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A Technique of a Stability Domain Determination for Nonlinear Discrete Polynomial Systems

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Abstract: This paper is devoted to the asymptotic stability region estimation for nonlinear discrete polynomial systems. An algebraic method is derived for the enlargement of a guaranteed stability region in which the asymptotic stability is ensured. The advantages of the proposed method are the accuracy of determination of the largest stability boundary, its numerical and theoretical robustness and its applicability to wide classes of dynamical discrete systems. A numerical example illustrates the proposed method.

Keywords: Autonomous polynomial system; Asymptotic stability region; Reversing trajectory method; Discrete time systems; Nonlinear systems.

1. INTRODUCTION

The problem of estimating the stability domain of an equilibrium point is well known in the area of nonlinear system analysis and control (Liberzon and Morse [1999], Loccufier and Noldus [2000]). In fact a given initial state lies within such a region is a question of practical importance in many engineering application as the synthesis of preferment nonlinear feedback control (Chiang and Thorp [1989], Noldus and Loccufier [1995]).

The majority of studies concerned with this object are setting in Lyapunov theory of stability and so they are called Lyapunov methods which are essentially applied to continuous power systems (Peleties and De Carlo [1991], Tesi et al. [1996], Peterfreund and Baram [1998]). Other works present an interesting "non Lyapunov techniques based on the reversing trajectory method which lead to satisfactory results by reaching the global asymptotic stability domain for the continuous nonlinear dynamical systems (Genesio et al. [1985], Bacha et al. [1997]).

In spite of the fact that more than few methods and studies are made and developed to estimate the regions of asymptotic stability for continuous dynamical nonlinear systems, one may notice that much less attention has been paid to the possibility of estimating the regions of asymptotic stability for nonlinear discrete time system (Ye et al. [1996]).

In our previous works we have developed several techniques allowing estimating the Region of Asymptotic Stability (RAS) of polynomial discrete time systems (Benhadj braiek [1996b], Bacha et al. [2006a], Bacha et al. [2006b], Bacha et al. [2007a], Bacha et al. [2007b]). The main limitation of the proposed techniques consists on the validity of the numerical inversion of the discrete state equation and so that the exactness of the obtained RAS.

In this paper we consider a new approach of estimating a large asymptotic stability domain for discrete time nonlinear polynomial system. Based on the Kronecker product (Benhadj braiek [1996a], Benhadj braiek [1996b]) and the Grownwell-bellman lemma for the estimation of a guaranteed region of stability, the proposed method permits to improve previous results in this field of research.

This paper is organized as follows: after the description of the studied systems in the second section, a guaranteed stability region (GSR) is characterized in the third section. Then in the next section a technique of enlargement of this region is developed. A simulation example illustrating the proposed approach is presented in the fifth section.

2. DESCRIPTION OF THE STUDIED SYSTEMS

We consider in this paper the discrete nonlinear polynomial systems described by a state equation of the following form

$$X(k+1) = F(X(k)) = \sum_{i=1}^{q} A_i X^{[i]}(k). \qquad (1)$$

where k is the discrete time variable, $X(k) \in \Re^n$ is the state vector, $X^{[i]}(k)$ designates the $i-th$ Kronecker power of the vector $X(k)$ and A_i, $i = 1, ..., q$ are $(n \times n^i)$ matrices.

The system (1) can also be written in the following form:

$$X(k+1) = M(X(k)).X(k) \qquad (2)$$

where:

$$M(X(k)) = A_1 + \sum_{j=2}^{q} A_j (I_n \otimes X^{[j-1]})(X(k))$$

$\otimes$ designates the Kronecker product (Benhadj braiek [1996b], Bacha et al. [2006a]).

Assumption 1: the linear part of the discrete system (1) is asymptotically stable i.e. all the eigenvalues of the matrix are of module little than 1.

3. GUARANTEED STABILITY REGION

Our purpose is to determine a sufficient domain Ω_0 of the initial conditions variation, in which the asymptotic stability of the equilibrium point $X = 0$ of the system (1) is guaranteed, according to the following definition:

$$\forall X_0 \in \Omega_0, \forall k \in \aleph, X(k, k_0, X_0) \in R$$
$$and \lim_{k\to\infty} X(k, k_0, X_0) = 0 \tag{3}$$

where $X(k, k_0, X_0)$ designates the solution of the nonlinear recurrent equation (1) with the initial condition $X(k_0) = X_0$.

The stability domain that we propose is considered as a ball of radius R_0 and of center the origin $X = 0$ i.e.,

$$\Omega_0 = \{X_0 \in \Re^n ; \|X_0\| < R_0\} \tag{4}$$

the radius R_0 is called the stability radius of the system (1).

A simple domain ensuring the stability of the system (1) is defined by the following theorem (Benhadj braiek [1996a]).

Theorem 1. Consider the discrete system (1) satisfying the assumption 1, and let c and α the positive numbers verifying $\alpha \in]0, 1[$,

$$\|A_1^{k-k_0}\| \leq c\alpha^{k-k_0} \forall k \geq k_0 \tag{5}$$

Then this system is asymptotically stable on the domain Ω_0 defined in (4) with R_0 the unique positive solution of the following equation:

$$\sum_{k=2}^{q} \gamma_k R_0^{k-1} - \frac{1-\alpha}{c} \tag{6}$$

where $\gamma_k, k = 2, ..., q$ denote:

$$\gamma_k = c^{k-1} \|A_k\| \tag{7}$$

Furthermore the stability is exponentially.

Proof. The equation (1) can be written as:

$$X(k+1) = A_1 X(k) + h(X(k))X(k) \tag{8}$$

with

$$h(X(k+1)) = \sum_{j=2}^{q} A_j (I_n \otimes X^{[j-1]}(k)) \tag{9}$$

Let us consider that:

$$\forall k \geq k_0, \|X(k)\| \leq R \tag{10}$$

then, we have using the matrix norm property of the kronecker product.

$$\|h(X(k))\| \leq \lambda(R) \tag{11}$$

with:

$$\lambda(R) = \sum_{j=2}^{q} \|A_j\| R^{j-1} \tag{12}$$

By using the lemma 1 (see the appendix), we have:

$$\|X(k)\| \leq \ c(\alpha + c\lambda(R))^{k-k_0} \|X(k_0)\| \tag{13}$$

with $g(X) = h(X)X$, we have $\|g(X)\| \leq \lambda(R)\|X\|$.

Then, if:

$$\lambda(R) < \frac{1-\alpha}{c} \tag{14}$$

the system is exponentially stable.

Now, to ensure the hypothesis (10) it is sufficient to have (from (6)):

$$c\|X(k_0)\| \leq R_1 or \|X(k_0)\| \leq R_0 = \frac{R_1}{c} \tag{15}$$

R_1 satisfies the equation (14) implies that R_0 satisfies the equation (6) of the theorem 1.

4. ENLARGEMENT OF THE GUARANTEED STABILITY REGION (GSR)

Our object in this section is to enlarge the Guaranteed Stability Region Ω_0 characterized in the section 3. For this goal, we consider the boundary Γ_0 of the obtained GSR of radius R_0. Let X_0^i be a point belonging in Γ_0, and X_k^i the image of X_0^i by the $F(.)$ function characterizing the considered system, k times.

$$X_k^i = F^k(X_0^i) \tag{16}$$

X_k^i is then a point belonging in the stability domain Ω_0

$$\|X_k^i\| < R_0, \|F^k(X_0^i)\| < R_0 \tag{17}$$

To enlarge the GSR, we will look for a radius $r_{0,i}$ such that for any initial state X_0 verifying

$$\|X_0 - X_0^i\| \leq r_{0,i}$$

one has

$$X_k = F^k(X_0) \in \Omega_0 \tag{18}$$

and the fact that after k iterations the state of the system attends the domain Ω_0 ensures that X_0 is a state belonging in the stability domain.

Let us note:

$$\delta X_0 = X_0 - X_0^i \tag{19}$$

and for $k \geq 1$

$$\delta X_k = X_k - X_k^i = F^k(X_0) - F^k(X_0^i) \tag{20}$$

δX_k can be expressed in terms of δX_0 as a polynomial function of degree $s = q^k$ where q is the degree of the $F(.)$ polynomial characterizing the system:

$$\delta X_k = E_1.\delta X_0 + E_2.\delta X_0^{[2]} + ... + E_s.\delta X_0^{[s]}; s = q^k \tag{21}$$

$E_1, E_2, ..., E_s$ are matrices depending on k and X_0^i and they can easily expressed in terms of A_i and X_0^i.

In the particular case where $q=3$ and $k=1$ one has:

$$\begin{aligned}\delta X_k &= X_1 - X_1^i = F(X_0) - F(X_0^i)\\ &= D_1.\delta X_0 + D_2.\delta X_0^{[2]} + D_3.\delta X_0^{[3]}\end{aligned} \tag{22}$$

where

$$\begin{cases} D_1 = \begin{bmatrix} A_1 + A_2\left(X_0^i \otimes I_n + I_n \otimes X_0^i\right) + \\ A_3\left(\left(X_0^i \otimes I_n\right)\otimes X_0^i + \left(I_n \otimes X_0^i\right)\otimes X_0^i\right) + \\ A_3\left(X_0^{i[2]} \otimes I_n\right)\end{bmatrix} \\ D_2 = \begin{bmatrix} A_2 + A_3\left(\left(X_0^i \otimes I_n\right)\otimes I_n + \left(I_n \otimes X_0^i\right)\otimes I_n\right) \\ +A_3\left(I_n^{[2]} \otimes X_0^i\right)\end{bmatrix} \\ D_3 = A_3 \end{cases}$$

From the relation:

$$X_k = \delta X_k + F^k(X_0^i) \tag{23}$$

one has:

$$\|X_k\| \leq \|\delta X_k\| + \|F^k(X_0^i)\| \tag{24}$$

From (21) we have:

$$\|\delta x_k\| \leq e_1.\|\delta X_0\| + e_2.\|\delta X_0^{[2]}\| + ... + e_s.\|\delta X_0^{[s]}\| \tag{25}$$

with:

$$e_j = \|E_j\|, j = 1, 2, ..., s$$

Hence, we have:

$$\begin{aligned}\|x_k\| &\leq \sum_{j=1}^{s} e_j.\|\delta X_0^{[j]}\| + \|F(X_0^i)\|\\ &\leq \sum_{j=1}^{s} e_j.r_{0,i}^j + \|F(X_0^i)\|\end{aligned} \tag{26}$$

Since it is desired that:

$$\|X_k\| \leq R_0; (X_k \in \Omega_0) \tag{27}$$

it will be sufficient to have:

$$\begin{aligned}&\sum_{j=1}^{s} e_j.r_{0,i}^j = R_0 - \|F^k(X_0^i)\|\\ &e_1.r_{0,i} + e_2.r_{0,i}^2 + ... + e_s.r_{0,i}^s = R_0 - \|F^k(X_0^i)\| > 0\end{aligned} \tag{28}$$

which yields:

$$\|\delta X_0\| = \|X_0 - X_0^i\| \leq r_{0,i} \tag{29}$$

where $r_{0,i}$ is the unique positive solution of the polynomial equation:

$$e_1.r_{0,i} + e_2.r_{0,i}^2 + ... + e_s.r_{0,i}^s = R_0 - \|F^k(X_0^i)\| \tag{30}$$

and this result can be stated in the following theorem.

Theorem 2. Let the following polynomial discrete system described by:

$$X_{k+1} = F(X_k) = A_1.X_k + A_2.X_k^{[2]} + ... + A_q.X_k^{[q]} \tag{31}$$

and let Ω_0 the GSR of radiusR_0 given by theorem 1, and Γ_0 the boundary of the GSR, then:
For any point $X_0^i \in \Gamma_0$, the ball Ω_i centred on X_0^i and of radius $r_{0,i}$ the unique positive solution of the equation (30) is also a domain of stability of the considered system.

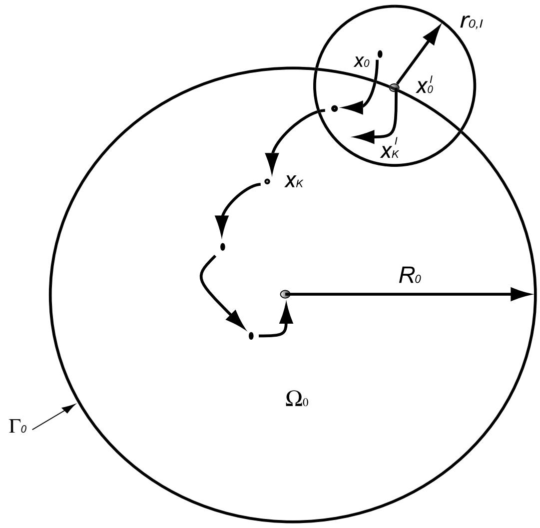

Fig. 1. Illustration of the proposed method principle.

In the particular case where $k=1$, one has the following corollary.

Corollary1 The ball Ω_i of radius $r_{0,i}$ solution of the equation:

$$\begin{aligned}\|D_1\|.r_{0,i} + \|D_2\|.r_{0,i}^2 + ... + \|D_q\|.r_{0,i}^q =\\ R_0 - \|F(X_0^i)\|\end{aligned} \tag{32}$$

is a domain of asymptotic stability of the considered system.

After considering all the points $X_0^i \in \Gamma_0$ (varying i), a new domain of stability is obtained by collecting all the little balls Ω_i to Ω_0:

$$\Omega = \bigcup_i \Omega_i \tag{33}$$

This idea is illustrated in Fig

5. SIMULATION RESULTS:APPLICATION TO VAN DER POOL MODEL

Let us consider the following discrete polynomial Van Der Pool model obtained from the Newton-Raphson approximation: (Jenning and McKeown [1992])

$$X_{k+1} = A_1.X_k + A_3.X_k^{[3]} \tag{34}$$

where $X_k = \begin{bmatrix} x_{1k} \\ x_{2k}\end{bmatrix}$

$$A_1 = \begin{pmatrix} 0.9988 & -0.0488 \\ 0.0488 & 0.950 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & -0.0012 & \\ & & 0_{2\times 6} \\ 0 & 0.0488 & \end{pmatrix}$$

Equation (34) has a linear asymptotically stable matrix A_1, which verifies the inequality (5) with $c = 1.7$ and $\alpha = 0.65$. Then, we may conclude that the origin is exponentially stable for each initial state X_0 included in the disc Ω_0 centered in the origin and of radius $R_0 = 0.33$.

The Fig. 2 shows the guaranteed stability domain Ω_0 obtained by the application of the theorem1, and the enlarged region resulting from the application of the theorem

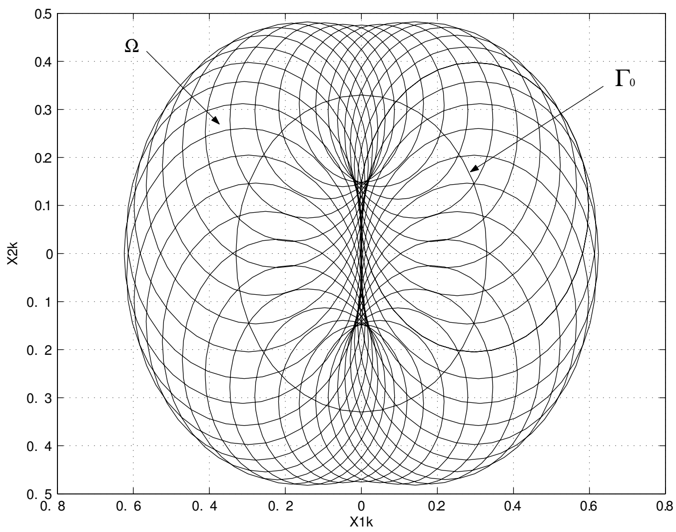

Fig. 2. Enlargement of a guaranteed *RAS* estimate of Van Der Pool discrete model.

2 for one iteration ($k = 1$), and for 22 points X_0^i on the boundary Γ_0. It comes out that the new result stated in the theorem 2 leads to an important enlargement of the guaranteed stability domain.

6. CONCLUSION

An advanced discrete algebraic method has been developed to determine and enlarge the region of asymptotic stability for autonomous nonlinear polynomial discrete time systems.

The exactness of the obtained RAS in this case constitutes the main advantage of the proposed approach.

The proposed method is proved theoretically and tested via numerical simulation on the discrete polynomial Van Der Pool model. This original technique can be considered as the equivalent discrete version of the reversing trajectory method which is used to estimate the RAS for the continuous systems.

Further research will be focused on the development and the implementation of an optimal numerical tool which allows to reach the larger region of asymptotic stability for discrete nonlinear systems.

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Appendix A. LEMMA1

Let a discrete nonlinear system defined by the state equation :

$$X(k+1) = A_1.X(k) + g(k, X(k)) \tag{A.1}$$

where the linear part satisfies the assumption 1, and the nonlinear part $g(k, X(k))$ verifies the following inequality:

$$g(k, X(k)) \leq \beta \|X(k)\| \tag{A.2}$$

where β is a positive constant.
Let $\Phi(k, k_0)$ denotes the transition matrix of the linear part of the discrete system (A.1):

$$\Phi(k, k_0) = A_1^{k-k_0} \tag{A.3}$$

and let c and α the positive numbers verifying $\alpha \in]01[$,

$$\|\Phi(k, k_0)\| \leq c\alpha^{k-k_0} \forall k \geq k_0 \tag{A.4}$$

Then the solution $X(k)$ of the system (A.1) verifies the following inequality:

$$\|X(k)\| \leq c(\alpha + c\beta)^{k-k_0} \|X(k_0)\| \tag{A.5}$$

So if $\beta < 1 - \frac{\alpha}{c}$, the system (A.1) is exponentially stable.