition 1. (robust stabilizing simple multi-period repetitive controller)
We call the controller $C(s)$ the robust stabilizing simple multi-period repetitive controller, if following expressions hold true:

(1) The controller $C(s)$ works as a multi-period repetitive controller. That is, the controller $C(s)$ is written by (7), where $C_0(s) \in R(s)$, $C_i(s) \neq 0 \in R(s)(i = 1, \ldots, N)$ and $q_i(s) \in R(s)(i = 1, \ldots, N)$ satisfies $\sum_{i=1}^{N} q_i(0) = 1$.
(2) When $\Delta(s) = 0$, the controller $C(s)$ makes the transfer function from the periodic reference input r to the output y in (1) and that from the disturbance d to the output y in (1) have a finite number of poles.
(3) The controller $C(s)$ satisfies the robust stability condition in (5).

3. THE PARAMETRIZATION OF ALL ROBUST STABILIZING SIMPLE MULTI-PERIOD REPETITIVE CONTROLLERS

In this section, we give the parametrization of all robust stabilizing simple multi-period repetitive controllers defined in Definition 1.

In order to obtain the parametrization of all robust stabilizing simple multi-period repetitive controllers, we must see that the robust stabilizing controllers hold (5). The problem of obtaining the controller $C(s)$, which is not necessarily a multi-period repetitive controller, satisfying (5) is equivalent to the following H_∞ problem. In order to obtain the controller $C(s)$ satisfying (5), we consider the control system shown in Fig. 1. $P(s)$ is selected such

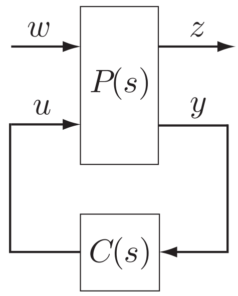

Fig. 1. Block diagram of H_∞ control problem

that the transfer function from w to z in Fig. 1 is equal to $T(s)W_T(s)$. The state space description of $P(s)$ is, in general,

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) = C_1 x(t) + D_{12} u(t), \\ y(t) = C_2 x(t) + D_{21} w(t) \end{cases} \tag{10}$$

where $A \in R^{n\times n}$, $B_1 \in R^n$, $B_2 \in R^n$, $C_1 \in R^{1\times n}$, $C_2 \in R^{1\times n}$, $D_{12} \in R$, $D_{21} \in R$. $P(s)$ is called the generalized plant. $P(s)$ is assumed to satisfy the standard assumption in (Doyle *et al.*, 1989). Under these assumptions, according to (Doyle *et al.*, 1989), following lemma holds true.

Lemma 1. If controllers satisfying (5) exist, both

$$\begin{aligned} &X\left(A - B_2 D_{12}^{\dagger} C_1\right) + \left(A - B_2 D_{12}^{\dagger} C_1\right)^T X \\ &+X\left(B_1 B_1^T - B_2\left(D_{12}^T D_{12}\right)^{-1} B_2^T\right) X \\ &+\left(D_{12}^{\perp} C_1^T\right)^T D_{12}^{\perp} C_1^T = 0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} &Y\left(A - B_1 D_{21}^{\dagger} C_2\right)^T + \left(A - B_1 D_{21}^{\dagger} C_2\right) Y \\ &+Y\left(C_1^T C_1 - C_2^T\left(D_{21} D_{21}^T\right)^{-1} C_2\right) Y \\ &+B_1 D_{21}^{\perp}\left(B_1 D_{21}^{\perp}\right)^T = 0 \end{aligned} \tag{12}$$

have solutions $X \geq 0$ and $Y \geq 0$ such that

$$\rho(XY) < 1 \tag{13}$$

and both

$$A - B_2 D_{12}^{\dagger} C_1 + \left(B_1 B_1^T - B_2\left(D_{12}^T D_{12}\right)^{-1} B_2^T\right) X$$

and

$$A - B_1 D_{21}^{\dagger} C_2 + Y \left(C_1^T C_1 - C_2 \left(D_{21} D_{21}^T \right)^{-1} C_2 \right)$$

have no eigenvalue in the closed right half plane. Using X and Y, the parametrization of all controllers satisfying (5) is given by

$$\begin{aligned} &C(s) \\ &= C_{11}(s) + C_{12}(s) Q(s) (I - C_{22}(s) Q(s))^{-1} C_{21}(s), \end{aligned} \tag{14}$$

where

$$\begin{bmatrix} C_{11}(s) & C_{12}(s) \\ C_{21}(s) & C_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & D_{c22} \end{array} \right] \tag{15}$$

$$\begin{aligned} A_c &= A + B_1 B_1^T X - B_2 \left(D_{12}^{\dagger} C_1 + E_{12}^{-1} B_2^T X \right) \\ &\quad - (I - XY)^{-1} \left(B_1 D_{21}^{\dagger} + Y C_2^T E_{21}^{-1} \right) \\ &\quad \left(C_2 + D_{21} B_1^T X \right), \\ B_{c1} &= (I - XY)^{-1} \left(B_1 D_{21}^{\dagger} + Y C_2^T E_{21}^{-1} \right), \\ B_{c2} &= (I - XY)^{-1} \left(B_2 + Y C_1^T D_{12} \right) E_{12}^{-1/2}, \\ C_{c1} &= -D_{12}^{\dagger} C_1 - E_{12}^{-1} B_2^T X, \\ C_{c2} &= -E_{21}^{-1/2} \left(C_2 + D_{21} B_1^T X \right), \\ D_{c11} &= 0, \;\; D_{c12} = E_{12}^{-1/2}, \;\; D_{c21} = E_{21}^{-1/2}, \\ D_{c22} &= 0, \;\; E_{12} = D_{12}^T D_{12}, \;\; E_{21} = D_{21} D_{21}^T \end{aligned}$$

and the free parameter $Q(s) \in H_\infty$ is any function satisfying $\|Q(s)\|_\infty < 1$ (Doyle *et al.*, 1989).

Using Lemma 1, the parametrization of all robust stabilizing simple multi-period repetitive controllers is given by following theorem.

Theorem 1. If simple multi-period repetitive controllers satisfying (5) exist, both (11) and (12) have solutions $X \geq 0$ and $Y \geq 0$ satisfying (13) and both

$$A - B_2 D_{12}^{\dagger} C_1 + \left(B_1 B_1^T - B_2 \left(D_{12}^T D_{12} \right)^{-1} B_2^T \right) X$$

and

$$A - B_1 D_{21}^{\dagger} C_2 + Y \left(C_1^T C_1 - C_2 \left(D_{21} D_{21}^T \right)^{-1} C_2 \right)$$

have no eigenvalue in the closed right half plane. Using X and Y, the parametrization of all robust stabilizing simple repetitive controllers satisfying (5) is given by

$$\begin{aligned} &C(s) \\ &= C_{11}(s) + C_{12}(s) Q(s) \left(I - C_{22}(s) Q(s) \right)^{-1} C_{21}(s), \end{aligned} \tag{16}$$

where $C_{ij}(s)(i = 1, 2; j = 1, 2)$ are given by (15) and the free parameter $Q(s) \in H_\infty$ is any function satisfying $\|Q(s)\|_\infty < 1$ and written by

$$Q(s) = \frac{Q_{n0}(s) + \sum_{i=1}^{N} Q_{ni}(s) e^{-sT_i}}{Q_{d0}(s) + \sum_{i=1}^{N} Q_{di}(s) e^{-sT_i}}, \tag{17}$$

$$Q_{ni}(s) = G_d(s) \bar{Q}_i(s) (i = 1, \ldots, N) \tag{18}$$

and

$$\begin{aligned} Q_{di}(s) &= -\frac{1}{1 + C_{11}(s) G_m(s)} G_n(s) \bar{Q}_i(s) \\ &\quad (i = 1, \ldots, N), \end{aligned} \tag{19}$$

where $Q_{n0}(s) \in RH_\infty$, $Q_{d0}(s) \in RH_\infty$, $G_n(s) \in RH_\infty$ and $G_d(s) \in RH_\infty$ are coprime factors of $-C_{22}(s) + (-C_{11}(s) C_{22}(s) + C_{12}(s) C_{21}(s)) G_m(s)$ on RH_∞ satisfying

$$\begin{aligned} &\frac{G_n(s)}{G_d(s)} \\ &= -C_{22}(s) + (-C_{11}(s) C_{22}(s) + C_{12}(s) C_{21}(s)) \\ &\quad G_m(s), \end{aligned} \tag{20}$$

$\bar{Q}_i(s) \neq 0 \in RH_\infty (i = 1, \ldots, N)$ are any functions and

$$\sum_{i=1}^{N} \left\{ -\frac{Q_{di}(0) - C_{22}(0) Q_{ni}(0)}{Q_{d0}(0) - C_{22}(0) Q_{n0}(0)} \right\} = 1 \tag{21}$$

holds true.

(Proof) First, necessity is shown. That is, if the multi-period repetitive controller written by (7) stabilizes the control system in (1) robustly and makes the transfer function from the periodic reference input r to the output y in (8) and that from the disturbance d to the output y in (9) have a finite number of poles when $\Delta(s) = 0$, then $C(s)$ and $Q(s)$ are written by (16) and (17), respectively. From Lemma 1, the parametrization of all robust stabilizing controllers $C(s)$ for $G(s)$ is written by (16), where $\|Q(s)\|_\infty < 1$. In order to prove the necessity, we will show that if $C(s)$ written by (7) stabilizes the control system in (1) robustly and makes the transfer function from the periodic reference input r to the output y in (8) and that from the disturbance d to the output y in (9) have a finite number of poles when $\Delta(s) = 0$, then the free parameter $Q(s)$ in (16) is written by (17). Substituting $C(s)$ in (7) into (16), we have (17), where

$$\begin{aligned} Q_{n0}(s) &= \left(C_{11n}(s) C_{0d}(s) - C_{11d}(s) C_{0n}(s) \right) C_d(s) \\ &\quad q_d(s) C_{12d}(s) C_{22d}(s) C_{21d}(s), \end{aligned} \tag{22}$$

$$\begin{aligned} Q_{ni}(s) &= \{ (C_{0n}(s) C_d(s) C_{11d}(s) - C_{0d}(s) C_d(s) \\ &\quad C_{11n}(s)) \, q_{in}(s) - C_{0d}(s) C_{in}(s) q_d(s) \\ &\quad C_{11d}(s) \} \, C_{12d}(s) C_{22d}(s) C_{21d}(s) \\ &\quad (i = 1, \ldots, N), \end{aligned} \tag{23}$$

$$\begin{aligned} Q_{d0}(s) &= (-C_{0n}(s) C_{11d}(s) C_{12d}(s) C_{22n}(s) C_{21d}(s) \\ &\quad + C_{0d}(s) C_{11n}(s) C_{12d}(s) C_{22n}(s) C_{21d}(s) \\ &\quad - C_{0d}(s) C_{11d}(s) C_{12n}(s) C_{22d}(s) C_{21n}(s)) \\ &\quad C_d(s) q_d(s) \end{aligned} \tag{24}$$

and

$$\begin{aligned} & Q_{di}(s) \\ &= (C_{0n}(s)C_d(s)C_{11d}(s)C_{12d}(s)C_{22n}(s) \\ &\quad C_{21d}(s) - C_{0d}(s)C_d(s)C_{11n}(s)C_{12d}(s) \\ &\quad C_{22n}(s)C_{21d}(s) + C_{0d}(s)C_d(s)C_{11d}(s) \\ &\quad C_{12n}(s)C_{22d}(s)C_{21n}(s))\, q_{in}(s) - C_{0d}(s) \\ &\quad C_{in}(s)q_d(s)C_{11d}(s)C_{12d}(s)C_{22n}(s)C_{21d}(s). \\ &\quad (i = 1, \ldots, N) \end{aligned} \tag{25}$$

Here, $C_{0n}(s) \in RH_\infty$ and $C_{0d}(s) \in RH_\infty$ are coprime factors of $C_0(s)$ on RH_∞ satisfying

$$C_0(s) = C_{0n}(s)C_{0d}^{-1}(s), \tag{26}$$

$q_{in}(s) \in RH_\infty(i = 1, \ldots, N)$, $q_d(s) \in RH_\infty$, $C_{in}(s) \in RH_\infty(i = 1, \ldots, N)$, $C_d(s) \in RH_\infty$, $C_{ijn}(s) \in RH_\infty(i = 1, 2; j = 1, 2)$ and $C_{ijd}(s) \in RH_\infty(i = 1, 2; j = 1, 2)$ are coprime factors satisfying

$$q_i(s) = \frac{q_{in}(s)}{q_d(s)}(i = 1, \ldots, N), \tag{27}$$

$$C_i(s) = \frac{C_{in}(s)}{C_d(s)}(i = 1, \ldots, N) \tag{28}$$

and

$$C_{ij}(s) = \frac{C_{ijn}(s)}{C_{ijd}(s)}(i = 1, 2; j = 1, 2). \tag{29}$$

From (22)∼(25), all of $Q_{n0}(s)$, $Q_{ni}(s)(i = 1, \ldots, N)$, $Q_{d0}(s)$ and $Q_{di}(s)(i = 1, \ldots, N)$ are included in RH_∞. Thus, we have shown that if $C(s)$ written by (7) stabilize the control system in (1) robustly, $Q(s)$ in (16) is written by (17). Since $\sum_{i=1}^{N} q_i(0) = 1(i = 1, \ldots, N)$, (21) holds true.

The rest to prove necessity is to show that when $\Delta(s) = 0$, if $C(s)$ in (7) make the transfer function from the periodic reference input r to the output y and the disturbance d to the output y have a finite number of poles, then $Q_{ni}(s)$ and $Q_{di}(s)$ are written by (18) and (19), respectively. From (17), when $\Delta(s) = 0$, the transfer function from the periodic reference input r to the output y and that from the disturbance d to the output y are written by

$$\frac{y}{r} = \frac{G_{ryn}(s)}{G_{ryd}(s)} \tag{30}$$

and

$$\frac{y}{d} = \frac{G_{dyn}(s)}{G_{dyd}(s)}, \tag{31}$$

respectively, where

$$\begin{aligned} & G_{ryn}(s) \\ &= [\{C_{11}(s)Q_{d0}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s) \\ &\quad C_{21}(s))\, Q_{n0}(s)\} + \sum_{i=1}^{N} \{C_{11}(s)Q_{di}(s) \\ &\quad + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s))\, Q_{ni}(s)\} \\ &\quad e^{-sT_i}]\, G_m(s), \end{aligned} \tag{32}$$

$$\begin{aligned} & G_{ryd}(s) \\ &= [(Q_{d0}(s) - C_{22}(s)Q_{n0}(s)) + \{C_{11}(s)Q_{d0}(s) \\ &\quad + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s))\, Q_{n0}(s)\} \\ &\quad G_m(s)] + \sum_{i=1}^{N} [Q_{di}(s) - C_{22}(s)Q_{ni}(s) + \{C_{11}(s) \\ &\quad Q_{di}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s)) \\ &\quad Q_{ni}(s)\}\, G_m(s)]\, e^{-sT_i}, \end{aligned} \tag{33}$$

$$\begin{aligned} & G_{dyn}(s) \\ &= Q_{d0}(s) - C_{22}(s)Q_{n0}(s) + \sum_{i=1}^{N} (Q_{di}(s) \\ &\quad + C_{22}(s)Q_{ni}(s))\, e^{-sT_i} \end{aligned} \tag{34}$$

and

$$\begin{aligned} & G_{dyd}(s) \\ &= [(Q_{d0}(s) - C_{22}(s)Q_{n0}(s)) + \{C_{11}(s)Q_{d0}(s) \\ &\quad + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s))\, Q_{n0}(s)\} \\ &\quad G_m(s)] + \sum_{i=1}^{N} [Q_{di}(s) - C_{22}(s)Q_{ni}(s) + \{C_{11}(s) \\ &\quad Q_{di}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s)) \\ &\quad Q_{ni}(s)\}\, G_m(s)]\, e^{-sT_i}. \end{aligned} \tag{35}$$

From the assumption that the transfer function from the periodic reference input r to the output y in (30) and that from the disturbance d to the output y in (31) have a finite number of poles, (33) and (35),

$$\begin{aligned} & Q_{di}(s) - C_{22}(s)Q_{ni}(s) + \{C_{11}(s)Q_{di}(s) \\ &\quad + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s))\, Q_{ni}(s)\}\, G_m(s) \\ &= 0 \quad (i = 1, \ldots, N) \end{aligned} \tag{36}$$

is satisfied. Using (20), this equation is rewritten by

$$Q_{di}(s) = -\frac{1}{1 + C_{11}(s)G_m(s)} \frac{G_n(s)}{G_d(s)} Q_{ni}(s) \quad (i = 1, \ldots, N). \tag{37}$$

Since $Q_{ni}(s) \in RH_\infty(i = 1, \ldots, N)$ and $Q_{di}(s) \in RH_\infty(i = 1, \ldots, N)$, $Q_{ni}(s)$ and $Q_{di}(s)$ are written by (18) and (19), respectively, where $\bar{Q}_i(s) \in RH_\infty(i = 1, \ldots, N)$. From the assumption that $C_i(s) \neq 0(i = 1, \ldots, N)$ and from (23) and (25), $\bar{Q}_i(s) \neq 0(i = 1, \ldots, N)$ hold true. We have thus proved necessity.

Next, sufficiency is shown. That is, if $C(s)$ and $Q(s) \in H_\infty$ are settled by (16) and (17), respectively, then the controller $C(s)$ is written by the form in (7), $\sum_{i=1}^{N} q_i(0) = 1$ holds true and the transfer function from the periodic reference input r to the output y and that from the disturbance d to the output y have a finite number of poles. Substituting (17) into (16), we have (7), where, $C_0(s)$, $C_i(s)(i = 1, \ldots, N)$ and $q_i(s)(i = 1, \ldots, N)$ are denoted by

$$C_0(s) = \frac{C_{11}(s)Q_{d0}(s) + (C_{12}(s)C_{21}(s)}{Q_{d0}(s) - C_{22}(s)Q_{n0}(s)}$$

$$\frac{-C_{11}(s)C_{22}(s))Q_{n0}(s)}{}, \tag{38}$$

$$C_i(s) = \frac{C_{11}(s)Q_{di}(s) + (-C_{11}(s)C_{22}(s)}{Q_{d0}(s) - C_{22}(s)Q_{n0}(s)} \frac{+C_{12}(s)C_{21}(s))\,Q_{ni}(s)}{} + C_0(s)q_i(s) \quad (i = 1, \ldots, N) \tag{39}$$

and

$$q_i(s) = -\frac{Q_{di}(s) - C_{22}(s)Q_{ni}(s)}{Q_{d0}(s) - C_{22}(s)Q_{n0}(s)} \quad (i = 1, \ldots, N). \tag{40}$$

We find that if $C(s)$ and $Q(s)$ are settled by (16) and (17), respectively, then the controller $C(s)$ is written by the form in (7). From $\bar{Q}_i(s) \neq 0 (i = 1, \ldots, N)$ and (39), $C_i(s) \neq 0 (i = 1, \ldots, N)$ holds true. Substituting (21) into (40), we have $\sum_{i=1}^{N} q_i(0) = 1$. In addition, from (18) and (19) and easy manipulation, we can confirm that when $\Delta(s) = 0$, the transfer function from the periodic reference input r to the output y and that from the disturbance d to the output y have a finite number of poles.

We have thus proved Theorem 1. ∎

4. CONCLUSION

In this paper, we proposed the parametrization of all robust stabilizing simple multi-period repetitive control systems such that the controller works as a robust stabilizing multi-period repetitive controller and the transfer function from the periodic reference input to the output and that from the disturbance to the output have a finite number of poles.

REFERENCES

Doyle, J.C., K. Glover, P.P. Khargonekar and B.A. Francis (1989). State-space solution to standard H_2 and H_∞ control problems. *IEEE Trans. on Automatic Control* **AC-34**(), 831–847.

Glaria, J.J. and G.C. Goodwin (1994). A parameterization for the class of all stabilizing controllers for linear minimum phase system. *IEEE Trans. on Automatic Control* **AC-39-2**, 433–434.

Gotou, M., S. Matsubayashi, F. Miyazaki, S. Kawamura and S. Arimoto (1987). A Robust System with an Iterative Learning Compensator and a Proposal of Multi-Period Learning Compensator. *J. of The Society of Instrument and Controll Engneers* **No. 5**, 367–374.

Hara, S. and Y. Yamamoto (1986). Stability of Multivariable Repetitive Control Systems - Stability Condition and Class of Stabilizing Controllers. **22-12**, 1256–1261.

Hara, S., Y. Yamamoto, T. Omata and M. Nakano (1988). Repetitive Control System: A New Type Servo System for Periodic Exogenous Signals. *IEEE Trans. on Automatic Control* **AC-33-7**, 659–668.

Ikeda, M. and M. Takano (1990). Repetitive Control for Systems with Nonzero Relative Degree. In: *Proc. 29th CDC*. pp. 1667–1672.

Inoue, T., M. Nakano, T. Kubo, S. Matsumoto and H. Baba (1981). High accracy control of a proton synchrotron magnet power supply. *Proc. IFAC 8th World Congress*.

Katoh, H. and Y. Funahashi (1996). A Design Method of Repetitive Controlles. *Trans. of the Society of Instrument and Control Engineers* **32-12**, 1601–1605.

Kucera, V. (1979). *Discrete linear system, The polynomial eqnarray approach. Wiley.*

Nakano, T., T. Inoue, Y. Yamamoto and S. Hara (1989). *Repetitive Control*. SICE Publications. Tokyo.

Omata, T., S. Hara and M. Nakano (1987). Nonlinear Repetitive Control with Application to Trajectory Control of Manipulators. *J. of Robotic Systems* **4-5**, 631–652.

Satoh, K., K. Yamada and Y. Mei (2006). The parametrization of all robust stabilizing multi-period repetitive controllers. *Theoretical and Applied Mechanics* **55**(), 125–132.

Vidyasagar, M. (1985). *Control System Synthesis - A factorization approach -*. MIT Press.

Watanabe, K. and M. Yamatari (1986). Stabilization of Repetitive Control System-Spectral Decomposition Approach. *Trans. of the Society of Instrument and Control Engineers* **22-5**, 535–541.

Weiss, G. (1997). Repetitive Control Sysems: Old and New Ideas. In: *Systems and Control in the Twenty-First Century* (C.I. Byrnes, B. N. Datta, C.F. Martin, D.S. Gilliam, Ed.). ed.. Vol. . pp. 389–404. Birkhäuser. Boston.

Yamada, K. and T. Okuyama (2000). A parametrization of all stabilizing repetitive controllers for linear minimum phase systems. *Trans. of the Society of Instrument and Control Engineers* **36**, 328–334.

Yamada, K., K. Satoh and T. Arakawa (2004). The parametrization of all stabilizing multi-period repetitive controllers. In: *The International Conference on Cybernetics and Information Technologies, System and Applications*. Vol. II. pp. 358–363.

Yamada, K., K. Satoh and T. Okuyama (2002*a*). The parameterization of all stabilizing repetitive controllers for a certain class of non-minimum phase systems. In: *Preprints of the 15th IFAC World Congress CD-ROM*.

Yamada, K., K. Satoh, N. Iida and T. Okuyama (2002*b*). Control structure of all stabilizing repetitive controllers for the non-minimum phase systems. In: *Proceedings of the 4th Asian Control Conference*.

Yamamoto, Y. (1993). Learning control and related problems in infinite-dimensional systems. In: *Essays on control:Perspectives in the theory and its applications* (H. L. Trentelman and J. C. Willems, Ed.). ed.. Vol. . pp. 191–222. Birkhaüser. .

Yamamoto, Y. and S. Hara (1988). Relationships between internal and external stability with applications to a servo problem. *IEEE Transactions on Automatic Control* **33**(), 1044–1052.

Yamamoto, Y. and S. Hara (1992). Internal and external stability and robust stability condition for a class of infinite-dimensional systems. *Automatica* **28**(), 81–93.

Youla, D.C., H. Jabr and J.J. Bongiorno (1976). Modern Wiener-Hopf design of optimal controllers. Part I. *IEEE Trans. on Automatic Control* **AC-21**, 3–13.