Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Transition to Complex Behavior in Networks of Coupled Dynamical Systems ⋆

J. G. Barajas-Ramírez * R. Femat *

* *Laboratorio para Biodinámica y Sistemas Alineales, División de Matemáticas Aplicadas, IPICYT, Apdo. Postal 3-90, Tangamanga, CP 78231, San Luis Potosí, S.L.P., México (Tel: 52-444-8342000x7222; e-mail: {jgbarajas,rfemat}@ipicyt.edu.mx).*

Abstract: The emergence of complex behavior is studied in a network of coupled dynamical systems whose trajectories converge to a stable equilibrium point. The effects of network topology on the stability of its synchronized behavior is measured in terms of transverse Lyapunov exponents. By choosing a suitable coupling configuration, the transverse Lyapunov exponents are made positive, which may lead to the emergence of complex behavior. Moreover, the effects of the coupling configuration can lead to unbounded trajectories. The relationship between the transverse Lyapunov exponents and the eigenvalues of the connectivity matrix is used to establish upper and lower limit values for the transition to complex behavior. The transition criteria are expressed in terms of coupling strength and the number of nodes in the network. There are two main contributions on this manuscript: (1) the analytical derivation of the relationship between the local Lyapunov exponents and those of the entire network as the number of nodes increases, and (2) to show the existence of an interval of coupling strength values for the transition into complex behavior in networks with homogeneous connectivity, which becomes smaller as the number of nodes in the network growths. Additionally, we also show that in networks with heterogeneous connectivity, the trajectories transit directly from stable equilibrium to unbounded behavior. These results are illustrated with numerical simulations.

Keywords: Networks, Coupled Dynamics, State Transition.

1. INTRODUCTION

Over the last couple of decades, the study of networks has attracted a great deal of attention from the scientific community [Boccaletti et al. (2006); Newman et al. (2007); Strogatz (2001); Wang and Chen (2003)]. The dynamical analysis of coupled systems is a particular aspect of network research. Typically, dynamical networks are studied under the assumptions that every node is identical and that they are coupled in regular structures, such as rings, chains, or lattices. The studies on phase coupled oscillators [Kuramoto (1975)], coupled map lattices (CML) [Ding and Yang (1997)], and cellular neural networks (CNN) [Chua (1998)] are examples of this approach. Recently, the discovery of the small-world [Watts and Strogatz (1998)] and scale-free [Barabasi and Albert (1999)] topologies has shifted the research emphasis towards the inclusion of topological complexity in the dynamical analysis of networks. The majority of the studies along this research line, are mainly concern with synchronization and control of dynamical networks in various complex topologies, different specific scenarios have been considered, from uniformly coupled small-world networks [Barahona and Pecora (2002)], to adaptively weighted scale-free networks [Zhou and Kurths (2006)]; among many others [Boccaletti et al. (2006); Li (2005); Lü et al. (2004); Motter et al. (2005); Wang and Chen (2003)].

Unlike the previously cited works, in this paper the objective is to investigate: How the topology of a network, constructed by coupling nodes that in isolation converge to an equilibrium point, can result on transitions from a common stable equilibrium state to bounded complex behavior, and finally to unbounded behavior. To this end, the dynamics of the entire network can be characterized in terms of Lyapunov exponents [Ott (1993)]. In our analysis is assume that the network connections are fixed and can be modeled as the linear combination of state variables of the nodes. In particular, in this paper we consider the case of diffusively coupled networks where all nodes are identical. Under these conditions, the network has a common solution, which corresponds to the dynamical behavior of an isolated node and describes the so-called synchronization manifold [Barahona and Pecora (2002); Boccaletti et al. (2006); Wang and Chen (2003)]. As presented in Section 2, the emergence of complex behavior can be described with regards to the synchronization manifold, from this description a direct relationship can be found between: The transverse Lyapunov exponents (*tLe*s) of the entire network [Ding and Yang (1997); Chen et al. (2003); Rangarajan and Ding (2002)], the Lyapunov exponents of an isolated node, and the eigenvalues of its connectivity matrix. Then, properly choosing the topolog-

⋆ This work was supported in part by CONACYT under Grant No. 48307-R.

ical characteristics of the network, some of its *tLe*s become positive [Li et al. (2004); Li and Chen (2005); Zhang et al. (2006)]. The emergence of complex behavior requires that some of the *tLe*s be positive. However, the presence of positive Lyapunov exponents is only a necessary condition. In order to avoid unbounded trajectories, we must have that the overall sum of *tLe*s is negative or at least equal to zero. Moreover, if the positive *tLe*s dominate the sum, the network transits to an unbounded state. The analytical relation between these transitions to complex behavior and network topology is one of the contributions of this paper. Although the intuition that as the number of nodes on a network growths instabilities can be expected, has been known for a long time [Motter et al. (2005); Zhou and Kurths (2006)], the results presented in Section 3 give an analytical validation to this assumption. We show that the requirements for emergence of bounded complex behavior can be express as limit values for the coupling strength and number of nodes in the network, this fact permits to investigate the effects of different topologies on the conditions for transition towards complex and eventually towards unbounded behavior. In Section 4, two simple networks are used to determine the effects of homogeneous and heterogeneous connectivity on the transitions between behaviors and is shown that for a network with homogeneous connectivity is possible to have complex behaviors while for heterogeneous connectivity the network transits directly from a stable equilibrium point towards an unbounded state. These results are commented and conclusions are given in the final Section of this paper.

2. TRANSITION FROM STABLE EQUILIBRIUM

Along the text, let us consider a network of N linearly and diffusively coupled identical nodes where each one is a m-dimensional dynamical system, which before been coupled into the network has an asymptotically stable equilibrium point. For simplicity, we will assume that when two nodes are coupled all their internal states are connected, then the dynamics of the entire network are given by the following state equations [Rangarajan and Ding (2002); Lü et al. (2004); Zhang et al. (2006)]:

$$\dot{x}_i(t) = f(x_i(t)) - \sum_{j=1}^{N} c_{ij} x_j(t), \qquad i = 1, 2, ..., N \qquad (1)$$

where $x_i(t) = [x_{i1}, x_{i2}, ..., x_{im}]^\top \in \mathbf{R}^m$ are the state variables of node i; $f : \mathbf{R}^m \to \mathbf{R}^m$ describes the dynamics of an isolated node; and the constant $c_{ij} \geq 0$ is the coupling strength between node i and node j. The coupling configuration of the network in (1) is assumed to satisfy the following:

(I) The entries of the connectivity matrix are symmetric ($c_{ij} = c_{ji}, \forall\ i,j$);

(II) The sum by rows and by columns are null ($\sum_{j=1}^{N} c_{ij} = \sum_{j=1}^{N} c_{ji} = 0, \forall i$); such that the diagonal elements c_{ii} satisfy the following equation

$$c_{ii} = - \sum_{j=1, j\neq i}^{N} c_{ij} = - \sum_{j=1, j\neq i}^{N} c_{ji}, \qquad i = 1, 2, ..., N \quad (2)$$

Then, if there are no isolated nodes the connectivity matrix $\mathcal{C} = \{c_{ij}\} \in \mathbf{R}^{N\times N}$ is irreducible with zero as an eigenvalue of multiplicity one and all others strictly negative [Wu and Chua (1995); Chen et al. (2003)]. Therefore, the eigenvalues of $\mathcal{C}$ can be ordered as follows:

$$0 = \varphi_1 > \varphi_2 \geq \varphi_3 \geq ... \geq \varphi_N \qquad (3)$$

The Lyapunov exponents of an isolated node ($\dot{x} = f(x)$) can be obtained using the following definition [Ott (1993)]:

$$h_i = \lim_{t\to+\infty} \frac{1}{t} |J(t, x_0) u_i|, \qquad i = 1, 2, ..., m \qquad (4)$$

where h_i is the Lyapunov exponent of an isolated node along the direction u_i; $J(t, x_0)$ is the Jacobian matrix of f evaluated at a randomly selected initial condition x_0; and $\{u_1, u_2, ..., u_m\}$ is a set of orthonormal vectors in the tangent space of the system. Since before been coupled into the network each node has a stable equilibrium point that attracts its trajectories, the m Lyapunov exponents of a node in isolation are strictly negative and can be ordered as follows:

$$0 > h_1 \geq h_2 \geq ... \geq h_m \qquad (5)$$

In regards to network (1), its dynamical behavior can be characterized in terms of its transverse Lyapunov exponents [Chen et al. (2003); Ding and Yang (1997); Li and Chen (2005); Rangarajan and Ding (2002)]. These exponents measure the average divergence between the network trajectories and the synchronized solution $\bar{x}(t) = x_i(t) = x_j(t), \forall\ i,j$. This solution describes a diagonal manifold in the mN-dimensional state space of the network called the synchronization manifold [Barahona and Pecora (2002); Boccaletti et al. (2006); Wang and Chen (2003)], which by construction $\bar{x}$ is a solution for the entire network. The stability of the synchronized behavior can be determine evaluating the sign of the transverse Lyapunov exponents, if they are all negative, the network will synchronize to $\bar{x}$. The stability analysis starts linearizing the synchronization errors ($\xi_i(t) = x_i(t) - \bar{x}(t)$) around the synchronized state, the following variational equation is obtained

$$\dot{\xi}_i(t) = J(x(t))\xi_i(t) - \sum_{j=1}^{N} c_{ij}\xi_j(t), \;\; i = 1, 2, ..., N \qquad (6)$$

which in vector form becomes

$$\dot{\mathcal{X}}(t) = J(x(t))\mathcal{X}(t) - \mathcal{X}(t)\mathcal{C}^\top \qquad (7)$$

where $\mathcal{X}(t) = [\xi_1, \xi_2, ..., \xi_N] \in \mathbf{R}^{m\times N}$.

Since the connectivity matrix is symmetric and irreducible, it is diagonalizable and satisfies the equation

$$\mathcal{C} = \Gamma\Lambda\Gamma^{-1} \qquad (8)$$

where $\Gamma = [\gamma_1, \gamma_2, ..., \gamma_N] \in \mathbf{R}^{N\times N}$; and $\Lambda = diag(\varphi_1, \varphi_2, ..., \varphi_N) \in \mathbf{R}^{N\times N}$; with γ_i the i-th eigenvector of $\mathcal{C}$ and λ_i its corresponding eigenvalue. From (8) expressing the variational equation (7) in terms of the eigenvectors of $\mathcal{C}$, one gets

$$\dot{\nu}_k(t) = J(x(t))\nu_k(t) - \varphi_k\nu_k(t), \quad k = 1, 2, ..., N \qquad (9)$$

where $\nu_k(t) = \mathcal{X}(t)\gamma_k \in \mathbf{R}^m$. Then, applying the definition in (4) to the variational equation (9), the entire spectrum of transverse Lyapunov exponents ($\mu_i(\varphi_k)$) of (1) is given by:

$$\mu_i(\varphi_k) = h_i - \varphi_k, \qquad (10)$$

for $i = 1, 2, ..., m$, and $k = 1, 2, ..., N$.

From (3) and (10) one can see that the contribution of the coupling structure to the transverse Lyapunov exponents makes them more positive. Notice that when the nodes are not connected ($\varphi_k = 0$, $k = 1, ..., N$), the mN transverse Lyapunov exponents of the network are all negative and the entire network has a common stable equilibrium point. For complex behavior to emerge the coupling structure be such that at least one of the transverse Lyapunov exponents becomes positive after the nodes are coupled [Li and Chen (2005); Zhang et al. (2006)]. However, the presence of positive exponents implies instability. Then, for the network's trajectories to remain bounded is necessary that the overall sum of transverse Lyapunov exponents be negative or at the most zero. Otherwise, the network's trajectories become unbounded growing along the directions of its positive transverse Lyapunov exponents.

Given that, the eigenvalues of the connectivity matrix and the Lyapunov exponents of an isolated node are ordered as in (3) and (5), respectively. Then, the spectrum of transverse Lyapunov exponents is ordered as follows:

$$\begin{aligned} &\mu_i(\varphi_N) \geq \mu_i(\varphi_{N-1}) \geq ... \geq \mu_i(\varphi_2) \geq \mu_i(\varphi_1), \\ &\qquad\qquad\qquad\qquad\qquad i = 1, 2, ..., m \\ &\mu_1(\varphi_k) \geq \mu_2(\varphi_k) \geq ... \qquad \geq \mu_{m-1}(\varphi_k) \geq \mu_m(\varphi_k), \\ &\qquad\qquad\qquad\qquad\qquad k = 1, 2, ..., N \end{aligned} \tag{11}$$

The largest nonzero transverse Lyapunov exponent, $\mu_1(\varphi_N)$, is the first one that can be made positive by coupling. To have an upper limit to the number of positive transverse exponents, is necessary to require that $\mu_\tau(\varphi_T)$ be the first negative exponent, for a given τ $(1 < \tau < m)$ and T $(1 < T < N)$. Then, from (10) and taking into account that h and φ are all not positive, the conditions to have at most τT positive transverse Lyapunov exponents are:

$$|h_1| < |\varphi_N| \tag{12}$$

$$|\varphi_T| < |h_\tau| \tag{13}$$

Satisfying (12) and (13) is a necessary condition for bounded complex behavior. However, this is not sufficient, to avoid unbounded trajectories the sum of positive (σ^+) and negative (σ^-) transverse Lyapunov exponents must satisfy

$$\sigma^+ + \sigma^- < 0 \tag{14}$$

The conditions for the transition from a stable fixed-point to a complex behavior presented above are expressed in terms of the eigenvalues of $\mathcal{C}$. A clearer way of understanding the implications of these conditions is to express them as bounds on the entries of the connectivity matrix and the number of nodes in the network. To this end, the following procedure can be used:

Consider a symmetric matrix $\Omega = \{\omega_{ij}\}$, with its entries defined as: $\omega_{ii} = -(N-1)\alpha$, for $i = 1, 2, ..., N$; and $\omega_{ij} = \alpha$, for $\forall\ i,\ j, i \neq j$; where α is a positive constant. By construction, the eigenvalues of Ω are

$$0 = \psi_1 > \psi_k = -N\alpha, \quad k = 2, 3, ..., N \tag{15}$$

Defining the matrix $P = \Omega - \mathcal{C}$, and examining its entries the following can be established:

(a) The diagonal element of P are given by $p_{ii} = \omega_{ii} - c_{ii} = \sum_{j=1,j\neq i}^{N} c_{ij} - (N-1)\alpha$. To have $p_{ii} > 0$ the following conditions must be satisfied

$$c_{ij} > \alpha, \quad \forall\ i, j, i \neq j \tag{16}$$

(b) The absolute value of the off-diagonal elements of P are given by $|p_{ij}| = |\alpha - c_{ij}|$, for $\forall\ i,\ j, j \neq i$. Then, if (16) is satisfied, one has

$$|p_{ii}| \geq \sum_{j=1,j\neq i}^{N} |p_{ij}|, \quad i = 1, 2, ..., N \tag{17}$$

If (16) and (17) hold, P is a positive semidefinite matrix. It follows that $\Omega \geq \mathcal{C}$, which can be expressed in terms of their eigenvalues as $\psi_k \geq \varphi_k$, for $k = 1, 2, ..., N$ [Horn and Johnson (1985)]. Using (15) we have $\varphi_k \leq -N\alpha$, $(k = 2, 3, ..., N)$, and from (12) $\varphi_N \leq -N\alpha < h_1$. Then, one has $\alpha > \frac{|h_1|}{N}$, which is only valid if (16) holds. Combining both restrictions the lower bound on the entries of the connectivity matrix is found to be $\frac{|h_1|}{N} < c_{ij}$. Next, defining the matrix $Q = \mathcal{C} - \Omega$ and following a similar procedure, an upper bound for the entries of $\mathcal{C}$ is found such that (13) is satisfied: $c_{ij} < \alpha < \frac{|h_\tau|}{N}$. Hence, the bounds on c_{ij}, such that at most τT transverse Lyapunov exponents of the network in (1) are made positive are given by

$$\frac{|h_1|}{N} < c_{ij} < \frac{|h_\tau|}{N} \tag{18}$$

Is important to keep in mind that the condition on (18) is a necessary condition for the emergence of complex behavior on the network, but is not sufficient, since the trajectories may become unstable along the positive directions. To have bounded trajectories, the condition on (14) must be also satisfied by the same connectivity entries. Although, the criterion on (18) is only a necessary condition, it indicates two aspects of the transition between behaviors: (i) The interval of values for the connectivity entries such that complex behavior can be observed is limited by the distance between the Lyapunov exponents of an isolated node; (ii) As the network growths, the interval of values or c_{ij} for which complex behavior can be observed, is reduced as an inverse function of the number of nodes in the network.

The conditions for the transition from one behavior to another depend on the eigenvalues of the connectivity matrix. Since different topologies have different eigenvalue spectrums, in the following sections the effect of different network topologies on the transition towards complex behavior is analyzed.

3. THE EFFECT OF TOPOLOGY

A commonly studied version of the network in (1) is a uniformly coupled network, which consists on an special case of the connectivity matrix, $\mathcal{C} = c\{a_{ij}\} \in \mathbf{R}^{N\times N}$, where $c > 0$ is a constant uniform coupling strength and a_{ij} is assigned as follows: If there is a connection between node i and node j $(i \neq j)$, $a_{ij} = a_{ji} = 1$, and otherwise $a_{ij} = a_{ji} = 0$, with $a_{ii} = -\sum_{i=1,i\neq j}^{N} a_{ij} = -\sum_{i=1,i\neq j}^{N} a_{ji}$. In this case, the network on (1) becomes

$$\dot{x}_i(t) = f(x_i(t)) - c\sum_{j=1}^{N} a_{ij}x_j(t), \qquad i = 1, 2, ..., N \quad (19)$$

Following a similar procedure as in the previous section, the entire spectrum of *tLe* for the network in (19) is given by

$$\mu_i(\lambda_k) = h_i - c\lambda_k, \quad i = 1, 2, ..., m; \quad k = 1, 2, ..., N \quad (20)$$

where h_i are the Lyapunov exponents of an isolated node; and λ_k are the eigenvalues of the matrix $\mathcal{A} = \{a_{ij}\} \in \mathbf{R}^{N\times N}$, which by construction are ordered as $0 = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$.

In order to make positive, at most τT of the *tLe* in (20), the uniform coupling strength must be within the following interval

$$\frac{|h_1|}{|\lambda_N|} < c < \frac{|h_\tau|}{|\lambda_T|} \quad (21)$$

Then, to have bounded trajectories, the overall sum of *tLe* must not be positive, that is

$$\Upsilon^+ + \Upsilon^- < 0 \quad (22)$$

where $\Upsilon^+ = \sum_{p=1}^{\tau-1}\sum_{q=0}^{T-1}\mu_p(\lambda_{(N-q)})$ and

$$\Upsilon^- = \sum_{p=1}^{\tau-1}\sum_{q=T}^{N}\mu_p(\lambda_{(N-q)}) + \sum_{p=\tau}^{m}\sum_{q=0}^{N-1}\mu_p(\lambda_{(N-q)}).$$

In what follows, the emergence of complex behavior on three commonly used regular coupling configurations is analyzed.

3.1 Globally Coupled Networks

The simplest coupling scheme that can be used to construct the network in (19), consists on connecting each node to every other one. The resulting network will have a connectivity matrix in the form

$$\mathcal{A}_{gc} = \begin{pmatrix} -(N-1) & 1 & ... & 1 \\ 1 & -(N-1) & ... & 1 \\ ... & & ... & ... \\ 1 & 1 & ... & -(N-1) \end{pmatrix}$$

where the eigenvalues are found to be $\lambda_{gc,1} = 0$, and $\lambda_{gc,k} = -N$, for $k = 2, 3, ..., N$.

From the results above, if the network in (19) is globally coupled and the coupling strength satisfies $\frac{|h_1|}{N} < c < \frac{|h_\tau|}{N}$. Then, at most τT transverse Lyapunov exponents become positive.

For a globally coupled network with a fixed coupling strength two situations arise as the network growths: (1) The lower limit on the coupling strength such that a *tLe* becomes positive goes to zero ($\frac{|h_1|}{N} \to 0$, as $N \to \infty$). Then, positive *tLe* are found even for a very small coupling strength. (2) From the other side, the upper limit will also go towards zero ($\frac{|h_\tau|}{N} \to 0$, as $N \to \infty$). Then, the number of positive *tLe* becomes mN.

Since as the network growths, the number of positive *tLe* also growths ($T \to N$), there is a critical value of N, after which the condition on (22) is no longer satisfied. Then, the network becomes unbounded.

3.2 Star Coupled Networks

The coupling scheme for a star network consists on connecting every node only to a central hub node. In this case, the connectivity matrix has the following form

$$\mathcal{A}_{sc} = \begin{pmatrix} -(N-1) & 1 & 1 & ... & 1 \\ 1 & -1 & 0 & ... & 0 \\ ... & ... & & ... & ... \\ 1 & 0 & 0 & ... & -1 \end{pmatrix}$$

where the eigenvalues are $\lambda_{sc,1} = 0$, $\lambda_{sc,k} = -1$, for $k = 2, 3, ..., N-1$, and $\lambda_{sc,N} = -N$.

For a network with a star coupling structure, the condition for making some of its *tLe* positive becomes $\frac{|h_1|}{N} < c < |h_\tau|$. Then, if the coupling strength is fixed, as the network growths only the lower limit will go to zero. This means that the *tLe* corresponding to the central hub node becomes positive, even for an extremely small coupling strength.

From the results above, for a sufficiently large network ($N > \frac{|h_1|}{c}$) all the *tLe* corresponding to the central hub node becomes positive and, as a consequence, this node becomes unbounded even if the majority of the nodes remain stable. For this coupling configuration, the network transits directly from a stable fixed-point behavior to an unstable state.

4. NUMERICAL SIMULATIONS

Consider a network in the form (19) where each node is a Lorenz system described by the following equations:

$$\begin{aligned} \dot{x}_1(t) &= -a(x_1(t) - x_2(t)) \\ \dot{x}_2(t) &= rx_1(t) - x_1(t)x_3(t) - x_2(t)) \\ \dot{x}_3(t) &= x_1(t)x_2(t) - bx_3(t) \end{aligned} \quad (23)$$

which, for the parameter values $a = 10$, $b = \frac{8}{3}$, and $r = 0.5$, has a single asymptotically stable equilibrium point at $\bar{x}$=(0, 0, 0). The Jacobian matrix of (23) is given by: $J(x) = \left(-10, 10, 0;\ 0.5 - x_3, -1, -x_1;\ x_2(t), x_1(t), -\frac{8}{3}\right)$. Then, the Lyapunov exponents of an isolated node can be estimate from the eigenvalues of the $J(x(t))$ evaluated around $\bar{x}$. In this example, the Lyapunov exponents of an isolated node are assigned to be $h_1 \doteq -0.5$, $h_2 \doteq -3$ and $h_3 \doteq -12$.

To illustrate the results presented above, the network (19) is constructed by coupling Lorenz systems with a uniform coupling strength. Then, the transition from regularity is analyzed for different coupling configurations as the network growths.

First, the network is constructed with a globally coupled configuration. According to (21), for a network with three nodes and a fixed uniform coupling strength $c = 0.1$, none of the *tLe* are be positive, and the network has a stable fixed-point behavior (see Figure 1(a)). As the number of nodes increases, the threshold value $\frac{|h_1|}{c}$ becomes smaller that $c = 0.1$. Then, some of the *tLe* become positive, but since the overall sum remains non positive, the trajectories

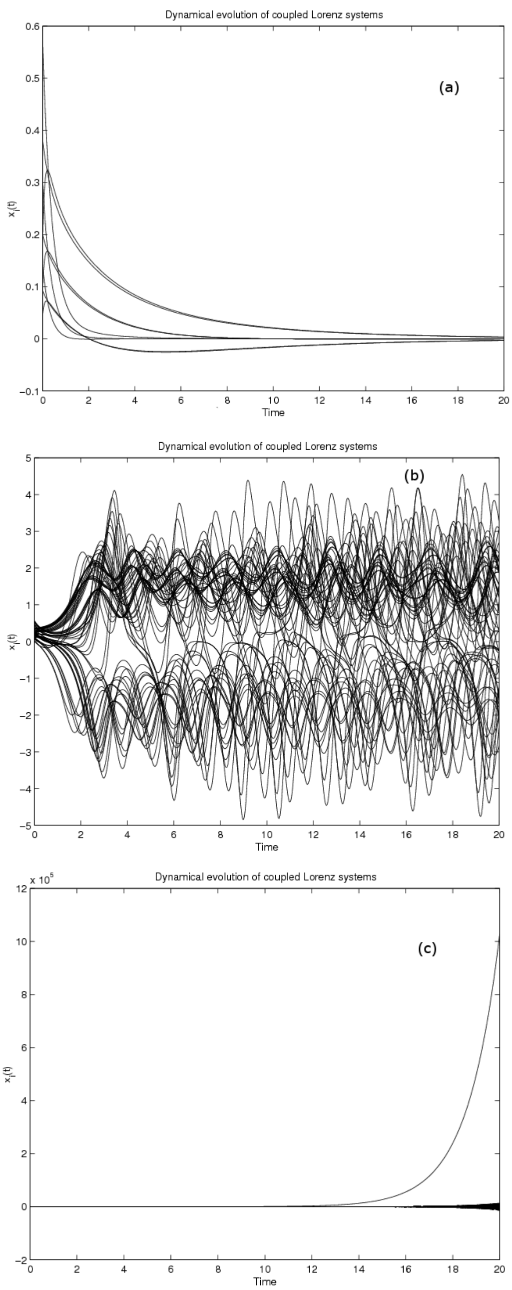

Fig. 1. Dynamical evolution of (a) three, (b) twenty-one and (c) thirty-five Lorenz systems in a globally coupled network with a uniform coupling strength $c = 0.1$

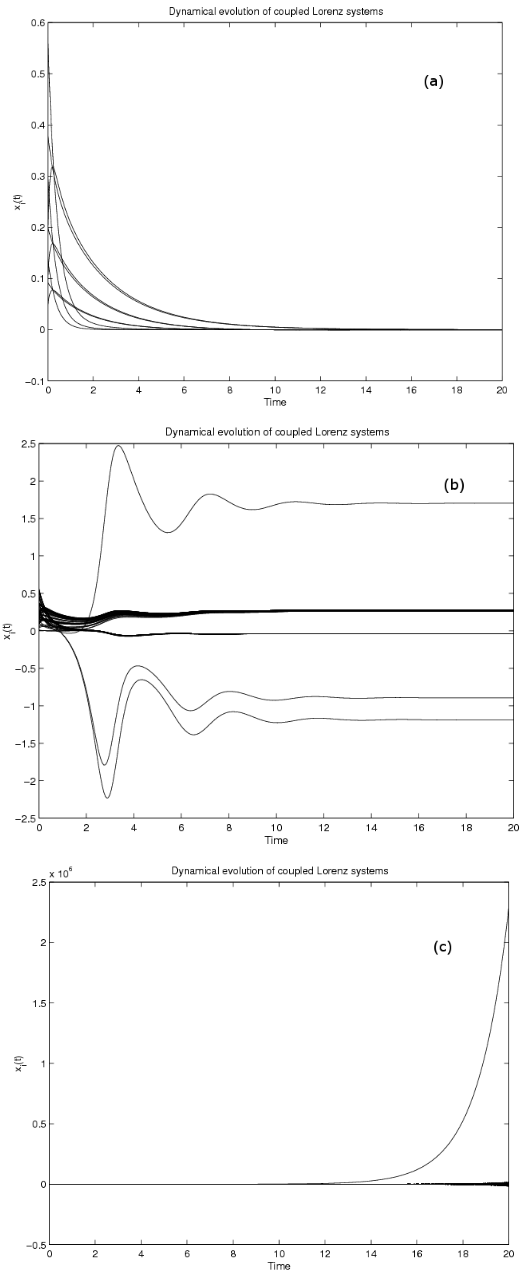

Fig. 2. The time evolution of (a) three, (b) twenty-one and (c) thirty-five Lorenz nodes in a start coupled network with a uniform coupling strength $c = 0.1$

remain bounded, and the network presents complex behavior (see Figure 1(b)). As more nodes are added to the network, the critical value $\frac{|h_\tau|}{c}$ is reached. At this point, the positive tLe dominate the overall sum exponents and condition (22) is no longer satisfied. Then, some trajectories become unbounded and the network becomes unstable (see Figure 1(c)).

The experiment was repeated for a network of the Lorenz systems with a star shaped coupling configuration. Again, from (21) it can be established that for a network with three nodes and a coupling strength of $c = 0.1$, every node has the same stable fixed-point behavior (see Figure 2(a)). As the network growths, the lower limit of (21) decreases until a positive tLe is generated. If the overall sum of tLe remains non positive, the network's trajectories remain bounded (see Figure 2(b)). However, as more nodes are added, the condition on (22) is no longer be satisfied, at that point the network becomes unstable (see Figure 2(c)).

The results shown by Figures 1 and 2 suggest that for networks with homogeneous connectivity, there is a window of values for the network size, such that complex behavior can be observed. While for networks with a heterogeneous connectivity, the network transits directly from a stable to an unstable state.

5. COMMENTS AND CONCLUSION

In this paper, two main contributions are presented: (1) the analytical derivation of the relationship of the local Lyapunov exponents and those of the entire network as the network growths. This analytical relation between the emergence of complex behavior and the structural characteristics of the network serves as an analytical result that validates the long held intuition that as the network size growths unbounded behavior is to be expected [Motter et al. (2005)]. and (2) we show that the existence of an interval of coupling strength values for the transition into complex behavior in networks with homogeneous connectivity, which becomes smaller as the network growths. While, in networks with heterogeneous connectivity, the network transits directly from regular to unstable behavior. These results are illustrated with numerical simulations. Here our contribution is to give an analytical justification for the known fragility of the stability of networks with heterogeneous connectivity [Zhou and Kurths (2006)]. We give sufficient conditions for the emergence of complex behavior, expressing these conditions in terms of coupling strength and network size. We also show that, in general, as the network growths, there is a critical network size after which the network's trajectories become unbounded. The coupling configuration of the network is a crucial characteristic in the emergence of complex behavior. From the results presented in this contribution, it can be concluded that for networks where the connectivity is homogeneous, there is an interval of values for the coupling strength and network size, such that positive *tLe* can be generated while its overall sum remains non positive. In this case, complex behavior can be observed. However, for networks with heterogeneous connectivity, the *tLe* corresponding to the hub nodes become positive long before the remaining *tLe*. As a consequence, the trajectories along these directions become unbounded and the network transit directly from a stable to an unstable state.

REFERENCES

Barabási, A. L., Albert, R. "Emergence of scaling in random networks," Science, 286, 509-512, 1999

Barahona, M., Pecora, L. M. "Synchronization in small-world systems," Phys Rev E, 89(5), 054101, 2002

Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., Hwang, D. U. "Complex Networks: Structure and Dynamics," Phys Rep,424, 175-308, 2006

Chen, Y., Rangarajan, G., Ding, M. "General stability analysis of synchronized dynamics in coupled systems," Physc Rev E, 67, 026209, 2003

Chua, L. O. "*CNN: A paradigm for complexity,*" World Scientific, Singapure, 1998

Ding, M., Yang, W. "Stability of synchronous chaos and on-off intermittency in coupled map lattices," Phys Rev E, 56(4), 4009-4016, 1997

Horn, R. A., Johnson, C. R.,"*Matrix Analysis*,", Cambridge University Press, Cambridge, 1985

Kuramoto, Y. "Self-entrainment of a population of coupled nonlinear oscillators," on *Int. Symp. on Mathematical problems in theoretical physics,* Ed. H. Araki, Springer, New York, Vol.39 of Lect. N. Phys, 420-422, 1975

Li, X., Chen, G., Ko, K. T. "Transition to chaos in complex dynamical networks," Physica A, 338, 367-378, 2004

Li, X. "Sync in complex dynamical networks: Stability, evolution, control and application," Int. J. Comp. Cognition, 3(4), 16-26, 2005

Li, X., Chen, G. "Transition from regularity to Li-Yorke chaos in coupled logistic networks," Phys Lett A, 338, 472-478, 2005

Lü, J., Yu, X., Chen, G. "Chaos synchronization of general complex dynamical networks," Phys A, 334, 281-302, 2004

Motter, A. E., Zhou, C. S., Kurths, J. "Enhancing complex-network synchronization," Europhys. Lett., 69(3), 334-340, 2005

Newman, M. E. J. "The structure and function of complex networks," SIAM reviews, 45(2), 167-256, 2003

Newman, M., Barabasi, A. L., Watts D. J. *The Structure and Dynamics of Networks,* Princeton University Press, New Jersey, USA, 2006

Ott, E. *Chaos in Dynamical Systems,* Cambridge University Press, Cambridge, Great Britain, 1993

Rangarajan, G., Ding, M. "Stability of synchronized chaos in coupled dynamical systems," Phys Lett A, 296, 204-209, 2002

Strogatz, S. H. "Exploring Complex Networks," Nature, 410, 268-276, 2001

Wang, X. F., Chen, G. "Complex Networks: Small-world, scale-free and beyond," IEEE Control Syst Maggazine, 6-20, 2003

Watts, D. J., Strogatz, S. H. "Collective dynamics of 'small world' networks," Nature, 393, 440-442, 1998.

Wu, C. W., Chua, L. O. "Synchronization in an array of linearly coupled dynamical systems," IEEE Trans. Circ. Syst. I., 42(8), 430-447, 1995

Zhang, H.-F., Wu, R.-X., Fu, X.-C. "The emergence of chaos in complex dynamical networks," Chaos, Solitons and Fractals, 28, 472-479, 2006

Zhou, C., Kurths, J. "Dynamical weights and enhanced synchronization in adaptive complex networks," Phys Rev E, 96, 164102, 2006