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Optimal Control of Switched Systems: A Polynomial Approach ⋆

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Abstract: A polynomial approach to solve the optimal control problem of switched systems is presented. It is shown that the representation of the original switched problem into a continuous polynomial systems allow us to use the method of moments. With this method and from a theoretical point of view, we provide necessary and sufficient conditions for the existence of minimizer by using particular features of the minimizer of its relaxed, convex formulation. Even in the absence of classical minimizers of the switched system, the solution of its relaxed formulation provide minimizers. *Copyright © 2008 IFAC*

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1. INTRODUCTION

In this paper we deal with optimal control problem of switched systems, i.e., continuous systems with switching signals. Recent efforts in switched systems research have been typically focused on the analysis of dynamic behaviors, such as stability, controllability and observability, etc.,(e.g., Sun and Ge [2005], Lin and Antsaklis [2005], Liberzon [2003]). Although there are several studies facing the problem of optimal control of switched systems (both from theoretical and from computational point of view Spinelli et al. [2006], Riedinger et al. [2003], Bengea and DeCarlo [2005], Xu and Antsaklis [2004]), there are still some problems not tackled, especially in issues where the switching mechanism is a design variable. There, we see how these difficulties arise, and how tools from nonsmooth calculus and optimal control can be combined to solve optimal control problems.

Previously, the approach based on convex analysis have been treated in Riedinger et al. [2003], and further developed in Bengea and DeCarlo [2005] considering an optimal control problem for a switched system, these approaches do not take into account assumptions about the number of switches nor about the mode sequence, because they are given by the solution of the problem. The authors use a switched system that is embedded into a larger family of systems and the optimal control problem is formulated for this family. When the necessary conditions indicate a bang-bang-type of solution, they obtain a solution to the original problem. However, in the cases when a bang-bang-type solution does not exist, the solution to the embedded optimal control problem can be approximated by the trajectory of the switched system generated by an appropriate switching control. On the other hand, in Riedinger et al. [2003] and in Patiño et al. [2007], the authors determine the appropriated control law by finding the singular trajectory along some time with non null measure.

The nonlinear, non-convex form of the control variable, prevents us from using the Hamilton equations of the maximum principle and nonlinear mathematical programming techniques on them. Both approaches would entail severe difficulties, either in the integration of the Hamilton equations or in the search method of any numerical optimization algorithm. Consequently, we propose to convexify the control variable by using the method of moments in the polynomial expression in order to deal with this kind of problems.

In this paper we present a method for solving optimal control for an autonomous switched systems problem based on the method of moments developed recently in Meziat et al. [2007] for optimal control, and in Lasserre [2001] for global optimization. We propose an alternative approach for computing effectively the solution of nonlinear, optimal control problems. This method works properly when the control variable (i.e., the switching signal) can be expressed as polynomials. The essential of this paper is the transformation of a nonlinear, non-convex optimal control problem (i.e., the switched system) into an equivalent optimal control problem with linear and convex structure, which allows us to obtain an equivalent convex formulation more appropriate to be solved by high performance numer-

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ical computing. To this end, first of all, it is necessary to transform the original switched system into a continuous non-switched system for which the theory of moments is able to work. Namely, we relate with a given controllable switched system, a controllable continuous non-switched polynomial system.

This paper is organized as follows. In Section 2 we present the problem formulation. A polynomial approach for the switched system is developed in Section 3. The new polynomial optimal control problem is described in Section 4 with the main elements of the theory of moments, which is presented to solve the polynomial problem in Section 5. Finally, in Section 6 some conclusions are drawn.

2. PROBLEM FORMULATION

A switched system is a system that consists of several continuous-time systems with discrete switching events. A switched system may be obtained from a hybrid system by neglecting the details of the discrete behavior and instead considering all possible switching patterns from a certain class. This represents a significant departure from hybrid systems, especially at the analysis stage Liberzon [2003]. Switched systems have many application examples, such as power electric circuits, automotive controllers, chemical processes, etc.

The mathematical model can be described by

$$\dot{x}(t) = f_{\sigma(t)}(x, u, t), \tag{1}$$

where $f_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^n$ are vector fields, the exogenous input $u \in \mathbb{R}^m$ and $\sigma : [0, t_f] \to \mathcal{Q} \in \{0, 1, 2, ..., q\}$ is a *piecewise constant* function of time. Every mode of operation corresponds to a specific subsystem $\dot{x}(t) = f_i(x, u, t)$, for some $i \in \mathcal{Q}$, and the *switching signal* σ determines which subsystem is followed at each point in time, into the interval of time $[0, t_f]$ with t_f as the final time . The control inputs, σ and u, are both measurable functions. No assumptions about the number of switches nor about the mode sequence are made. In addition, we consider non Zeno behavior, i.e., we exclude an infinite switching accumulation points in time. The state of the system described does not undergo into jump discontinuities. Further, for the interval $[t_0, t_f]$, the control functions must be chosen such that the initial and final conditions are satisfied. Define the optimization functional to be minimized

$$\begin{aligned} J &= \varphi(x(t_0), x(t_f)) + \int_{t_0}^{t_f} L_{\sigma(t)}(t, x, u)dt, \\ \text{s.t.} \quad & \dot{x}(t) = f_{\sigma(t)}(x, u, t) \\ & x(t_0) = x_0 \\ & \sigma(t) \in \mathcal{Q} \end{aligned} \tag{2}$$

where φ is a real-valued function, and $L_{\sigma(t)} : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are continuously differentiable.

In order to solve this problem, we use a continuous polynomial expression that is able to represent the original problem. The main idea is to transform the original problem into a problem represented by a differential-algebraic system in a polynomial form, as it is shown in the next section.

3. THE POLYNOMIAL APPROACH

In this section, we show how the optimal control problem (2) can be reformulated. For that, it is necessary to transform (1) into a polynomial expression that mimics the behavior of the original system. Then, with this polynomial expression, we can apply the relaxation based on several tools from nonlinear optimal control theory. In particular, the relaxation based on the moments problem.

3.1 Polynomial Expression

The polynomial expression that is able to mimic the behavior of the switched system is developed using a new variable s, which works as a control variable. The starting point is to rewrite (1) as a continuous non-switched control system in its more general case.

First, we define a drift vector field $\mathbf{F}(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

$$\mathbf{F}(x, u) = [f_0(x, u), f_1(x, u), ..., f_q(x, u)] \tag{3}$$

where $f_i(x, u)$, $i \in \mathcal{Q}$ is the function for each subsystems of the switched systems given in (1). Now, in order to find the polynomial expression we need for each $i \in \mathcal{Q} = \{0, 1, ..., q\}$, a quotient l_k with the property that $l_k(i) = 0$ when $i \neq k$ and $l_k(k) = 1$.

Let $\mathbf{L}$ be the vector of Lagrange polynomial interpolation quotients Burden and Faires [1985] defined with the new variable s, i.e.,

$$\mathbf{L}(s) = [l_0(s), l_1(s), ..., l_q(s)]^T \tag{4}$$

where

$$l_k(s) = \prod_{\substack{i=0 \\ i \neq k}}^{q} \frac{(s - i)}{(k - i)} \tag{5}$$

On the other hand, we can use a complementary polynomial which is used to constrain s to take only integer values. Let $Q(s)$ be the constraint polynomial so that

$$Q(s) = \prod_{k=0}^{q} (s - k) = 0 \tag{6}$$

A related continuous polynomial system of the switched system (1) is constructed in the following representation.

Consider a switched system of the form given in (1) with a drift vector field which are in the form given in (3). Then, there exists a unique polynomial P of order q with the property of

$$f_i(x, u) = P(x, u, i) \text{ for each } i \in \mathcal{Q}$$

This polynomial is given by

$$\begin{aligned} P(x, u, s) &= \mathbf{F}(x, u)\mathbf{L}(s) \\ &= \sum_{k=0}^{q}(s) f_k(x, u) l_k \end{aligned} \tag{7}$$

where $s \in \mathbb{R}$, and $l_k(s)$ is as in (5). Now, the system (1) takes the constrained continuous form of a nonlinear differential algebraic equation

$$\begin{aligned} \dot{x} &= P(x, u, s) \\ 0 &= Q(s) \end{aligned}$$

for each $i \in \mathcal{Q} = \{0, 1, ..., q\}$ subject to the constraint polynomial $Q(s)$ given in (6). For instance, if $q = 1$, the system (7) has the same form of the convex combination of two subsystems. Note that the trajectories of the original switched system (1) correspond to piecewise constant controls taking values in the set $\sigma \in \{0, 1, ..., q\}$. In the same way we define a polynomial representation for $L_{\sigma(t)}$ using the Lagrange's quotients as follow,

$$\mathcal{L}(x, t, u, s) = \sum_{k=0}^{q} (s) L_k(x, u) l_k \tag{8}$$

3.2 Polynomial Optimal Control Problem

Now, based on the above reformulation we can define an optimal control problem based on this representation. For that, we take the original problem (2) with the additional restriction, $Q(s) = 0$, i.e.,

$$\begin{aligned} & J = \varphi(x(t_0), x(t_f)) + \int_{t_0}^{t_f} \mathcal{L}(x, t, u, s) dt \\ \text{s.t.} \quad & \dot{x}(t) = P(x, u, s) = \left(\sum_{k=0}^{q} f_k(x, u) l_k(s) \right) \\ & Q(s) = 0 \\ & x(t_0) = x_0 \\ & x \in \mathbb{R}^n, s \in \mathbb{R}, u \in \mathbb{R}^m \end{aligned} \tag{9}$$

where $l_k(s)$, $Q(s)$, and $\mathcal{L}$ as above. Note that the system (9) is a continuous nonlinear differential-algebraic equation system (DAEs).

Although the representation presented above has been developed in general form, from now on we focus on a specific case. In particular, we are concerned with the case where there is no external control variable u, called autonomous case. We use s as a control variable.

The Hamiltonian for the optimal control problem (9) has the form

$$H(x, t, s, \lambda) = \mathcal{L}(x, t, s) + \lambda^T P(x, s) + \gamma Q(s) \tag{10}$$

Let $Q(s)$, $P(x, s)$, and $\mathcal{L}(x, t, s)$ be polynomials shown in (6), (7) and (8). λ and γ are the Lagrange multiplier. All of these polynomials can be expressed as polynomials in the control variable s. In general we have:

$$\begin{aligned} \mathfrak{L}(x, t, s) &= \sum_{k=0}^{q} a_k(x, t) s^k \\ P(x, s) &= \sum_{k=0}^{q} p_k(x, t) s^k \\ Q(s) &= \sum_{k=0}^{q+1} c_k s^k \end{aligned} \tag{11}$$

With these polynomials, the Hamiltonian H must have a polynomial form in the control variable s. From (10) and (11), it can be shown that we have the Hamiltonian as

$$H(x, t, s, \lambda) = \sum_{k=0}^{q+1} \alpha_k(x, t, \lambda, \gamma) s^k \tag{12}$$

Thus, the global minimization of H in s has the form

$$\min_{s(t)} H(s) = \sum_{k=0}^{q+1} \alpha_k(x, t, \lambda, \gamma) s^k \tag{13}$$

This is a problem well suited to be solved by the method of moments (Meziat et al. [2007], Lasserre [2001], Ben-Tal and Nemirovsky [2001]) as we see in the next section.

4. CONVEXIFICATION OF THE POLYNOMIAL EXPRESSION AND THE THEORY OF MOMENTS

In this section we present an alternative method for computing effectively the solution of nonlinear, optimal control problems. This method works properly when the nonlinearities in the control variable can be expressed as polynomials. The essential of this approach is the transformation of a nonlinear, non-convex optimal control problem into an equivalent optimal control problem with linear and convex structure. First, we present some basic concepts about the moments problems and its relationship with polynomial expressions.

4.1 Moment Problems and Polynomials

In Berg [1994], an interesting historical introduction is presented about the moment problem introduced by Stieltjes, where it is also introduced what is now known as the Stieltjes integral with respect to an increasing function ϕ, the latter describing a distribution of mass (a measure μ) via the convention that the mass in an interval $(a, b]$ is $\mu((a, b]) = \phi(b) - \phi(a)$. This integral was used to solve the following problem which is called the Hamburger moment problem:

Given a sequence $m_0, m_1, ...m_n$ of real numbers. Find necessary and sufficient conditions for the existence of a measure μ on $\mathbb{R}$ so that

$$m_n = \int_{\mathbb{R}} x^n d\mu(x) \text{ for } n = 0, 1, ...$$

The number m_n is called the n'th moment of μ, and the sequence (m_n) is called the moment sequence of μ.

Now, some concepts related with the moment problem and its relation with polynomials are presented. Let $m := \{m_k\}$ with first element $m_0 = 1$ be the vector of moments of some probability measure μ_m. Let $M_q(m)$ be the moment matrix of dimension q, which is composed of all the vectors in $\mathbb{R}^{q+1}$ whose entries form a positive semidefinite Hankel matrix (Ben-Tal and Nemirovsky [2001]),(Lasserre [2001]):

$$M_q(m) = \{(m_{i+j}(t)), i, j = 0, 1, ..., q \text{ with } m_0 = 1\}$$

For instance, if we have $q = 1$, the Hankel matrix is as follows:

$$M_{q=1}(m) = \begin{bmatrix} 1 & m_1 \\ m_1 & m_2 \end{bmatrix}$$

The theory of moments identifies those sequences m with $M_q(m) \succeq 0$ that correspond to moments of some probability measure μ_m on $\mathbb{R}^n$.

4.2 Convexification of the Polynomial Expression

For solving non-convex polynomial programs like (13) we can use the convex hull of the graph of the polynomial H, so that it would be a coercive function, i.e., $\alpha_k > 0$. We can describe such convex set as,

$$\mathbf{co}(\text{graph}(H)) = \left\{ \int_{\mathbb{Q}} H(s)d\mu(s) : \mu \in P_{\mathbb{Q}} \right\} \tag{14}$$

where $P_{\mathbb{Q}}$ stands for the family of all probability Borel measures with support contained in $\mathbb{Q}$. Here $\mathbb{Q}$ can be seen as a compact set, not necessary convex, defined by polynomial inequalities or equalities, as in (6), and **co** is the convex hull. Then, we can state the global optimization problem (13) as an optimization problem defined in probability measures,

$$\min_{\mu \in P_{\mathbb{Q}}} \int_{\mathbb{Q}} H(s)d\mu(s) \tag{15}$$

whose solution is the family of all probability measures supported in $\arg\min(H)$. For an extended overview of the theory, see Lasserre [2001].

Theorem 1. (Lasserre [2001],Meziat [2003]). When H is coercive, the set of solutions of (15) is the set of all probability measures supported in the set of global minima of H, i.e., $\arg\min H$.

Corollary 2. (Meziat [2003]). When $\arg\min(H)$ is the singleton $\{s^*\}$, the Dirac measure: $\mu^* = \delta_{s^*}$ is the unique solution of (15).

We use the polynomial structure of the objective function H in order to transform the optimization problem (15) into the mathematical problem: where $M_q(m)$ is moment matrix, whose entries are the algebraic moments of a probability measure $m = \{m_k\}$ supported in $\mathbb{Q}$.

$$\begin{aligned} &\min_m \sum_{k=0}^{q+1} \alpha_k m_k(t) \\ \text{s.t.}\quad & M_q(m) \succeq 0, \quad \text{with } m_0 = 1 \\ & \sum_{k=0}^{q+1} c_k m_k(t) = 0 \end{aligned} \tag{16}$$

which has the form of a semidefinite program and $\sum_{k=0}^{q+1} c_k m_k$ is the polynomial (11) in a linear form, it means that each s^k is replaced by $\{m_k\}$, for all $k = 0, 1, ..., q$.

When $H(s)$ is a coercive polynomial with a unique global minimum s^*, the semidefinite program (16) has a unique solution $m^* \in \mathbb{R}^{q+1}$, composed by the algebraic moments of Dirac measure δ_{s^*}. Thus, $m_k^* = (s^*)^k$ for $k = 0, ..., q$. Hence, the global minimization of the Hamiltonian H can be formulated as

$$\begin{aligned} &\min_m \widetilde{H} = \min_m \sum_{k=0}^{q+1} \alpha_k m_k(t) \\ \text{s.t.}\quad & M_q(m) \succeq 0, \quad \text{with } m_0 = 1 \\ & \sum_{k=0}^{q+1} c_k m_k(t) = 0 \end{aligned} \tag{17}$$

Therefore, if $\arg\min(H)$ is the singleton $\{s^*\}$, the optimal control can be expressed as,

$$s^*(x, t, \lambda) = m_1^*(x, t, \lambda)$$

because the entries of $m^*(x, t, \lambda)$ are the moments of the Dirac measure δ_{s^*}. In this way, we can solve the non-convex problem (9) by working out its convex relaxation

$$\begin{aligned} \min_{m(t)}\ & \varphi(x(t_0), x(t_f)) + \int_{t_0}^{t_f} \sum_{k=0}^{q} a(x,t) m_k dt \\ \text{s.t.}\quad & \dot{x} = \sum_{k=0}^{q} p(x,t) m_k(t) \\ & \sum_{k=0}^{q+1} c_k m_k(t) = 0 \\ & M_q(m) \succeq 0, \quad \text{with } m_0 = 1 \\ & x(t_0) = x_0 \end{aligned} \tag{18}$$

The convex relaxation opens many possibilities to solve the nonlinear problem. In the next section we present an analysis of the convexified problem in order to deal with a computational solution.

5. THE CONVEXIFIED OPTIMAL CONTROL PROBLEM

With the concepts presented above, we can present the optimal control problem in a convexified form, which is tractable from a computational point of view due to its linear structure and its related semidefinite program.

5.1 Analysis of the problem

We emphasize the fact that the semidefinite program (17) corresponds to the optimization of the Hamiltonian of the convex formulation (18). In this way we have

$$\widetilde{H} = \widetilde{\mathcal{L}} + \lambda^T \widetilde{P} + \gamma \widetilde{Q} = \sum_{k=0}^{q+1} \alpha_k(x, t, \lambda) m_k(t)$$

where $\widetilde{\mathcal{L}} = \sum_{k=0}^{q} a_k(x,t) m_k$, $\widetilde{P} = \sum_{k=0}^{q} p_k(x,t) m_k$, and $\widetilde{Q} = \sum_{k=0}^{q+1} c_k m_k$ are in linear form in the control variable m.

Indeed, this is precisely the relaxation in moments of the global optimization of the Hamiltonian $H(x, t, \lambda, s)$ when the variable s is transformed into the vector m. Thus, every minimizer of the convex formulation (18) attains the minimum value of the nonlinear optimal control problem (9) and for this reason those minimizer attains the minimum value of the switched optimal control problem (2). The next theorem is a variation of Meziat et al. [2007].

Theorem 3. Let us assume that $s^*(t)$ is a minimizer of the optimal control problem (9). Then, the control vector $m^*(t)$ given as,

$$m_k^*(t) = (s^*(t))^k \quad \forall k = 0, ..., q+1 \tag{19}$$

is a minimizer of the formulation (18). Therefore, it is a minimizer of the (2).

Proof. Since $\sigma^*(t)$ is an optimal control for (2), in the form (9), the maximum principle claims that $\sigma^*(t)$ satisfies the global minimization problem, i.e.,

$$H(x^*, t, \lambda^*, \sigma^*) = \min_{\sigma \in \mathbb{Q}} H(x^*, t, \lambda^*, \sigma) \tag{20}$$

where x^* and λ^* satisfy the boundary value problem:

$$\begin{aligned}\dot{x} &= P(x,t,s)\\ \dot{\lambda} &= -\frac{\partial H}{\partial x}(x,t,s(t)^*)\\ x(t_0) &= x_0,\end{aligned}\tag{21}$$

On the other hand, the Hamiltonian function H has the polynomial form (10) and $s(t)^*$ solves the global minimization problem (20), therefore the vector $m^*(t)$ given as

$$m_k^*(t) = (s^*)^k \ k = 0, ..., q+1$$

solves the semidefinite program:

$$\min_{m \in M_q} H(x^*, t, \lambda^*, m)\tag{22}$$

where the functions x^*, λ^* in (22) come from the solution of the boundary value problem (21). Since f and $\frac{\partial H}{\partial x}$ in (21) have a polynomial form in the variable $s^*(t)$, and every appearance of the k-th power of $s^*(t)$ can be replaced by $m_k^*(t)$, then the boundary value problem (21) can be expressed as

$$\begin{aligned}\dot{x}^* &= P\ (x^*, t, m^*(t))\\ \dot{\lambda} &= -\frac{\partial \widetilde{H}}{\partial x}(m(t)^*, x^*, t, \lambda^*)\\ x(t_0) &= x_0,\end{aligned}\tag{23}$$

and as $m(t)^*$ solves (21) and x^*,λ^* satisfy (23) we have

$$\widetilde{H}(x^*, t, \lambda^*, m^*(t)) = \min_{m \in M_q} H(x^*, t, \lambda^*, m)\tag{24}$$

which is the maximum principle's necessary condition of the convex formulation (18). Since the relaxation (18) is convex, the maximum principle's necessary conditions are also sufficient to guarantee optimality. Thus, $m^*(t)$ is a minimizer of (18). □

The theorem above states that the component $m_1^*(t)$ of the moments vector is, in fact, the optimal switched law for the switched system (2). If $m^*(t)$ is a minimizer of (18), satisfying (18), $m_1^*(t)$ is a minimizer of (2). Namely, $m_1^*(t)$ is the $\sigma^*(t)$ (i.e., the optimal switched law). If all minimizers of (18) fail in satisfying the expression (19), then the problem (2) lacks of minimizers, i.e., an optimal $\sigma^*(t)$ does not exist. On the other hand, if the Hamiltonian H is coercive, and it has a unique global minimum in s irrespective of the values of x, t, and λ, then every minimizer of the formulation (18) has the form (19). Hence, (2) has at least a minimizer.

5.2 Autonomous Two-Switched System Case

In this section, we apply the theory of moments to transform the problem (2) into a semidefinite program. Consider for a specific case, the polynomial (7) when $q = 1$ and all the vector fields are linear: $f_i(x, u, t) = A_i x$, where $A_i \in \mathbb{R}^{n \times n}$. This yields a linear autonomous switched system and of the form,

$$\dot{x}(t) = A_{\sigma(t)} x(t)\tag{25}$$

Using the polynomial transformation we have,

$$\begin{aligned}P(x,u,s) &= \sum_{k=0}^{1} L_k(s) f_k(x,u) = L_0 A_0 x(t) + L_1 A_1 x(t)\\ \dot{x}(t) &= P(x,u,s) = \left(\sum_{k=0}^{1} L_k(s) A_k\right) x(t)\end{aligned}\tag{26}$$

with $L_0 = (1-s)$, $L_1 = s$. Combining (25) and (26), we obtain the dynamics given by

$$\begin{aligned}\dot{x}(t) &= (A_0(1-s) + A_1 s)\, x(t)\\ Q(s) &= s(s-1) = 0\end{aligned}\tag{27}$$

Note that the trajectories of the original switched system (8) correspond to piecewise constant controls taking values in the set $\{0,1\}$. In particular, $\dot{x}(t) = A_0 x(t)$ results by setting $s = 0$ in (9), while $\dot{x}(t) = A_1 x(t)$ result by setting $s = 1$. The switching behavior is defined by the constrained polynomial $Q(s)$. This can be generalized for any $q > 1$. For illustration and to clarity this idea, consider the regulator problem with $t_0 = 0$, $t_f = 10$, $x_0 = (5, -5)$, and $\mathbf{R} = \mathbf{I}_{2\times 2}$.

$$A_0 = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \ A_1 = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix}$$

Using the theory of moments, we can change the polynomial variables for moments, i.e., $s = m_1$, $s^2 = m_2$. The relaxed problem based on the approach developed above, can be reformulated as,

$$\begin{aligned}\min_{u} \quad & \int_0^{t_f} x^T(t)\mathbf{R}x(t)dt\\ \text{s.t.} \quad & \dot{x}(t) = (A_0(1-m_1) + A_1 m_1)\, x(t)\\ & \widetilde{Q} = m_2 - m_1 = 0\\ & \begin{bmatrix} 1 & m_1 \\ m_1 & m_2 \end{bmatrix} \geq 0\end{aligned}\tag{28}$$

This particular convex relaxation can be solved by high-performance, numerical methods for convex mathematical programs as in Sturn [1999], and Toh et al. [1999]. In Fig. 1 we can see the system response and the switching signal for the regulator. It is clear that the system switches between both subsystem, in order to stabilize the system to zero from the initial condition. When the switching signal is '1' the active subsystem is A_1, and when the signal is '0' the second subsystem is active. This numerical example, using the MATLAB optimization toolbox, let us confirm that the first moment, i.e., m_1, corresponds with the polynomial variable s, and therefore with the switching signal σ.

6. CONCLUSION

In this paper, we have developed a new method for solving the optimal control problem of switched systems based on a polynomial approach. First, we transform the original problem into a polynomial system, that is able to mimic the switching behavior but with a continuous differential-algebraic nonlinear representation. After that, we transform the polynomial problem into a relaxed convex problem, through the method of moments. From a theoretical point of view, we provide necessary and sufficient conditions for the existence of minimizer by using particular features of the minimizer of its relaxed, convex formulation. Even in the absence of classical minimizers of the

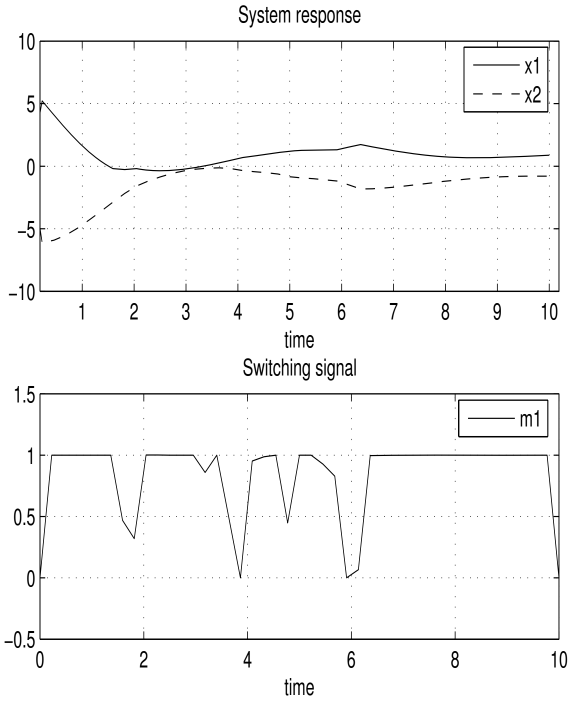

Fig. 1. System response and switching signal for initial condition x=(5,-5)

switched system, the solution of its relaxed formulation provide minimizers.

Further directions of this work can be focused on the development a computational tool to solve the convex relaxed problem in general cases, i.e., nonlinear vector fields, and to prove the computational efficiency of the method proposed. On the other hand, we have several tools to optimal control of the switched system with this polynomial representation, different of the method of moments. And it opens several possibilities for the system analysis, as stability analysis by using sum of squares Parrilo [2000], and some other analysis as controllability, observability among others. Some results on controllability of switched systems related with non-switched polynomial system have been presented in Perera and Dayawansa [2004]. It means that with this approach, we have the possibility of analysis and control for nonlinear switched systems in a consistent way.

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