IFAC

A controller for swinging-up and stabilizing the inverted pendulum

J. Aracil, * J.Á. Acosta * and F. Gordillo *

* Dept. de Ingeniería de Sistemas y Automática, Escuela Técnica
Superior de Ingenieros, Universidad de Sevilla, Camino de los
Descubrimientos s/n, 41092, Sevilla, Spain
(e-mail: {aracil,jaar,gordillo}@esi.us.es)

Abstract: The hybrid solution to the pendulum swinging-up and stabilizing problem introduced by Åström and Furuta is based in two steps: an energy injection and a linear stabilization around the desired inverted position. However the energy injection stage only considers the pendulum, and not the motion of the pivot. Furthermore, for the stabilization stage linear law, only a very small basin of attraction can be guaranteed. In this paper the energy controller is enlarged to cope with the pivot dynamics and a nonlinear controller is introduced for the stabilization stage with a larger basin of attraction. The approach proposed allows to cope both with the pendulum on a cart and the Furuta one. Experiments with a laboratory Furuta pendulum are included.

Keywords: Hamiltonian systems; pendular systems; swing–up.

1. INTRODUCTION

As it is well known, the inverted pendulum presents two main problems: swinging up the pendulum to the upright position Åström and K. Furuta [2000], Gordillo et al [2003], Lozano et al [2000] and stabilizing it in this position once it is approached. These problems have traditionally been treated as two separate ones, and solved with a switching controller which commutes between two stages, see Zhao and Spong [2001]: energy injection and stabilization. The contributions of this paper are related to both stages of the problem.

With respect to the swing-up controller, it is well known that the solution of Åström and Furuta can be interpreted as an application of Fradkov's speed-gradient method, see Fradkov and Pogromsky [1998], Shiriaev et al [2000, 2001]. In the current paper this last method is used to cope not only with the pendulum, but also with the pivot (cart or arm). In this way both the pendulum on a cart and the Furuta pendulum are treated. However we will be more concerned with the Furuta case, which is more complex. The approach here is close to the one of Acosta et al [2001], Gordillo et al [2003] but simpler. The simplicity comes from the fact that here a model of the pendulum after partial linearization is used. In this paper we are only concerned with stopping the pivot, and not with stopping it at some prescribed position.

The goal of the controller is to stabilize the pendulum at its open-loop unstable upright equilibrium. The method for designing the controller proposed here is based on the stabilization of a set containing the desired equilibrium. This set is obtained from the system invariants for the free or unforced system $(u = 0)$. One of these invariants

is the energy, so the method aims to stabilize a specified energy level. The other one is a trivial one, as we shall see later.

As pointed out above this approach can be applied both to the pendulum on a cart and to the Furuta pendulum. The main differences between the pendulum on a cart and the Furuta one are 1) that the former may have problems with the restricted cart track length and 2) the dynamics are much more complicated in the latter due the rotating forces.

With respect to the stabilization problem, in this paper the usual LQR controller is not used. The reason for this is the following. In order to implement the switching strategy it is necessary to have an estimation of the domain of attraction (DOA) of the desired point when the stabilizing controller is used. The authors have noticed that the use of the LQR cost function as Lyapunov funtion yields extremely small estimations of the DOA, at least for our particular implementation of the Furuta pendulum. If this estimation is used for the switching strategy then the controller would not work properly, since even sensor noise would make the controller to continuously switch to the swing-up law.

In this paper a new, nonlinear, local stabilizing controller is designed using forwarding for both, the pendulum on a cart and the Furuta pendulum. The corresponding DOA is estimated yielding practical estimations a the DOA (at least for our pendulum).

Experimental results are also presented showing the good behavior of the resultant hybrid controller.

The paper is organized as follows. In the next Section the energy injection to swing up the pendulum proposed by Åström and Furuta is extended to the case where the pivot of the pendulum is also considered. Then, in Section 3

a switching controller is presented where the global law found in Section 2 is complemented with a local nonlinear controller which gives a wider domain of attraction than the usual linear one. Section 4 deals with the stability analysis and the estimation of the DOA for the nonlinear local controller. A Section with experimental work on a laboratory Furuta pendulum is also included.

2. GLOBAL CONTROLLER DESIGN

The model of the pendulum on a cart after partial linearization Spong [1998] is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha \sin x_1 - \beta \cos x_1 u \\ \dot{x}_3 &= u, \end{aligned} \tag{1}$$

where x_1 is the angular position of the pendulum with the origin at the upright position, and x_2 and x_3 are the velocities of the pendulum and the cart respectively. Parameters α and β include all the physical parameters of the system.

The energy of the simple unforced pendulum (with $u = 0$), disregarding the pivot, is given by

$$E = \alpha(\cos x_1 - 1) + \frac{x_2^2}{2}. \tag{2}$$

It should be realized that the energy (2) is an invariant or constant quantity for the unforced system (1).

To swing up the pendulum from any position, including the hanging one, energy should be injected to the system. Åström and K. Furuta [2000] have proposed a controller for this energy injection that can be interpreted using the speed-gradient Fradkov method Fradkov and Pogromsky [1998], taking as objective function $Q = (E - E^*)^2/2$, where E^* is the system energy at the desired equilibrium point; that is, $E^* = 0$. Then the controller

$$u = kE x_2 \cos x_1 \tag{3}$$

is obtained. This is the classical solution of Åström and K. Furuta [2000] to the swing-up problem. This controller leads to the stabilization around a homoclinic orbit. In this way, the system will eventually approach the desired equilibrium but without achieving local stability on it since, due to small disturbances, the system will go away from this point. Then it oscillates in a homoclinic orbit. One explanation of this behavior is that the desired position is a saddle point and the (attractive) homoclinic orbit is its stable manifold.

This controller has not only the problem of leading to oscillations, but also the one associated with the fact that it does not cope with the cart. Thus, the controller only deals with the pendulum that goes to the upright position, but the cart does not stop and has a remanent drift. In the case Åström and K. Furuta [2000] the cart is stopped in the stabilization stage, but not in the energy injection one.

However the cart can be driven to zero velocity in the energy stage if we take into account that there is another trivial invariant of the unforced system given by x_3 since, for $u = 0$, $\dot{x}_3 = 0$. Therefore the speed-gradient objective function becomes

$$Q = k_1 \frac{E^2}{2} + k_2 \frac{x_3^2}{2} \tag{4}$$

which leads to the control law

$$u = -g^T \nabla_x Q = k_1 E \beta x_2 \cos x_1 - k_2 x_3. \tag{5}$$

It can be checked by simulation the improvement of the system behavior if controller (5) instead of (3) is used. The cart tends now to rest in the energy injection stage. Therefore, it improves the classical Åström-Furuta law.

This approach can be applied to the Furuta pendulum in a similar way to Gordillo et al [2003], Acosta et al [2001]. Here we reproduce the controller derived there in lagrangian coordinates. Thus, after partial linearization Spong [1998] the normalized system model for the Furuta pendulum is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha \sin x_1 + \sin x_1 \cos x_1 x_3^2 - \beta \cos x_1 u \\ \dot{x}_3 &= u, \end{aligned} \tag{6}$$

Notice that the natural energy of the pendulum is no longer an invariant of (6). However, here the corresponding "energy" of (6) is given by the hamiltonian

$$H_1 = \frac{x_2^2}{2} + \alpha(\cos x_1 - 1) + \frac{\cos 2x_1}{4} x_3^2, \tag{7}$$

because we have $\dot{H}_1 = 0$ for the unforced system of (6). The other invariant is again the trivial one x_3.

Now, following the speed–gradient algorithm, thoroughly described in Gordillo et al [2003] for this case, we can define the Fradkov objective function

$$Q = k_1 \frac{(H_1 - H_1^*)^2}{2} + k_2 \frac{x_3^2}{2}. \tag{8}$$

for the Furuta system. Notice that in the desired inverted position $(x_1, x_2, x_3) = (0, 0, 0)$ and hence $H_1^* = 0$.

The controller according the speed-gradient Fradkov method yields

$$\begin{aligned} u &= -g^T \nabla_x Q \\ &= k_1 H_1 \left(\beta x_2 \cos x_1 - \frac{\cos 2x_1}{2} x_3 \right) - k_2 x_3. \end{aligned} \tag{9}$$

Another possibility is

$$u = -\phi(g^T \nabla_x Q) = \phi(y), \tag{10}$$

where ϕ is such that $y\phi(y) \le 0$ and

$$y = k_1 H_1 \left(-\beta x_2 \cos x_1 + \frac{\cos 2x_1}{4} x_3 \right) + k_2 x_3. \tag{11}$$

One advantage of (10) is that it permits to take into account the saturation effects of the controller actuator.

These results can be reformulated applying passivity. The system (6) with output (11) is a passive system with storage function Q, given by (8).

With controllers (9) or (10) it is clear that $Q \to 0$, and therefore $H_1 \to H_1^*$ and $x_3 \to 0$; that is, the trajectory tends to the homocline defined by $H_1 = H_1^*$ and $x_3 = 0$. Hence, the system evolves towards a stable set containing the desired equilibria; which, in this case, is an invariant set of the unforced ($u = 0$) system. This means that the system oscillates passing in every oscillation close to the origin in the cylindrical state space where the system (6) is defined. This fact is used in the next Section to capture the trajectory near the origin and to switch the controller to a nonlinear one that drives the system to the desired upright position.

3. LOCAL CONTROLLER DESIGN

Usually, the stabilization problem is solved using linear methods. However the resultant hybrid law even if it works well in experimental settings, has a big theoretical problem regarding the domain of attraction of the linear part, which can be very small and therefore the robustness of the law can be only hardly guaranteed. To enlarge the domain of attraction, in this paper a nonlinear law for the local controller is proposed.

3.1 A motivational example

The nonlinear local controller proposed here is based on a variant of forwarding Mazenc and Praly [1996]. In this section a motivational example of this approach is included. To that end consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_1 x_2^2 - u \\ \dot{x}_2 &= u. \end{aligned} \tag{12}$$

To put this system in cascade form suitable for applying forwarding the following precontroller is applyed

$$u = 2x_1 + x_1 x_2^2 + v,$$

and then the system becomes

$$\begin{aligned} \dot{x}_1 &= -x_1 - v \\ \dot{x}_2 &= x_1(2 + x_2^2) + v. \end{aligned} \tag{13}$$

The upper unforced subsystem has a Lyapunov function $V_1 = \frac{x_1^2}{2}$. To find an invariant of the full unforced system the following PDE has to be solved

$$-x_1 \frac{\partial \nu}{\partial x_1} + x_1(2 + x_2^2) \frac{\partial \nu}{\partial x_2} = 0, \tag{14}$$

which gives the invariant

$$\nu = x_1 + \frac{1}{\sqrt{2}} \arctan \frac{x_2}{\sqrt{2}}. \tag{15}$$

Then a Lyapunov function for the system (13) is

$$V = \frac{x_1^2}{2} + k \frac{\nu^2}{2} \tag{16}$$

and therefore

$$\dot{V} = -x_1^2 - \left(x_1 - k\nu \left(-1 + \frac{1}{2 + x_2^2} \right) \right) v,$$

hence the controller is given by

$$v = x_1 - k\nu \left(-1 + \frac{1}{2 + x_2^2} \right). \tag{17}$$

3.2 Pendulum-on-a-cart case

The model of the pendulum on a cart system after partial linearization Spong [1998] is given by (1).

First of all, using the ideas of Åström et al [2005], it is desired to transform the first two equations of (1) in a Hamiltonian system with energy

$$H_2 = \frac{x_2^2}{2} + \alpha(\cos x_1 - a\cos^2 x_1), \tag{18}$$

with $a > 0.5$. This is achieved with the energy-shaping controller $u = \frac{2a\alpha}{\beta} \sin x_1 + v$ yielding

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha \sin x_1 - 2a\alpha \sin x_1 \cos x_1 - \beta \cos x_1 v \\ \dot{x}_3 &= \frac{2a\alpha}{\beta} \sin x_1 + v. \end{aligned} \tag{19}$$

An invariant ν for system (19) is given by the PDE

$$x_2 \frac{\partial \nu}{\partial x_1} + \alpha \sin x_1 (1 - 2a\cos x_1) \frac{\partial \nu}{\partial x_2} + \frac{2a\alpha}{\beta} \sin x_1 \frac{\partial \nu}{\partial x_3} = 0, \tag{20}$$

which has the following solution

$$\nu = x_3 + \sqrt{2a\alpha} \ln \left(-\frac{\alpha}{\sqrt{2}} + \sqrt{2a\alpha} \cos x_1 + \sqrt{a\alpha} x_2 \right) - \frac{\sqrt{2a\alpha}}{2\beta} \ln(a\alpha). \tag{21}$$

This solution can only exist in region Ω defined by

$$\Omega \triangleq \{ (x_1, x_2) \in \mathcal{S} \times \mathbb{R} : \alpha(2a\cos x_1 - 1) + x_2\sqrt{2\alpha a} > 0 \} \tag{22}$$

In region Ω we can define the Lyapunov function

$$V = \frac{x_2^2}{2} + \alpha \left(\cos x_1 - a\cos^2 x_1 \right) + \frac{k_1}{2} (\nu - \nu_0)^2. \tag{23}$$

Then

$$\dot{V} = \left(-\beta x_2 \cos x_1 + k_1(\nu - \nu_0) \left(-\beta \cos x_1 \frac{\partial \nu}{\partial x_2} + \frac{\partial \nu}{\partial x_3} \right) \right) v,$$

which leads to the controller

$$v = -k_2 \phi_1 \left(-x_2 \beta \cos x_1 + k_1(\nu - \nu_0) \left(-\beta \cos x_1 \frac{\partial \nu}{\partial x_2} + \frac{\partial \nu}{\partial x_3} \right) \right), \tag{24}$$

where ϕ_1 satisfies $z\phi_1(z) \geq 0$. In this way, $\dot{V} \leq 0$.

In order to define the switching strategy, an estimation for the origin domain of attraction (DOA) is needed. This estimation can be obtained looking for the largest closed level surface of V that is included in Ω. For brevity, this step is omitted here but a similar and more complex case will be discussed below for the Furuta pendulum.

Now we can build a switching controller. When the state of the system is outside the estimation of the DOA the nonlinear global law (9) is applied until the DOA is reached. Then the controller is switched to law (24). This switch occurs only once, unlike in sliding controllers.

3.3 Furuta case

The system model, after partial linearization Spong [1998], is given by (6). After applying the energy-shaping controller

$$u = \frac{\sin x_1}{\beta} (2a\alpha + x_3^2) + v, \tag{25}$$

we obtain

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha(\sin x_1 - 2a\sin x_1 \cos x_1) - \beta \cos x_1 v \\ \dot{x}_3 &= \frac{\sin x_1}{\beta} (2a\alpha + x_3^2) + v, \end{aligned} \tag{26}$$

where the two first equations are as (19) and therefore they give rise to a Hamiltonian system with H_2 given by (18).

An invariant ν for system (26) is given by a solution for the partial differential equation

$$x_2 \frac{\partial \nu}{\partial x_1} + \alpha \sin x_1 (1 - 2a\cos x_1) \frac{\partial \nu}{\partial x_2} + \frac{\sin x_1}{\beta} (2a\alpha + x_3^2) \frac{\partial \nu}{\partial x_3} = 0. \tag{27}$$

One such a solution is

$$\nu = 2 \ln \left(-\frac{\alpha}{\sqrt{2}} + \sqrt{2a\alpha} \cos x_1 + x_2\sqrt{a\alpha} \right)$$
$$+ 2\beta \arctan \left(\frac{x_3}{\sqrt{2a\alpha}} \right) - \ln(a\alpha) + \nu_0, \tag{28}$$

with the constant ν_0 such that $\nu(0,0,0) = 0$ and so

$$\nu_0 = 2\ln\left(\alpha\left(-\frac{1}{\sqrt{2}} + \sqrt{2}a\right)\right) - \ln(a\alpha).$$

The invariant ν exists in the region Ω defined by (22). Actually we are only concerned with a subregion of Ω defined around the origin. The controller is now

$$v = -\phi\left(-\beta x_2 \cos x_1 + b\nu\left(-\beta \cos x_1 \frac{\partial\nu}{\partial x_2} + \frac{\partial\nu}{\partial x_3}\right)\right), \tag{29}$$

where ϕ satisfies $z\phi(z) \geq 0$. The corresponding Lyapunov function is

$$W = H_2 + \frac{1}{2}b\nu^2, \tag{30}$$

with b some positive constant and H_2 given by (18).

4. STABILITY ANALYSIS

In this Section only the Furuta pendulum is treated with some detail. The pendulum on cart case is a simpler version of the Furuta case, and can be solved in the same way as this last.

Proposition 1. Consider the system (6) with the composite controller given by (25)–(29) with the constants $a > 1/2$ and $b > 0$. Then there exits a value $c_{max} > 0$ such that the sub–level sets $W = c$ of (30), for all $0 < c \leq c_{max}$ are compact. Moreover, all the trajectories starting in $W = c$ are bounded.

Proof. After a geometric study of function W, which is omitted here for brevity, it can be seen that the level curves for W are compact for $0 < c \leq c_{max}$ with $c_{max} = \min_{x_2} \varphi(x_2)$, where function $\varphi(x_2)$ is given by

$$\varphi(x_2) = \frac{1}{2}x_2{}^2 + \frac{b}{2}\left(2\ln\left(\frac{\alpha(2a-1) + x_2\sqrt{2\alpha a}}{\alpha(2a-1)}\right) + \pi\beta\sqrt{\alpha a}\right)^2.$$

To prove the boundedness of trajectories, since the sub–level sets $W \leq c_{max}$ are compact then we use the positive definite function W as a candidate to Lyapunov function. Thus, the derivative of W along the system trajectories reads

$$\dot{W} = \dot{H}_2 + b\nu\dot{\nu}$$
$$= -\left(-\beta x_2 \cos x_1 + b\nu\left(-\beta \cos x_1 \frac{\partial\nu}{\partial x_2} + \frac{\partial\nu}{\partial x_3}\right)\right)^2$$
$$\leq 0, \tag{31}$$

and therefore all trajectories starting in $W \leq c_{max}$ are bounded.

Proposition 2. Consider the proposition 1 taking effect, then for all the trajectories starting in $W = c$ with $c \leq c_{max}$, the zero equilibrium is asymptotically stable.

Proof. By proposition 1 we know that all trajectories are bounded in any sub–level set $W = c$ with $c \leq c_{max}$. From (31) we know that $\dot{W}$ is semi–definite negative then we use LaSalle's Invariance Principle to prove that the largest sub–level set inside $W = c$, with $c \leq c_{max}$, is the zero equilibrium. Thus, from (26) and forcing $W = c$ the residual dynamics become

$$\dot{x}_1 = x_2 \tag{32}$$
$$\dot{x}_2 = -\alpha \sin x_1(2a\cos x_1 - 1) \tag{33}$$
$$\dot{x}_3 = \frac{\sin x_1}{\beta}(2a\alpha + x_3{}^2), \tag{34}$$

subject to the constraint $W \equiv 0$ given by

$$F \triangleq \left(-\beta x_2 \cos x_1 + b\nu\left(-\beta \cos x_1 \frac{\partial\nu}{\partial x_2} + \frac{\partial\nu}{\partial x_3}\right)\right) \equiv 0. \tag{35}$$

Now, the equations (32)–(33) are the corresponding to the Hamiltonian conservative two-dimensional system, and then in the residual dynamics we know additionally that H_2 given by (18) is constant too, which means that

$$\dot{H}_2 = \frac{\partial H_2}{\partial x_1}(x_1, x_2)x_2 - \frac{\partial H_2}{\partial x_2}(x_1, x_2)V'(x_1) \equiv 0. \tag{36}$$

Thus, the fact that H_2 is constant together with the definition of W imply that ν is also constant in these residual dynamics, say $\nu = \nu^0$. Now, the time derivative of F becomes

$$\dot{F} = \frac{\partial F}{\partial x_1}(x_1, x_2, \nu^0)x_2 - \frac{\partial F}{\partial x_2}(x_1, x_2, \nu^0)V'(x_1) \equiv 0. \tag{37}$$

It is easy to see that functions H_2, ν and F are linearly independent. Thus we have three independent invariants on a third-order dynamics and therefore, the residual trajectory is a fixed point. From the residual dynamics (32)–(34) the only fixed points have the form $(x_1, x_2, x_3) = (0, 0, x_3^0)$. Now, the value of ν in these points reads

$$\nu^0 \triangleq \nu(0, 0, x_3^0) = 2\beta \arctan\left(\frac{x_3^0}{\sqrt{2\alpha a}}\right),$$

and using the fact that the change of coordinates $(x_1, x_2, x_3) \rightsquigarrow (x_1, x_2, \nu)$ is a (local) diffeomorphism in the compact set $W \leq c$ then, it only remains to prove that either ν^0 or x_3^0 are zero. Further, evaluating (35) in these fixed points becomes

$$\nu^0\left((2a-1)\cos^2\left(\frac{\nu^0}{2\beta}\right) - 2a\right)cst = 0,$$

which has as unique solution $\nu^0 = 0$ for any positive a, concluding the proof.

5. EXPERIMENTS ON A LABORATORY FURUTA PENDULUM

The implementation of the Furuta pendulum used in this article is thoroughly described in Acosta et al [2001], Gordillo et al [2003]. The laboratory electro-mechanical system consists of: a DC motor (15 Nm / 2000 rpm) with tachometer that measures the speed of the arm; a power supply (50 VA); a PWM servo-amplifier; a pendulum; an encoder that measures the angle of the pendulum and a slip ring that drives the signal to the base. The control system is composed by: a monitor PC with a target (DS1102) for control based on DSP (TMS320C31)and a software (DSPACE) for control, monitoring and supervisor. The friction in the actuated coordinate was compensated with a non–linear compensator based on the LuGre model Canudas et al [1995], to dominate the friction forces of the arm of the pendulum. The full control system is shown in Fig. 1. The physical values were $\alpha = 52.55$ and $\beta = 1.26$ and, the controller parameters for the experiment $a = 1$ and $b = 1$. With these parameters the bound $c_{max} \approx 6.4$ can be obtained solving the minimization problem presented in the proof of Proposition. For the experiment we choose $c = 6 < c_{max}$. The successful experimental results are shown in Fig. 2. Figure 2 is split into two parts: the left one is the response with the global controller; and the
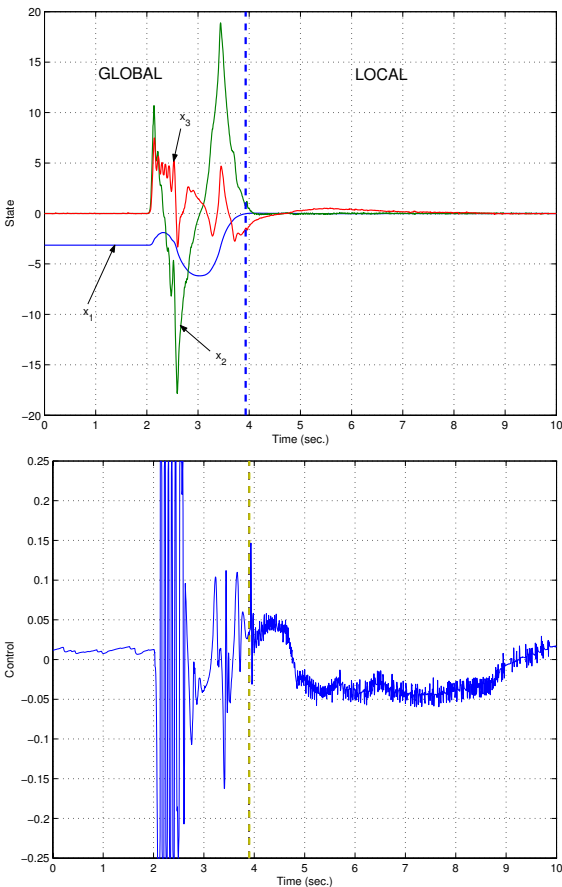
Fig. 1. Laboratory Furuta pendulum.

Fig. 2. Experimental results.

right one the response with the local controller (26)–(29). The switching algorithm is:

$$u = \begin{cases} u_{local} & \text{if } W < c \\ u_{global} & \text{if } W \geq c \end{cases}$$

In order to compare this estimate of the DOA and the one obtained when an LQR is used, we compute an LQR

controller with the following parameters [1]

$$Q = 10^3 \text{ diag}\{8808, 23.45, 0.281\} \quad \text{and} \quad R = 1.$$

By numerical inspection, the maximum level curve $V_{LQR} = 1/2 x^\top P x$ for which $\dot{V} \leq 0$ is $c_{max} \approx 4.43$, and the value of V_{LQR} at the time of switching [2] is equal to 794.12, meaning that the system is far from entering in the estimated DOA for the LQR controller. On the other hand, at this time, the system is entering in the estimated DOA for the presented controller, because $W \approx 5.93$. This is a sign that the presented estimate of the DOA is much larger than the one of the LQR controller. Notice also that, the control action saturates when the global controller is being applied and, this fact does not affect to the stability result because speed–gradient controllers are independent of the saturation limit. Otherwise, when the local controller is acting the control action does not saturate.

REFERENCES

J.Á. Acosta, J. Aracil and F. Gordillo, On swinging the Furuta pendulum up based on Routh's reduction and SG method In *Proceedings of the 9th Mediterranean Conference on Control and Automation*, 2001.

K.J. Åström, J. Aracil and F. Gordillo, A new family of smooth strategies for swinging up a pendulum In *16th IFAC World Congress. Prague*, 2005.

K. J. Åström and K. Furuta, Swinging up a Pendulum by Energy Control *Automatica*, Vol. 36, pp. 287–295, 2000.

C. Canudas de Wit, H. Olsson, K. J. Åström, and P. Lischinsky, A new model for control of systems with friction. *IEEE Transactions on Automatic Control*, Vol. 40, pp. 419–425, 1995.

A.L. Fradkov and A.Y. Pogromsky, *Introduction to control of oscillations and chaos.* World Scientific River Edge (1998).

F. Gordillo, J.Á. Acosta, and J. Aracil, A new swing–up law for the Furuta pendulum *International Journal of Control*, Vol. 76, No. 8, pp. 836–844, 2003.

R. Lozano, I. Fantoni, and D.J. Block, Stabilization of the inverted pendulum around its homoclinic orbit *Systems & Control Letters*, Vol. 40, No. 3, pp. 197–204, 2000.

F.Mazenc and L.Praly, Adding integrations, saturated controls and global asymptotic stabilization for feedforward systems *IEEE Transactions on Automatic Control*, Vol. 41, No. 11, 1996.

A.S. Shiriaev, O. Egeland, H. Ludvigsen, and A.L. Fradkov, VSS version of energy based control for swinging up a pendulum Systems & Control Letters, Vol. 44, No. 1, pp. 41–56,2001.

A. Shiriaev, A. Progromsky, H. Lugvigsen, and O. Egeland, On global properties of pasivity-based control of an inverted pendulum International Journal of Robust and Nonlinear Control, Vol. 10, No. 4, pp. 283–300, 2000.

M. W. Spong, Underactuated mechanical systems. In B. Siciliano and K. Valavanis, editors, *Control Problems in Robotics and Automation*, Vol. 230, Springer-Verlag, 1998.

J. Zhao and M. W. Spong, Hybrid control for global stabilization of the cart–pendulum system *Automatica*, Vol. 37, pp. 1941–1951, 2001.

---

[1] Similar results can be obtained with other parameters values.

[2] Dashed vertical line across the entire Fig. 2.