Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

New LMI Characterizations for Discrete-Time Descriptor Systems and Application to Multiobjetive Control System Synthesis ⋆

Noboru Sebe *

* *Kyushu Institute of Technology*
Department of Artificial Intelligence
680-4, Kawazu, Iizuka, 820-8502 Japan
(e-mail: sebe@ai.kyutech.ac.jp)

Abstract: This paper presents new LMI characterizations for stability, H_2 and H_∞ norms of discrete-time descriptor systems. Based on these characterizations, an iterative design procedure for multiobjective and structurally constrained feedback control are proposed. The first key idea of the iterative design procedure is embedding the previously designed feedback gain $\hat{K}$ in the descriptor representation of the closed-loop system. The second key idea of the iterative design procedure is linearizing the products terms of the actual controller parameter K and the auxiliary variables by the assignment of variables instead of the 'change of variables' technique.

Keywords: multiobjective control, decentralized control, linear matrix inequalities, descriptor systems, dilation, Lyapunov variables.

1. INTRODUCTION

The aim of this paper is to propose LMI characterizations of discrete-time descriptor systems. This paper also proposes an iterative design procedure based on the proposed characterizations.

The dilated (or extended) LMI characterizations enable us to use parameter-dependent Lyapunov functions for robust system analysis and synthesis (de Oliveira *et al.*, 1999; Peaucelle *et al.*, 2000; Apkarian *et al.*, 2001; de Oliveira *et al.*, 2002) and independent Lyapunov functions for multiobjective control synthesis problems (Shimomura *et al.*, 2001; Ebihara and Hagiwara, 2004). These results have promoted the great advance of the control theory. It should be also mentioned that Chen (2004) has pointed out that the dilated LMI characterizations are the LMI characterizations for adequate descriptor representation with redundant state variables.

On the other hand, there have been many attempts to improve the performance by iterative designs (Arzelier and Peaucelle, 2002; Ebihara *et al.*, 2004; Shimomura and Fujii, 2005; Saeki, 2006). Although the iterative designs are effective, the procedures are rather specific for each combination of objective functions.

For continuous-time systems, the other dilated LMI characterizations for iterative design of multiobjective and structurally constrained feedback control are derived (Sebe, 2007). The key idea is dividing the system matrices into some pieces and reconstructing a descriptor system with the previously designed controller parameter. The proposed embedment of controller parameter ensures the improvement of achievable performance by iterative design. As the proposed procedure is based on the stability and performance characterizations of descriptor systems, any combination of the performance specifications and structural constraints on controllers can be dealt with. Contrary to the continuous-time case, the relation between the dilated LMI conditions for the stability and the stability (rigorously the admissibility) of descriptor systems is not so clear for the discrete-time systems. Therefor the iterative design procedure proposed by Sebe (2007) can not be applied for discrete-time systems.

In this paper, new LMI characterizations for the stability, H_2 and H_∞ norms of discrete-time descriptor systems are derived. Based on these characterizations, an iterative design procedure for multiobjective control and structurally constraint controller designs is proposed. Parallel to the continuous-time case, the proposed procedure embeds the previously designed feedback gain $\hat{K}$ in the descriptor representation of the closed-loop system. The proposed procedure linearizes the products terms of the actual controller parameter K and the auxiliary variables by the assignment of variables instead of the 'change of variables' technique (Scherer *et al.*, 1997; I. Masubuchi, 1998). The assignment of auxiliary variables for discrete-time systems is different from that for continuous-time systems. This paper also demonstrates the effectiveness of the proposed design procedure through numerical examples.

We use the following notations. I and O denote the identity and zero matrix, respectively. For a matrix M, M^{-1} and M^{T} are the inverse and transpose matrix of

⋆ This research was supported in part by the Ministry of Education, Culture, Sports, Science and Technology, Japan, under the Grant-in-Aid for Scientific Research (B) No.19360192.

M, respectively. He$\{M\}$ is a shorthand notation for $M + M^{\mathrm{T}}$. $\bar{\sigma}(M)$ is the maximum singular value of M. In some partitioned symmetric matrices, the symbol '$\star$' denotes each of its symmetric block.

2. STABILITY CONDITION FOR DISCRETE-TIME DESCRIPTOR SYSTEMS

2.1 Preliminaries

Let us consider a discrete-time state space system

$$x(k+1) = Ax(k) \tag{1}$$

where $x(k) \in \Re^n$. A well-known stability condition for the system (1) is given as follows.

Proposition 1. The discrete-time system (1) is stable if and only if there exists a matrix P such that

$$P - A^{\mathrm{T}}PA > O, \tag{2}$$

$$P = P^{\mathrm{T}} > O. \tag{3}$$

Here are the dilated LMI conditions proposed by Oliveira *et al.* (1999) and Peaucelle *et al.* (2000).

Proposition 2. (de Oliveira *et al.*, 1999) The discrete-time system (1) is stable if and only if there exist matrices G and P such that

$$\begin{bmatrix} P & -(GA)^{\mathrm{T}} \\ -GA & G + G^{\mathrm{T}} - P \end{bmatrix} > O, \tag{4}$$

$$P = P^{\mathrm{T}} > O. \tag{5}$$

Proposition 3. (Peaucelle *et al.*, 2000) The discrete-time system (1) is stable if and only if there exist matrices F, G and P such that

$$\begin{bmatrix} P - (FA + A^{\mathrm{T}}F^{\mathrm{T}}) & F - (GA)^{\mathrm{T}} \\ F^{\mathrm{T}} - GA & G + G^{\mathrm{T}} - P \end{bmatrix} > O, \tag{6}$$

$$P = P^{\mathrm{T}} > O. \tag{7}$$

In these propositions, the matrix P is the actual Lyapunov matrix for the system(1), and the matrices G and F are the auxiliary variables.

Let us consider a discrete-time descriptor system

$$\tilde{E}\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) \tag{8}$$

where $\tilde{x}(k) \in \Re^l$. If the matrix $\tilde{E}$ is singular, the system might have impulsive modes. Therefor the admissibility of the system i.e., the regularity, impulse-free property and stability, should be considered. The rigorous definitions of these properties are found in Hsiung and Lee (1999).

Proposition 4. (Hsiung and Lee, 1999; Xu and Yang, 1999) The system (8) is admissible if and only if there exists a symmetric matrix $\tilde{X} \in \Re^{l\times l}$ such that

$$\tilde{E}^{\mathrm{T}}\tilde{X}\tilde{E} - \tilde{A}^{\mathrm{T}}\tilde{X}\tilde{A} > O, \tag{9}$$

$$\tilde{E}^{\mathrm{T}}\tilde{X}\tilde{E} \geq O. \tag{10}$$

2.2 Relation between the dilated LMI characterization and descriptor system

The relation between the stability conditions with dilated LMIs and the stability condition for descriptor system is now considered. Let us consider a descriptor system

$$\begin{bmatrix} I & O \\ O & O \end{bmatrix} \tilde{x}(k+1) = \begin{bmatrix} O & I \\ A & -I \end{bmatrix} \tilde{x}(k) \tag{11}$$

where $\tilde{x}(k) = [x(k)^{\mathrm{T}} \;\; x(k+1)^{\mathrm{T}}]^{\mathrm{T}}$. This descriptor system is equivalent to the system (1). Let the Lyapunov matrix $\tilde{X} \in \Re^{2n\times 2n}$ be

$$\tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^{\mathrm{T}} & \tilde{X}_{22} \end{bmatrix} \tag{12}$$

where $\tilde{X}_{ij} \in \Re^{n\times n}$ and $\tilde{X}_{ii} = \tilde{X}_{ii}^{\mathrm{T}}$. From Proposition 4, the stability conditions of the descriptor system (11) are

$$\begin{bmatrix} \tilde{X}_{11} & O \\ O & O \end{bmatrix} - \begin{bmatrix} O & A^{\mathrm{T}} \\ I & -I \end{bmatrix} \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^{\mathrm{T}} & \tilde{X}_{22} \end{bmatrix} \begin{bmatrix} O & I \\ A & -I \end{bmatrix} > O, \tag{13}$$

$$\tilde{X}_{11} = \tilde{X}_{11}^{\mathrm{T}} \geq O. \tag{14}$$

Furthermore, the strict positivity of the $(1,1)$ block in (13) implies

$$\tilde{X}_{11} > A^{\mathrm{T}}\tilde{X}_{22}A \geq O. \tag{15}$$

If we choose

$$\tilde{X}_{11} = P > O, \quad \tilde{X}_{12} = G, \quad \tilde{X}_{22} = O, \tag{16}$$

then the conditions are reduced to those in Proposition 2. This implies that Proposition 2 is a sufficient condition for Proposition 4. Let us assign the matrices F, G and P as

$$P = \tilde{X}_{11} > O, \quad \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} O & \frac{1}{2}A^{\mathrm{T}} \\ I & -\frac{1}{2}I \end{bmatrix} \begin{bmatrix} \tilde{X}_{12} \\ \tilde{X}_{22} \end{bmatrix}, \tag{17}$$

then (6) and (7) hold. Thus, Proposition 4 is a sufficient condition for Proposition 3. As Propositions 2 and 3 are equivalent, Proposition 4 with (11) is also equivalent to those propositions.

The stability condition proposed by Peaucelle *et al.* (2000) has advantages:

- There does not exist product terms of the Lyapunov matrix P and the system matrix A. Thus the conditions with parameter dependent Lyapunov functions are readily applicable for the robust stability analysis and state feedback synthesis of systems with polytopic uncertainties.
- There are no indefinite quadratic terms of the system matrix A. This fact makes the controller design problem as LMIs.
- There are large number of decision variables, which may reduce the conservativeness in controller design.

From these reasons, the dilated LMI characterizations are powerful tools for controller designs of robust control, multiobjective control and structural constraint control.

Unfortunately, these dilated LMI characterizations are the stability conditions for the descriptor system (11), and are not for general descriptor systems. Thus, the conditions can not be applied to the iterative design procedure proposed by Sebe (2007).

2.3 Main results

In this subsection, the generalized stability condition for descriptor systems is given. The H_2 and H_∞ norm conditions for descriptor systems are also given. Let a given discrete-time descriptor system be

$$\tilde{E}\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}\tilde{w}(k), \tag{18a}$$

$$\tilde{z}(k) = \tilde{C}\tilde{x}(k) + \tilde{D}\tilde{w}(k) \tag{18b}$$

where $x(k) \in \Re^l$, $w(k) \in \Re^m$, $z(k) \in \Re^p$. The transfer function from $w(k)$ to $z(k)$ is defined by

$$G(z) = \tilde{C}(z\tilde{E} - \tilde{A})^{-1}\tilde{B} + \tilde{D}. \quad (19)$$

Assume that

$$\text{rank } \tilde{E} = r. \quad (20)$$

There always exist two non-singular matrices $\tilde{U}$ and $\tilde{V}$ such that

$$\tilde{U}\tilde{E}\tilde{V} = \text{diag } \{E, O\} \quad (21)$$

where $E \in \Re^{r\times r}$ is non-singular. Note that these two matrices $\tilde{U}$ and $\tilde{V}$ can be found by the singular value decomposition, which is numerically stable. Let us partition the matrices $\tilde{U}$ and $\tilde{V}$ as

$$\tilde{U} = \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix} \quad (22)$$

where $\tilde{U}_1 \in \Re^{r\times l}$, $\tilde{U}_2 \in \Re^{(l-r)\times l}$ $\tilde{V}_1 \in \Re^{l\times r}$, $\tilde{V}_2 \in \Re^{l\times(l-r)}$. Then, the following theorem holds.

Theorem 5. The given descriptor system (18) is admissible if and only if there exist matrices $P \in \Re^{r\times r}$ and $\tilde{F} \in \Re^{l\times(l-r)}$ such that

$$(\tilde{U}_1\tilde{E})^{\mathrm{T}}P(\tilde{U}_1\tilde{E}) - (\tilde{U}_1\tilde{A})^{\mathrm{T}}P(\tilde{U}_1\tilde{A}) + \mathrm{He}\{\tilde{F}\tilde{U}_2\tilde{A}\} > O, \quad (23)$$

$$P = P^{\mathrm{T}} > O. \quad (24)$$

Remark 6. If the matrices $\tilde{E}$ and $\tilde{A}$ are chosen as (11), Theorem 5 coincides with Proposition 3.

Theorem 7. The given descriptor system (18) is admissible and $\|G(z)\|_\infty < \gamma_\infty$ if and only if there exist matrices $P \in \Re^{r\times r}$ and $\tilde{F} \in \Re^{(l+m)\times(l-r)}$ such that

$$\left(\tilde{U}_1[\tilde{E}\ \tilde{O}]\right)^{\mathrm{T}} P \left(\tilde{U}_1[\tilde{E}\ \tilde{O}]\right) - \left(\tilde{U}_1[\tilde{A}\ \tilde{B}]\right)^{\mathrm{T}} P \left(\tilde{U}_1[\tilde{A}\ \tilde{B}]\right) + \mathrm{He}\left\{\tilde{F}\tilde{U}_2[\tilde{A}\ \tilde{B}]\right\} + \begin{bmatrix} O & \\ & \gamma_\infty^2 I \end{bmatrix} - [\tilde{C}\ \tilde{D}]^{\mathrm{T}}[\tilde{C}\ \tilde{D}] > O, \quad (25)$$

$$P = P^{\mathrm{T}} > O. \quad (26)$$

Remark 8. Theorems 5 and 7 do not use $\tilde{V}$. This fact implies that these theorems do not require the information of right null space of $\tilde{E}$.

Theorem 9. The given descriptor system (18) is admissible and $\|G(z)\|_2 < \gamma_2$ if and only if there exist matrices $P \in \Re^{r\times r}$, $Q \in \Re^{m\times m}$, $\tilde{F} \in \Re^{l\times(l-r)}$ and $\tilde{G} \in \Re^{(l+m)\times(l-r)}$ such that

$$\gamma_2^2 > \text{trace } Q \quad (27)$$

$$(\tilde{U}_1\tilde{E})^{\mathrm{T}}P\tilde{U}_1\tilde{E} - (\tilde{U}_1\tilde{A})^{\mathrm{T}}P\tilde{U}_1\tilde{A} + \mathrm{He}\{\tilde{F}\tilde{U}_2\tilde{A}\} - \tilde{C}^{\mathrm{T}}\tilde{C} > O \quad (28)$$

$$\tilde{V}_{2e}^{\mathrm{T}}\Big([O\ \ I]^{\mathrm{T}}Q[O\ \ I] - (\tilde{U}_1[\tilde{A}\ \ \tilde{B}])^{\mathrm{T}}P(\tilde{U}_1[\tilde{A}\ \ \tilde{B}]) - [\tilde{C}\ \ \tilde{D}]^{\mathrm{T}}[\tilde{C}\ \ \tilde{D}] + \mathrm{He}\{\tilde{G}\tilde{U}_2[\tilde{A}\ \ \tilde{B}]\}\Big)\tilde{V}_{2e} > O, \quad (29)$$

$$P = P^{\mathrm{T}} > O, \quad Q = Q^{\mathrm{T}} > O \quad (30)$$

where $\tilde{V}_{2e} = \mathrm{diag}\{\tilde{V}_2, I\}$.

Remark 10. Inequality (29) can be replaced by

$$[O\ \ I]^{\mathrm{T}}Q[O\ \ I] - (\tilde{U}_1[\tilde{A}\ \ \tilde{B}])^{\mathrm{T}}P(\tilde{U}_1[\tilde{A}\ \ \tilde{B}]) - [\tilde{C}\ \ \tilde{D}]^{\mathrm{T}}[\tilde{C}\ \ \tilde{D}] + \mathrm{He}\{\tilde{G}_1\tilde{U}_2[\tilde{A}\ \ \tilde{B}]\} + \mathrm{He}\{\tilde{G}_2\tilde{U}_1[\tilde{E}\ \ O]\} > O. \quad (31)$$

This condition (31) is independent to $\tilde{V}$, as in Theorems 5 and 7. On the other hand, the size of matrix inequality is larger than that of (29). Therefor, the condition (29) is preferable from the view point of computational cost.

Remark 11. Similar to the result in Masubuchi (2006), Theorems 7 and 9 do not require any conditions on the direct term $\tilde{D}$ like $\bar{\sigma}(\tilde{D}) < \gamma$ and are realization independent.

The outlines of the proofs are given in Appendix.

3. ITERATIVE CONTROLLER SYNTHESIS

3.1 Descriptor representation of closed-loop system

In this section, let us apply the idea in Sebe (2007) to the discrete-time H_2/H_∞ static feedback design. Note that fixed order dynamic controllers can be similarly designed with augmented matrices given in Iwasaki and Skelton (1994). Furthermore, the design procedure proposed here can also deal with multiobjective control problems with any combinations of objective functions and structural constraints on controllers.

Let us consider a generalized plant

$$\begin{aligned} x(k+1) &= Ax(k) + B_0w_0(k) + B_1w_1(k) + B_2u(k), \\ z_0(k) &= C_0x(k) + D_{00}w_0(k) + D_{01}w_1(k) + D_{02}u(k), \\ z_1(k) &= C_1x(k) + D_{10}w_0(k) + D_{11}w_1(k) + D_{12}u(k), \\ y(k) &= C_2x(k) + D_{20}w_0(k) + D_{21}w_1(k) + D_{22}u(k). \end{aligned} \quad (32)$$

Let T_i $(i = 0, 1)$ denote the transfer functions from w_i to z_i. Assume γ_∞ is a given scalar. Then, the problem is to find a static feedback $u = Ky$ which minimizes $\gamma_2 = \|T_0\|_2$ under the constraint $\|T_1\|_\infty < \gamma_\infty$.

As this paper aims to propose an iterative design procedure, let us assume that a previously designed feedback gain $\hat{K}$ is given. With this designed $\hat{K}$ and the feedback K to be designed, the explicit descriptor representation of the closed-loop system, which is used for controller designs, is proposed as follows:

$$\tilde{E}\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}\tilde{w}(k), \quad (33a)$$

$$\tilde{z}(k) = \tilde{C}\tilde{x}(k) + \tilde{D}\tilde{w}(k) \quad (33b)$$

where $\tilde{E} = \text{block diag}\{I, O\}$,

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] = \left[\begin{array}{cc|cc} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_{10} & \tilde{B}_{11} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_{20} & \tilde{B}_{21} \\ \hline \tilde{C}_{01} & \tilde{C}_{02} & \tilde{D}_{00} & \tilde{D}_{01} \\ \tilde{C}_{11} & \tilde{C}_{12} & \tilde{D}_{10} & \tilde{D}_{11} \end{array}\right] = \left[\begin{array}{cc|cc} A & B_{\mathrm{L}} & B_0 & B_1 \\ O & -I & O & O \\ \hline C_0 & O & D_{00} & D_{01} \\ C_1 & O & D_{10} & D_{11} \end{array}\right] + \left\{ \left[\begin{array}{c} B_2 \\ O \\ \hline D_{02} \\ D_{12} \end{array}\right] \hat{K} + \left[\begin{array}{c} O \\ B_{\mathrm{R}} \\ \hline D_{02} \\ D_{12} \end{array}\right] (K - \hat{K}) \right\} \times [\, C_2\ O | D_{20}\ D_{21} \,], \quad (33c)$$

$$B_2 = B_{\mathrm{L}}B_{\mathrm{R}}, \qquad \xi = B_{\mathrm{R}}u, \quad (33d)$$

$$\tilde{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \tilde{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \quad \tilde{z} = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}. \quad (33e)$$

The idea of the decomposition of B_2 in (33d) is first proposed in Saeki (2006) and is also discussed in Sebe (2007). Mostly the recommended decomposition is

$$B_{\mathrm{L}} = U_{B2}\Sigma_{B2}^{\frac{1}{2}}, \qquad B_{\mathrm{R}} = \Sigma_{B2}^{\frac{1}{2}}V_{B2}^{\mathrm{T}} \quad (34)$$

where $U_{B2}\Sigma_{B2}V_{B2}^{\mathrm{T}}$ is the singular value decomposition of B_2.

Applying Theorems 7 and 9 to the descriptor system (33), the H_2/H_∞ control problem can be formulated as an optimization problem below.

Problem 12. Find positive definite matrices P_2, P_∞, Q, and matrices K, $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ such that

$$\text{minimize } \gamma_2$$

$$\text{subject to } \ \gamma_2^2 \geq \text{trace } Q, \tag{35}$$

$$\begin{bmatrix} M_1 & \star \\ [\tilde{C}_{01} \ \ \tilde{C}_{02}] & I \end{bmatrix} > O, \tag{36}$$

$$\begin{bmatrix} M_2 & \star \\ [\tilde{C}_{02} \ \ \tilde{D}_{00}] & I \end{bmatrix} > O, \tag{37}$$

$$\begin{bmatrix} M_3 & \star \\ [\tilde{C}_{11} \ \ \tilde{C}_{12} \ \ \tilde{D}_{11}] & I \end{bmatrix} > O, \tag{38}$$

$$M_1 = \begin{bmatrix} P_2 & O \\ O & O \end{bmatrix} - \begin{bmatrix} \tilde{A}_{11}^{\mathrm{T}} \\ \tilde{A}_{12}^{\mathrm{T}} \end{bmatrix} P_2[\tilde{A}_{11} \ \ \tilde{A}_{12}] + \mathrm{He}\left\{\tilde{F}[\tilde{A}_{21} \ \ \tilde{A}_{22}]\right\}, \tag{39}$$

$$M_2 = \begin{bmatrix} O & O \\ O & Q \end{bmatrix} - \begin{bmatrix} \tilde{A}_{12}^{\mathrm{T}} \\ \tilde{B}_{10}^{\mathrm{T}} \end{bmatrix} P_2[\tilde{A}_{12} \ \ \tilde{B}_{10}] + \mathrm{He}\left\{\tilde{G}[\tilde{A}_{22} \ \ \tilde{B}_{20}]\right\}, \tag{40}$$

$$M_3 = \begin{bmatrix} P_\infty & O & O \\ O & O & O \\ O & O & \gamma_\infty^2 I \end{bmatrix} - \begin{bmatrix} \tilde{A}_{11}^{\mathrm{T}} \\ \tilde{A}_{12}^{\mathrm{T}} \\ \tilde{B}_{11}^{\mathrm{T}} \end{bmatrix} P_\infty[\tilde{A}_{11} \ \ \tilde{A}_{12} \ \ \tilde{B}_{11}] + \mathrm{He}\left\{\tilde{H}[\tilde{A}_{21} \ \ \tilde{A}_{22} \ \ \tilde{B}_{21}]\right\}. \tag{41}$$

Similar to the the conventional dilated LMI characterizations, there are no product terms of the Lyapunov variables P_* and the controller parameter K in the above conditions. On the other hand, the product terms of the auxiliary variables $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ and the controller parameter K exist. Thus, linearization is required to make the problem tractable.

Here, the conceptual iterative design procedure is summarized. The actual linearization will be mentioned in the next subsection.

Algorithm 1. Let $K^{(i)}$ be the feedback gain which is designed at the i-th design iteration.

(i) Find a static feedback K which satisfies $\|T_1\|_\infty < \gamma_\infty$. Set $K^{(0)} = K$ and $i = 1$.
(ii) Set $\hat{K} = K^{(i-1)}$ in (33). Solve Problem 12 with the linearized conditions of (36), (37) and (38). (We will mention the actual linearization in the next subsection.) Set $K^{(i)} = K$ where K is the solution.
(iii) If a stopping criterion is satisfied, exit. Otherwise, set $i = i + 1$ and go to Step (ii).

With this iterative design procedure, let us define some variables. Let $\gamma_{2\mathrm{g}}^{(i)}$ be the guaranteed upper bound, i.e., the optimized value γ_2 in Problem 12, and $\gamma_{2\mathrm{a}}^{(i)}$ be the actually achieved H_2 norm of the closed-loop system with the designed $K^{(i)}$. Evidently, $\gamma_{2\mathrm{a}}^{(i)} \leq \gamma_{2\mathrm{g}}^{(i)}$. The matrices $P_{2\mathrm{a}}^{(i)}$ and $P_{\infty\mathrm{a}}^{(i)}$ denote the Lyapunov variables which evaluate the H_2 and H_∞ norms of the closed-loop system with $K^{(i)}$, respectively. Note that these matrices are different from the solutions P_2 and P_∞ at the i-th design iteration.

3.2 Linearization by assignment of variables

In many controller design methods based on LMI characterizations, the 'change of variables' technique is used to linearize the product terms of the Lyapunov or auxiliary variables and the controller parameter at the expense of the unification of the Lyapunov or auxiliary variables. The unification induces conservative results. Instead of the 'change of variables' technique, the assignment of variables is proposed to linearize the product terms by Sebe (2007) for the continuous-time systems. Similar to the continuous-time case, linearization with the Lyapunov variables $P_{2\mathrm{a}}^{(i-1)}$ and $P_{\infty\mathrm{a}}^{(i-1)}$ is proposed in this subsection.

As mentioned before, there exist the product terms of the auxiliary variables $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ and the controller parameter K. Obviously, the problem becomes linear, if the auxiliary variables are fixed. Thus we now propose to linearize the conditions (36), (37) and (38) by fixing these auxiliary variables. Explicit assignments are given below.

$$\tilde{F} = -[\tilde{A}_{11} \ \ \tilde{A}_{12}]^{\mathrm{T}} P_{2\mathrm{a}}^{(i-1)} B_{\mathrm{L}}, \tag{42a}$$

$$\tilde{G} = -[\tilde{A}_{12} \ \ \tilde{B}_{10}]^{\mathrm{T}} P_{2\mathrm{a}}^{(i-1)} B_{\mathrm{L}}, \tag{42b}$$

$$\tilde{H} = -[\tilde{A}_{11} \ \ \tilde{A}_{12} \ \ \tilde{B}_{11}]^{\mathrm{T}} P_{\infty\mathrm{a}}^{(i-1)} B_{\mathrm{L}}. \tag{42c}$$

Then, the next theorem holds.

Theorem 13. Assume the conditions (36), (37) and (38) be linearized by the variable assignments (42). Then, the inequality $\gamma_{2\mathrm{g}}^{(i)} \leq \gamma_{2\mathrm{a}}^{(i-1)}$ holds.

Proof. Let $K = \hat{K} = K^{(i-1)}$, $P_2 = \hat{P}_2 = P_{2\mathrm{a}}^{(i-1)}$, $P_\infty = \hat{P}_\infty = P_{\infty\mathrm{a}}^{(i-1)}$ and substitute (42) into the constraints (36), (37), (38), then the inequalities become

$$\begin{bmatrix} P_2 - (*)^{\mathrm{T}} P_2(A + B_2\hat{K}C_2) & O & \star \\ O & B_{\mathrm{L}}^{\mathrm{T}}\hat{P}_2 B_{\mathrm{L}} & O \\ C_0 + D_{02}\hat{K}C_2 & O & I \end{bmatrix} > O, \tag{43}$$

$$\begin{bmatrix} B_{\mathrm{L}}^{\mathrm{T}}\hat{P}_2 B_{\mathrm{L}} & O & O \\ O & Q - (*)^{\mathrm{T}}(B_0 + B_2\hat{K}D_{20}) & \star \\ O & D_{00} + D_{02}\hat{K}D_{20} & I \end{bmatrix} > O, \tag{44}$$

$$\begin{bmatrix} N_3 & \star \\ [C_1 + D_{12}\hat{K}C_2 \ \ O \ \ D_{11} + D_{12}\hat{K}D_{21}] & I \end{bmatrix} > O, \tag{45}$$

$$N_3 = \begin{bmatrix} \hat{P}_\infty & O & O \\ O & B_{\mathrm{L}}^{\mathrm{T}}\hat{P}_2 B_{\mathrm{L}} & O \\ O & O & \gamma_\infty^2 I \end{bmatrix} - [\,*\,]^{\mathrm{T}}\hat{P}_\infty[A + B_2\hat{K}C_2 \ \ \ O \ \ \ B_1 + B_2\hat{K}D_{21}]. \tag{46}$$

The inequalities (43) and (44) are the H_2 constraints and the inequality (45) is the H_∞ constraint of the closed-loop system with $\hat{K}$, respectively. As the matrices $\hat{P}_2$ and $\hat{P}_\infty$ are the Lyapunov matrices which evaluate the H_2 and H_∞ norms, the optimal value is $\gamma_{2\mathrm{a}}^{(i-1)}$. □

3.3 Remarks on implementation

Similar to the continuous-time case in Sebe (2007), we can modify the term $B_{\mathrm{L}}^{\mathrm{T}} P_*^{(i-1)} B_{\mathrm{L}}$ in the assignments (42). The

$(2,1)$ blocks of the auxiliary variables $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ can be replaced by R_F, R_G, R_H, respectively, where

$$R_F > \frac{1}{2} B_{\mathrm{L}}^{\mathrm{T}} P_{2\mathrm{a}}^{(i-1)} B_{\mathrm{L}}, \tag{47a}$$

$$R_G > \frac{1}{2} B_{\mathrm{L}}^{\mathrm{T}} P_{2\mathrm{a}}^{(i-1)} B_{\mathrm{L}}, \tag{47b}$$

$$R_H > \frac{1}{2} B_{\mathrm{L}}^{\mathrm{T}} P_{\infty\mathrm{a}}^{(i-1)} B_{\mathrm{L}}. \tag{47c}$$

Even with these modifications, Theorem 13 still holds.

Let us consider more restrictive generalization

$$\tilde{F} = -[\tilde{A}_{11} \ \ \beta\tilde{A}_{12}]^{\mathrm{T}} P_{2\mathrm{a}}^{(i-1)} B_{\mathrm{L}} \quad (\beta > \frac{1}{2}) \tag{48}$$

and consider only $M_1 > O$ in H_2 constraint (36), i.e., the stability condition of the closed-loop system. With the assignment (48), (1,2) block of M_1 becomes

$$\beta\{B_2(K-\hat{K})C_2\}^{\mathrm{T}} \hat{P} B_{\mathrm{L}} - (A + B_2\hat{K}C_2)^{\mathrm{T}}(P - \hat{P})B_{\mathrm{L}}, \tag{49}$$

where $\hat{P} = P_2^{(i-1)}$ and $\hat{K} = K^{(i-1)}$. Roughly speaking, the smaller this off-diagonal block is, the less conservative design can be achieved. As discussed in Sebe (2007), the parameter β balances the magnitude of $(P - \hat{P})$ and $(K - \hat{K})$ and determines the convex approximation of the actual admissible parameter space. One of the recommended choice is

$$\beta = \frac{\bar{\sigma}((A + B_2\hat{K}C_2)^{\mathrm{T}} \hat{P} B_{\mathrm{L}})}{\bar{\sigma}((B_2\hat{K}C_2)^{\mathrm{T}} \hat{P} B_{\mathrm{L}})}. \tag{50}$$

If we ignore A, $\beta = 1$ is a simple and recommended choice.

Here, let us also mention about the relation to the method proposed in Arzelier and Peaucelle (2002). Roughly speaking, the procedure in Arzelier and Peaucelle (2002) sets $\hat{K} = O$ and alternately optimizes the performance

- with variables P, $\tilde{F}$ and fixed K,
- with variables P, K and fixed $\tilde{F}$.

Current semidefinite programing (SDP) solvers such as SeDuMi (Sturm, 1999) and LMI toolbox in MATLAB are based on the interior-point method. These SDP solvers tend to give very large auxiliary variables R_* in (47) when K is fixed in first step. The problem is that the large R_* prevent to update the controller parameter K in second step. As long as using SDP solvers based on the interior-point method, adequate assignments of R_* are important for convergence and achieved performance. It should be also mentioned that Arzelier and Peaucelle (2002) proposed to use $\tilde{A}_{22}$ in (33) as an optimization variable. This proposal seems to be effective and should be investigated in detail.

4. NUMERICAL EXAMPLES

Two examples demonstrate the efficiency of the proposed method. All the examples are carried out by Robust Control Toolbox in MATLAB (Release 2007a) on a PC (Pentium 4, 3.2GHz with 2GB RAM). For all examples, the decomposition of B_2 is $B_{\mathrm{L}} = B_2$ and $B_{\mathrm{R}} = I$, and the stopping criterion is $|\gamma_{*\mathrm{a}}^{(i-1)} - \gamma_{*\mathrm{a}}^{(i)}| < 1 \times 10^{-5}$.

Example 14. (H_2/H_∞ control). Let us consider the H_2/H_∞ control problem in Oliveira *et al.* (2002). The state space data of the generalized plant are

$$\begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & D_{22} \end{bmatrix} = \left[\begin{array}{ccc|cc|ccc|c} 2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0.5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right].$$

The problem is to find a strictly proper output feedback controller which minimizes $\|T_0\|_2$ under the H_∞ constraints $\|T_{1ii}\|_\infty < \gamma_\infty$ $(i = 1, 2, 3)$, where T_{1ii} are the diagonal elements of T_1 and γ_∞ is a given value.

Three design methods are used to design controllers for comparison, the common Lyapunov variable design by Scherer *et al.* (1997), the dilated LMI characterization by Oliveira *et al.* (2002) and the proposed method.

First, let us compare the three design methods from the viewpoint of the feasible H_∞ norm constraints. Without H_2 norm optimization, the problem becomes the simultaneous H_∞ optimization problem to find a controller which minimizes the worst H_∞ norm of T_{1ii}. Table 1 shows the minimum feasible γ_∞ for each design method. Please note that worst H_∞ norm achieved by the optimal (non-robust) H_2 controller is 21.757 ($=\bar{\gamma}$). The H_2 optimal controller is optimal for H_2/H_∞ control problem with $\gamma_\infty \geq \bar{\gamma}$. In other words, it is meaningless to specify H_∞ constraint with $\gamma_\infty \geq \bar{\gamma}$. The minimum feasible γ_∞ for Scherer's method is larger than $\bar{\gamma}$, and that for Oliveira's is slightly smaller than $\bar{\gamma}$. This fact implies that these two methods do not provide efficient design results for this example. For the proposed procedure, the controller designed by Oliveira's method is selected as an initial controller. Then, after 8 iterations, the minimum feasible γ_∞ for the proposed method is 10.865 and is much smaller than $\bar{\gamma}$. The state space data of the designed controller for $\gamma_\infty = 10.865$ is

$$\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = \left[\begin{array}{ccc|c} -1.2192 & -3.9646 & 0.2890 & -0.1017 \\ 0.9316 & -0.9926 & 0.0632 & 2.4815 \\ 0.0040 & 0.0387 & -0.5029 & -0.9873 \\ \hline -1.6088 & -3.9381 & 1.2711 & 0 \end{array}\right].$$

Next, let us examine the guaranteed and achieved H_2 norms. For the proposed method, the simultaneous H_∞ controller above is selected as an initial controller. Then, Figure 1 shows the relation between the H_∞ constraint γ_∞ and the guaranteed and achieved H_2 norms. As the guaranteed and achieved H_2 norms by the proposed method are same, only the guaranteed norm is shown. It is easy to see that the proposed method provides efficient design results. For example, only the proposed method can solve the problem with $\gamma_\infty = 15$, and the state space data of the designed controller is

$$\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = \left[\begin{array}{ccc|c} -0.7692 & 3.1366 & 0.1370 & 0.1377 \\ -0.9702 & -0.6705 & -0.0309 & 2.0843 \\ -0.0096 & -0.1001 & -0.5000 & -0.9205 \\ \hline 1.2565 & -3.4486 & 0.8083 & 0 \end{array}\right].$$

which achieves $\|T_0\|_2 = 32.434$. This controller is obtained after 16 iterations.

Table 1. Feasible H_∞ constraint and actual H_∞ norm (Example 14).

Method	feasible H_∞ constraint	actual H_∞ norm
Scherer *et al.*	29.081	16.854
Oliveira *et al.*	20.958	15.153
proposed	10.865	10.865
H_2 optimal	—	21.757

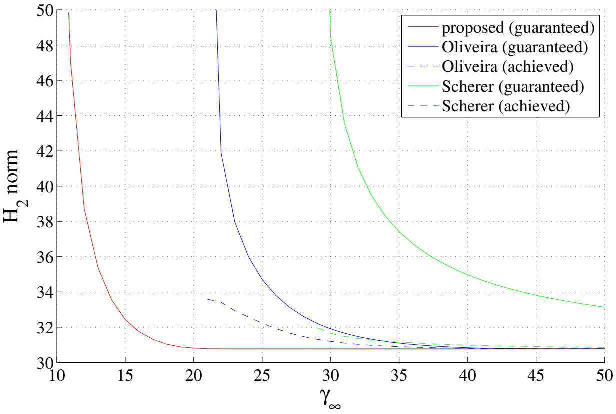

Fig. 1. Guaranteed and achieved H_2 norm (Example 14).

Example 15. (Decentralized H_2 control). The proposed method is applied to a decentralized H_2 controller design. The generalized plant is also borrowed from Oliveira *et al.* (2002) and its state space data are

$$A = \begin{bmatrix} 0.8189 & 0.0863 & 0.0900 & 0.0813 \\ 0.2524 & 1.0033 & 0.0313 & 0.2004 \\ -0.0545 & 0.0102 & 0.7901 & -0.2580 \\ -0.1918 & -0.1034 & 0.1602 & 0.8604 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.0953 \\ 0.0145 \\ 0.0862 \\ -0.0011 \end{bmatrix}, B_2 = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D_{02} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_2 = I, \quad D_{00} = O.$$

The problem is to find a decentralized state feedback which minimizes $\|T_0\|_2$. Let the initial decentralized state feedback $K^{(0)}$ be

$$K^{(0)} = \begin{bmatrix} -2.5841 & -5.0228 & 0 & 0 \\ 0 & 0 & 1.9632 & -8.6826 \end{bmatrix}$$

which is designed by Oliveira *et al.* (2002). After 11 iterations, we obtained a decentralized state feedback

$$K^{(11)} = \begin{bmatrix} -0.4723 & -0.3001 & 0 & 0 \\ 0 & 0 & -0.3782 & -0.1848 \end{bmatrix},$$

which attains $\gamma_2^{(11)} = 0.2728$. Figure 2 shows the convergence of the achieved H_2 norm.

Ebihara *et al.* (2004) propose an alternating projection method for structural constraint controller designs. Although their method is effective, their method does not ensure the monotonic decrease of the performance index. Table 2 shows the computational expense of Ebihara's and the proposed methods. The proposed method is much effective from the viewpoint of computational expense. Furthermore, Ebihara's method requires bisection method

Table 2. Achieved H_2 norm and computational expense (Example 15).

Method	H_2 norm	iteration	time [s]
centralized	0.2707	–	–
Oliveira *et al.*	0.6509	–	–
Ebihara *et al.*	0.2735	62	34.44
proposed	0.2728	11	1.82

(The results by Ebihara *et al.* are taken from their paper and they use PC with Pentium 4, 2.53GHz.)

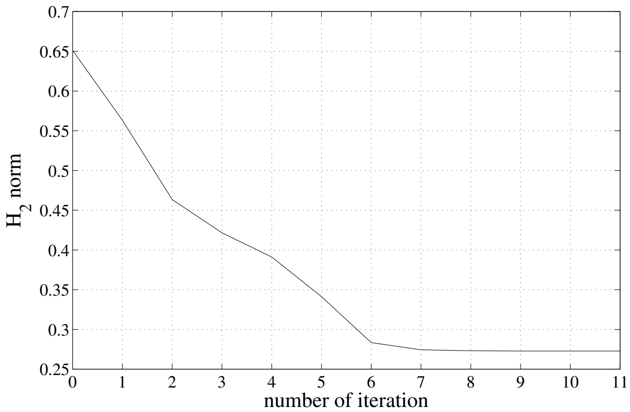

Fig. 2. Achieved H_2 norm (Example 15).

to obtain the optimal controller and it takes 62 iterations for a given $\gamma_2 = 0.2735$.

5. CONCLUSIONS

This paper introduces new dilated LMI characterizations for discrete-time descriptor systems. Based on the characterizations an iterative design procedure is proposed for multiobjective control and structurally constrained controller designs for discrete-time systems.

ACKNOWLEDGEMENTS

The author would like to thank Prof. Eitaku Nobuyama for the helpful discussions.

REFERENCES

Apkarian, P., H. D. Tuan and J. Bernussou (2001). Continuous-time analysis, eigenstructure assignment, and H_2 synthesis with enhanced linear matrix inequalities (LMI) characterizations. *IEEE Transactions on Automatic Control* **46**(12), 1941–1946.

Arzelier, D. and D. Peaucelle (2002). An iterative method for mixed H_2/H_∞ systhesis via static output-feedback. In: *Proceedings of the 41st IEEE Conference on Decision and Control*. Las Vegas. pp. 3464–3469.

Chen, G. (2004). System analysis using redundancy of descriptor representation. In: *2004 IEEE International Symposium on Computer Aided Control System Design*. Taipei. pp. 231–236.

de Oliveira, M. C., J. Bernussou and J. C. Geromel (1999). A new discrete-time robust stability condition. *Systems & Control Letters* **37**, 261–265.

de Oliveira, M. C., J. C. Geromel and J. Bernussou (2002). Extended H_2 and H_∞ norm characterizations and controller parametrizations for discrete-time systems. *International Journal of Control* **75**(9), 666–679.

Ebihara, Y. and T. Hagiwara (2004). New dilated LMI characterizations for continuous-time multiobjective controller synthesis. *Automatica* **40**(11), 2003–2009.

Ebihara, Y., K. Tokuyama and T. Hagiwara (2004). Structured controller synthesis using LMI and alternating projection method. *International Journal of Control* **77**(12), 1137–1147.

Hsiung, K.-L. and L. Lee (1999). Lyapunov inequality and bounded real lemma for discrete-time descriptor systems. *IEE Proceedings - Control Theory and Applications* **146**(4), 327–331.

I. Masubuchi, A. Ohara, N. Suda (1998). LMI-based controller synthesis: A unified formulation and solution. *International Journal Robust and Nonlinear Control* **8**(8), 669–686.

Iwasaki, T. and R. E. Skelton (1994). All controllers for the general H_∞ control problem — LMI existence conditions and state-space formulas. *Automatica* **30**(8), 1307–1317.

Kuijiper, M. (1994). *First-order Representations of Linear Systems.* Birkhäuser.

Masubuchi, I. (2006). Dissipativity inequalities for continuous-time descriptor systems with applications to synthesis of control gains. *Systems & Control Letters* **55**(2), 158–164.

Peaucelle, D., D. Arzelier, O. Bachelier and J. Bernussou (2000). A new robust $\mathcal{D}$-stability condition for real convex polytopic uncertainty. *Systems & Control Letters* **40**, 21–30.

Saeki, M. (2006). Static output feedback design for H_∞ control by descent method. In: *Proceedings of the 45th IEEE Conference on Decision and Control.* San Diego. pp. 5156–5161.

Scherer, C., P. Gahinet and M. Chilali (1997). Multiobjective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control* **42**(7), 896–911.

Sebe, N. (2007). A new dilated LMI characterization and iterative control system synthesis. In: *11th IFAC/IFORS/IMACS/IFIP Symposium on Large Scale Systems: Theory and Applications.* Gdańsk. 6 pages.

Shimomura, T. and T. Fujii (2005). Multiobjective control via successive over-bounding of quadratic terms. *International Journal of Robust and Nonlinear Control* **15**, 363–381.

Shimomura, T., M. Takahashi and T. Fujii (2001). Extended-space control design with parameter-dependent Lyapunov functions. In: *Proceedings of the 40th IEEE Conference on Decision and Control.* Orlando. pp. 2157–2162.

Sturm, J. F. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software* **11-12**, 625–653. Version 1.1 available from `http://sedumi.mcmaster.ca/`.

Stykel, Tatjana (2006). On some norms for descriptor systems. *IEEE transactions on automatic control* **51**(5), 842–847.

Xu, Shengyuan and Chengwu Yang (1999). Stabilization of discrete-time singular systems: a matrix inequalities approach. *Automatica* **35**, 1613–1617.

Appendix A. OUTLINES OF PROOFS

A.1 Proof of Theorem 5

Lemma 16. (Kuijiper, 1994) The given descriptor system (18) is regular and impulse-free, if and only if

$$\mathrm{Im}\tilde{E} + \mathrm{Im}\tilde{A}(\mathrm{Ker}\tilde{E}) = \Re^n. \tag{A.1}$$

Proof of Theorem 5. Necessity. From Proposition 4, there exists a symmetric matrix $\tilde{X}$ which satisfies

$$\tilde{E}^{\mathrm{T}}\tilde{X}\tilde{E} - \tilde{A}^{\mathrm{T}}\tilde{X}\tilde{A} > O, \tag{A.2}$$

$$\tilde{E}^{\mathrm{T}}\tilde{X}\tilde{E} \geq O. \tag{A.3}$$

if the given descriptor system (18) is admissible. Let us define $\tilde{Y}$ as

$$\tilde{Y} = \tilde{U}^{-\mathrm{T}}\tilde{X}\tilde{U}^{-1} = \begin{bmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} \\ \tilde{Y}_{12}^{\mathrm{T}} & \tilde{Y}_{22} \end{bmatrix}. \tag{A.4}$$

With this $\tilde{Y}$ and multiplying $\tilde{V}^{\mathrm{T}}$ and $\tilde{V}$, (A.2) and (A.3) become

$$\begin{bmatrix} E & \\ & O \end{bmatrix}^{\mathrm{T}} \tilde{Y} \begin{bmatrix} E & \\ & O \end{bmatrix} - (\tilde{U}\tilde{A}\tilde{V})^{\mathrm{T}}\tilde{Y}(\tilde{U}\tilde{A}\tilde{V}) > O, \tag{A.5}$$

$$E^{\mathrm{T}}\tilde{Y}_{11}E > O. \tag{A.6}$$

Similar to (15), the strict positivity of (A.6) can be shown. The inequality (A.5) can be rewritten as

$$\begin{aligned} &\begin{bmatrix} E^{\mathrm{T}} \\ O \end{bmatrix} \tilde{Y}_{11} \, [\, E \;\; O \,] - (\tilde{U}_1\tilde{A}\tilde{V})^{\mathrm{T}}\tilde{Y}_{11}(\tilde{U}_1\tilde{A}\tilde{V}) \\ &\quad - (\tilde{U}\tilde{A}\tilde{V})^{\mathrm{T}} \begin{bmatrix} O & \tilde{Y}_{12} \\ \tilde{Y}_{12}^{\mathrm{T}} & \tilde{Y}_{22} \end{bmatrix} (\tilde{U}\tilde{A}\tilde{V}) > O. \end{aligned} \tag{A.7}$$

Then, the assignments

$$P = \tilde{Y}_{11}, \quad \tilde{F} = (\tilde{U}\tilde{A})^{\mathrm{T}} \begin{bmatrix} \tilde{Y}_{12} \\ \frac{1}{2}\tilde{Y}_{22} \end{bmatrix} \tag{A.8}$$

accomplish the necessity.

Sufficiency. Multiplying $\tilde{V}^{\mathrm{T}}$ and $\tilde{V}$, the condition (23) becomes

$$\begin{aligned} &(\tilde{U}_1\tilde{E}\tilde{V})^{\mathrm{T}}P(\tilde{U}_1\tilde{E}\tilde{V}) - (\tilde{U}_1\tilde{A}\tilde{V})^{\mathrm{T}}P(\tilde{U}_1\tilde{A}\tilde{V}) \\ &\quad \mathrm{He}\{\tilde{V}^{\mathrm{T}}\tilde{F}\tilde{U}_2\tilde{A}\tilde{V}\} > O. \end{aligned} \tag{A.9}$$

Let

$$\tilde{U}\tilde{A}\tilde{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{A.10}$$

From Lemma 16, if the system is not impulse-free, there exist a vector $\boldsymbol{p}(\neq \boldsymbol{o})$ which satisfies $\tilde{A}_{22}\boldsymbol{p} = \boldsymbol{o}$. Multiplying $[\boldsymbol{o}^{\mathrm{T}} \quad \boldsymbol{p}^{\mathrm{T}}]^{\mathrm{T}}$ and its transpose to (A.9), the left hand side of the inequality becomes 0. This contradicts the negativity of (A.9). Thus the system is regular and impulse-free, and $\tilde{A}_{22}$ is non-singular. Applying the elimination lemma, the inequality (A.9) is equivalent to

$$(A_{11} - A_{12}A_{22}^{-1}A_{21})^{\mathrm{T}}P(A_{11} - A_{12}A_{22}^{-1}A_{21}) - E^{\mathrm{T}}PE < O. \tag{A.11}$$

This shows the stability of exponential modes of the system. □

A.2 Outlines of the other proofs

Once the conditions for the admissibility of discrete-time descriptor systems are obtained, the conditions for H_2 and H_∞ norm of the systems can be derived directly from the results in Stykel (2006) and Hsiung and Lee (1999) .