Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Combinatorial Vector Fields for Piecewise Affine Control Systems

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Abstract: This paper is intended to be a continuation of Habets and van Schuppen [2004] and Habets et al. [2006], which address the control problem for piecewise-affine systems on an arbitrary polytope or a family of these. Our work deals with the underlying combinatorics of the underlying discrete system. Motivated by Forman [1998], on the triangulated state space we define a combinatorial vector field, which indicates for a given face the future simplex. In the suggested definition we allow nondeterminacy in form of splitting and merging of solution trajectories. The combinatorial vector field gives rise to combinatorial counterparts of the flow lines. The main result is then an algorithm for synthesis of supervisory control.

1. INTRODUCTION

Although much advanced non-linear control theory has been developed for almost every kind of system, it is still desirable to make the process of design automatic. Previous examples of such efforts has been reported in Tabuada and Pappas [2003], where an approach based on temporal logics has been suggested. It, however, requires the system in question to be converted into Brunovsky canonical form. It furthermore has the shortcoming, that it only allows for the system trajectory to evolve on the highest dimension.

This paper deals with combinatorial formulation for piecewise-affine control systems. It is thought as a carry-over of Belta et al. [2002], Habets and van Schuppen [2004] and Habets et al. [2006], which address the control problem for piecewise-affine systems on an arbitrary polytope that forces the solution trajectories of the closed loop system to either leave it or stay in it for ever. This paper merely considers the underlying discrete system. Interest of the control community in systems defined on simplicial objects has been initiated by Sontag [1982, 1981]. Reachability and controllability on such systems has been studied before in Asarin et al. [2000], Bemporad et al. [2000]. Whereas previous methods have been based on the concept of a transition system, this paper focuses on its higher dimensional generalization, a simplicial complex.

In the paper, on the triangulated state space - a simplicial complex - we define a combinatorial vector field, a concept borrowed form algebraic topology, c.f Forman [1998]. The main motivation for doing this is that combinatorial vector field allows flow on the simplices of all dimensions. In particular one can study flow lines of simplices of the maximal dimension but also study traces of their faces, which is particularly important for obstacle avoidance control problem. The main result of the paper is formulation of the nondeterministic combinatorial vector field. We devise a notion of a combinatorial flow map - an explicit function which takes on a simplex and delivers its possible future simplices. Useability of the method is shown through an algorithm for synthesis of a supervisor.

Firstly, we define simplicial complexes in Section 2 and combinatorial manifolds in Section 3. A simplicial complex is a triangulation of the manifold in question preserving the original manifolds topology. We associate a piecewise-affine control system to each simplex, e.g. a point in 0-dimension, a line segment in dimension 1, a triangle in dimension 2, a tetrahedron in dimension 3. This comprises a definition of a combinatorial control system in Section 3. The control action on each simplex makes the combinatorial flow defined in Section 4 either transversal or tangential to faces of the simplex in question. Finally, when all the possible control actions have been found the supervisory control strategy in Section 5 is capable of selecting the shortest path through the combinatorial manifold and guarantees that this path is followed.

We use the following notation: $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of natural numbers, $\mathbb{Z}_+ = \{n \in \mathbb{Z}|\ n \geq 0\}$.

2. SIMPLICIAL COMPLEXES

A simplex is a convex hull of its vertices, they being independent points in a Euclidean space $\mathbb{R}^N$. A simplicial complex is a quotient space of collection of simplices obtained by identifying certain of their faces. The aim of this section is to make the above statement formal. More detailed information on simplicial complexes can be found e.g. in May [1992]. Having in mind particular application to piecewise-affine control and hybrid systems, the attention in the following exposition has been limited to polyhedra, i.e. subsets of $\mathbb{R}^N$ which can be expressed as a locally finite union of simplices.

Definition 1. (Definition IV.1.1 in Bredon [1997]). *Let* $\mathbb{R}^n$ *have standard basis* $e_0, ..., e_n$. *Then* ***standard*** n-***simplex*** *is*

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10.3182/20080706-5-KR-1001.3088

$$\triangle_n \equiv \left\{ x = \sum_{i=0}^{n} \lambda_i e_i \;\middle|\; \sum_{i=0}^{n} \lambda_i = 1,\ 0 \le \lambda_i \le 1 \right\}.$$

The λ_i are called **barycentric coordinates**.

Definition 2. (Definition IV.1.2 in Bredon [1997]). *Given $n \le N$ independent points $v_0, ..., v_n \in \mathbb{R}^N$, $[v_0, ..., v_n]$ is an affine map $\triangle_n \to \mathbb{R}^N$ defined by*

$$\sum_i \lambda_i e_i \mapsto \sum_i \lambda_i v_i.$$

We shall call $[v_0, ..., v_n]$ an ***affine*** *n***-simplex** *or just a simplex.*

The image of $[v_0, ..., v_n]$ is the convex span of the points v_i. We shall often identify an (affine) simplex with its image

$$\mathrm{Image}[v_0, ..., v_n] = \left\{ \sum_{i=0}^{n} \lambda_i v_i \;\middle|\; \sum_{i=0}^{n} \lambda_i = 1,\ 0 \le \lambda_i \le 1 \right\}.$$

The notation of putting a hat over one element of a group of similar symbols means that that one is omitted. Thus $[v_0, ..., \hat{v}_i, ..., v_n]$ denotes the $(n-1)$-simplex $[v_0, ..., v_{i-1}, v_{i+1}, ..., v_n]$.

Definition 3. (Definition IV.1.5 in Bredon [1997]). *The map $[e_0, ..., \hat{e}_i, ..., e_n] : \triangle_{n-1} \to \triangle_n$ is called the* ***face map*** *and it is denoted by F_i^n.*

We have that $F_i^n(e_j) = e_j$ for $j < i$, and $F_i^n(e_j) = e_{j+1}$ for $j \ge i$. Thus it is seen that for $i < j$

$$F_j^{n+1} \circ F_i^n = [e_0, ..., \hat{e}_i, ..., \hat{e}_j, ..., e_n] = F_i^{n+1} \circ F_{j-1}^n. \quad (1)$$

If $\sigma : \triangle_n \to \mathbb{R}^N$ is an n-simplex, then the ith face of σ is the $(n-1)$-simplex

$$\sigma^i = \sigma \circ F_i^n : \triangle_{n-1} \to \mathbb{R}^N.$$

In other words F_i^n induces the map d_i^n taking the simplex σ to σ^i. The following relation is a direct consequence of (1)

$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n \text{ for } i < j. \quad (2)$$

We introduce a relation on the set of simplices: $\gamma \prec \sigma$ if γ is a face of σ. We say that γ is a maximal face of σ if $\gamma \prec \sigma$ and $\dim \gamma + 1 = \dim \sigma$.

Definition 4. (Definition 2.1. in Lickorish [1999]). *A (finite)* ***simplicial complex*** *K is a finite collection of simplices, contained linearly in some $\mathbb{R}^N$, such that*

(1) $\sigma \in K$ and $\gamma \prec \sigma$ implies that $\gamma \in K$;
(2) $\sigma \in K$ and $\tau \in K$ implies that $\sigma \cap \tau$ is a face of σ and τ.

We denote the set of all n-simplices in a simplicial complex K by K_n,

$$K_n \equiv \{\sigma \in K |\ \sigma \text{ is an } n\text{-simplex}\}.$$

Notice that the data determining a simplicial complex is combinatorial, with no topology involved. Alternatively we may use the following purely combinatorial definition of a simplicial complex, c.f. May [1992].

Definition 5. (Definition 1.1, May [1992]). *An abstract simplicial complex (or $\triangle$-set) K is a family of sets $\{K_n |\ n \in \mathbb{Z}_+\}$ with face maps $d_i^n : K_n \to K_{n-1}$ $(0 \le i \le n)$ satisfying Eq. (2).*

In particular a directed graph is a $\triangle$-set consisting of a pair of sets $\{K_0, K_1\}$ with two face maps $d_0 \equiv d_0^1,\ d_1 \equiv d_1^1 : K_1 \to K_0$. This definition is equivalent to the standard definition of the transition system (V, E), where V is the set of vertices and $E \subset V \times V$ is the set of edges. The transition system (V, E) defines the directed graph $\{V, E\}$ with $d_i = \pi_i$, where π_0 is the projection on the first and π_1 on the second factor. Conversely, the directed graph $\{K_0, K_1\}$ with two face maps $d_i : K_1 \to K_0$, $i = 0, 1$ defines the transition system (K_0, E) with $E = \{(d_0 e, d_1 e)|\ e \in K_1\}$.

Definition 6. (Definition IV.1.5 in Bredon [1997]). *The n***-chain group** *$C_n(K)$ of the simplicial complex K is the free abelian group generated by n-simplices*

$$C_n(K) = \bigoplus_{\sigma \in K_n} \mathbb{Z},$$

which is equivalent to $\mathbb{Z}[\sigma_1] \times \mathbb{Z}[\sigma_2] \times \ldots$ where $\sigma_i \in K_n$. Thus an n-chain is a formal sum

$$c = \sum_{\sigma \in K_n} n_\sigma \sigma$$

of n-simplices σ with integer coefficients n_σ.

Each map $d_i^n : K_n \to K_{n-1}$ can be extended to $C_n(K)$ so as to be a homomorphism $d_i^n : C_n \to C_{n-1}$ by

$$d_i^n(\sum_\sigma n_\sigma \sigma) = \sum_\sigma n_\sigma d_i^n \sigma = \sum_\sigma n_\sigma \sigma^i.$$

Definition 7. *If $\sigma : \triangle_n \to \mathbb{R}^N$ is an n-simplex, then the* ***boundary map*** *is a homomorphism $\partial_n : C_n(K) \to C_{n-1}(K)$ defined by*

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^n.$$

In particular the boundary of σ is $\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma^i$. Notice that $\partial_n \circ \partial_{n+1} = 0$. An example of how to find the boundary map of a simplex, as depicted in Figure 1, using Definition 7, is as follows:

$$\partial[a, b, c] = [\hat{a}, b, c] - [a, \hat{b}, c] + [a, b, \hat{c}] = [b, c] - [a, c] + [a, b].$$

Notice the notion of direction: $[a, b] = -[b, a]$.

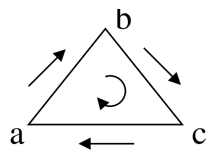


Fig. 1. Example of boundary map calculation.

3. PIECEWISE-AFFINE CONTROL SYSTEM ON COMBINATORIAL MANIFOLDS

In this section we formulate the notion of a combinatorial manifold M - a simplicial complex of particularly regular structure. We associate to each simplex of maximal dimension in M a piecewise affine control system.

Let σ be an n-simplex. We say that a control vector field $\zeta : \mathrm{Im}(\sigma) \times \mathbb{R}^m \to \mathbb{R}^n$ is piecewise-affine n-control system if ξ is defined by the piecewise-affine map

$$\zeta(x, u) = Ax + Bu + a,$$

where A is an n by n matrix, B is an n by m matrix and a is an n-vector.

Let K be a simplicial complex. We define some basic combinatorial operations on K. The star of a simplex β consists of all simplices that have β as a face

$$\text{St}\ \beta = \{\sigma \in K |\ \beta \preceq \sigma\}.$$

Star is not a complex in general, since condition 2. of Definition 4 may not be satisfied. We can make star into a complex by adding all its missing faces - the closed star. Thus the closed star $\overline{\text{St}}\ \beta$ is the smallest complex that contains St β. The link of β consists of all the simplices in $\overline{\text{St}}\ \beta$ that are disjoint from τ

$$\text{Lk}\ \beta = \{\tau \in \overline{\text{St}}\ \beta |\ \text{Image}(\tau) \cap \text{Image}(\beta) = \emptyset\}.$$

Definition 8. (Definition 2.2 in Lickorish [1999]). *A* ***combinatorial*** n***-ball*** *is a simplicial complex* B^n *piecewise linearly homeomorphic to* $\triangle^n$. *A* ***combinatorial*** n***-sphere*** *is a simplicial complex* S^n *piecewise linearly homeomorphic to* $\partial\triangle^{n+1}$. *A* ***combinatorial*** n***-manifold*** *is a simplicial complex* M *such that, for every vertex* v *of* M, Lk v *is a combinatorial* $(n-1)$*-ball or a combinatorial* $(n-1)$*-sphere.*

Definition 9. *An* ***combinatorial*** n***-control system*** *is a pair* (M, ξ), *where* $M = \{M_0, ..., M_n\}$ *is a combinatorial n-manifold, and* $\xi = \{\xi_\sigma |\ \sigma \in M_n\}$ *is a family of piecewise affine n-control systems.*

Let (M, ξ) be a combinatorial n-control system. A control objective for (M, ξ) is decomposed in Habets and van Schuppen [2004] and Habets et al. [2006] into two control problems posed for each n-simplex σ (in fact in Habets and van Schuppen [2004] the authors treat more general problem of control synthesis on a polytope):

Problem 1. (Problem 4.1 in Habets et al. [2006]). *Let* $\sigma \in M_n$. *Given a subset* S *of maximal faces of* σ *find a control law*

$$k_\sigma : \text{Im}(\sigma) \to \mathbb{R}^m,\ k_\sigma(x) = F_\sigma x + g_\sigma, \tag{3}$$

where F_σ *is an* m *by* n *matrix and* g_σ *is an* n*-vector, such that it guarantees that all flow lines of the closed-loop system*

$$\dot{x} = (A + BF)x + (a + Bg), \tag{4}$$

starting at a $p \in \text{Im}(\sigma)$ *leaves the simplex* σ *in finite time by crossing one of the faces in* S.

Problem 2. (Problem 4.2 in Habets et al. [2006]). *For a given* $\sigma \in M_n$ *find a control law (3) such that for any* $p \in \text{Im}(\sigma)$ *the flow line* $\phi_p(t)$ *of the closed-loop system (4) satisfies* $\phi_p(t) \in \text{Im}(\sigma)$ *for any* $t \geq 0$.

We say that the control law (3) **blocks** a maximal face γ of a simplex σ if the vector field ξ^c_σ of the closed loop system on σ - defined by the right hand side of equation (4) - satisfies the equality

$$\langle \xi^c_\sigma(x), n_\gamma \rangle \leq 0 \tag{5}$$

for any $x \in \text{Im}(\gamma)$, where n_γ is the outward normal vector to τ and $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^n$. Inequality (5) indicates that γ is an **exit face**.

Problems 1 and 2 are solved in Habets et al. [2006] by blocking maximal faces that are complementary to the set S. We observe that if $S' \subset S$ and the control law k_σ blocks all the faces in S then it also blocks the faces in S'; thus the more blocking faces the more restrictive control it is.

The focus in this work is on the combinatorial part of the control synthesis problem, i.e. on a supervisor that selects blocking faces of a combinatorial n-control system such that every trajectory of the closed loop system starting in an n-simplex σ_s reaches the target n-simplex σ_t in finite time. For a treatment of the necessary and sufficient conditions for guaranteeing control to a certain facet of a simplex for the PWA system the reader is referred to Habets et al. [2006] and the references therein.

4. COMBINATORIAL VECTOR FIELDS

In this section we introduce the central notion of this paper - a combinatorial vector field. The notion has been developed by R. Forman in Forman [1998] for studying topological invariants of CW complexes. The attention in this paper is restricted to geometrical properties of a combinatorial vector field. We extend the notion of a combinatorial vector field to encompass non-determinism in Definition 12. It is treated as a generator of flow. The notion of combinatorial flow lines is used in Section 5 for synthesis of supervisory control.

Definition 10. (Definition 1.2, Forman [1998]). *Let* K *be a simplicial complex. A* ***combinatorial vector field*** V *on* K *is a family* $\{V_n |\ n \in \mathbb{N}\}$ *of maps*

$$V_n : K_{n-1} \to K_n \cup \{0\}$$

that satisfies

(1) $V_n \circ V_{n-1} = 0$, *that is if* $\sigma \in \text{Image}(V_{n-1})$ *then* $V_n(\sigma) = 0$.
(2) For each $\sigma \in K_n$, *the number of elements of the preimage*
$$V_n^{-1}(\sigma) \equiv \{\alpha \in K_{n-1} |\ V_n(\alpha) = \sigma\}$$
is 0 *or* 1.

Alternatively a combinatorial vector field is a set $\bar{V}$ of pais of simplices $\langle \alpha, \sigma \rangle$, where α is a maximal face of σ, and for which no simplex is in more than one pair. It is helpful to picture a combinatorial vector field on K by arrows, where the tail is at α and the arrow at σ, see Figure 2.

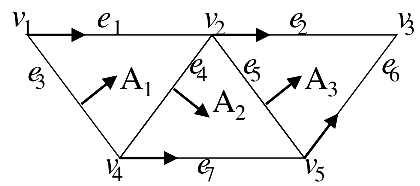
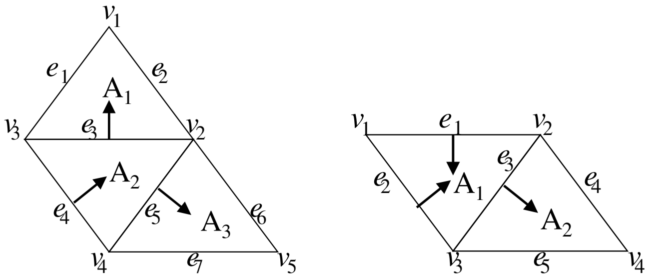


Fig. 2. A combinatorial vector field. The vertex v_3 is a rest point.

Intuitively condition 1. of Definition 10 means that the system is of the first order; geometrically it implies that the future simplices do not increase the dimension, see the definition of the flow map below. Condition 2. excludes merging the future cells. It is illustrated in Figure 3 that splitting of flow is allowed whereas merging is excluded. In Definition 12 of a nondeterministic combinatorial vector field we shall allow both situations.

Since no simplex is in more than one pair in $\bar{V}$, every cell σ of the simplicial complex K satisfies precisely one of the following conditions:



(a) Splitting (b) Merging

Fig. 3. Illustration of Definition 10. Merging on the right hand side is excluded in whereas splinting on the left hand side is allowed.

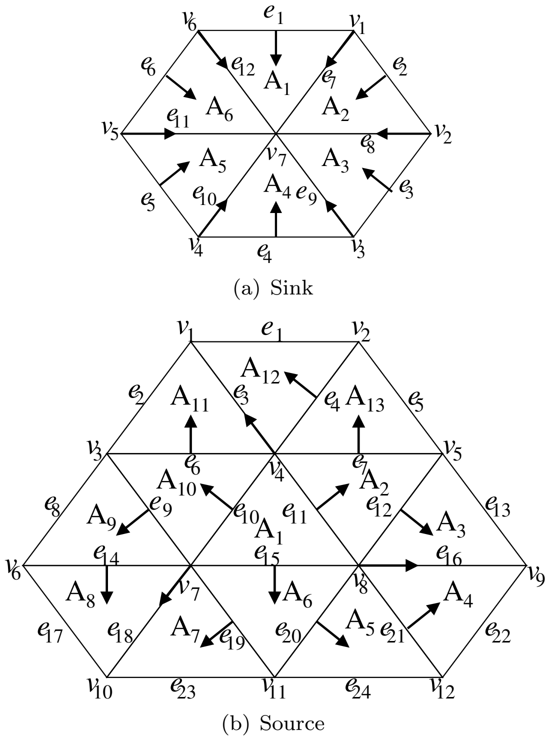
(1) σ is the tail of exactly one arrow;
(2) σ is the head of exactly one arrow;
(3) σ is neither the tail not the head of any arrow.

A simplex that satisfies condition 3. is called a rest point.

Definition 11. (Definition 1.3 of Forman [1998]). *Let V be a combinatorial vector field on K. We say that $\sigma \in K_n$ is a* ***rest point*** *of V of index n if*

(1) $V_{n+1}(\sigma) = 0$ *and*
(2) $\sigma \notin \mathrm{Image}(V_n)$.

Figure 4 illustrates a rest point of minimal index 0 - a sink and the maximal index n - a source.



(a) Sink

(b) Source

Fig. 4. (a) v_7 is a rest point of index 0; (b) A_1 is a rest point of index n.

Section 3 indicates that discrete behavior of a piecewise affine control system involves nondeterminacy induced by blocking more than one maximal faces of a simplex. It seems therefore natural to unleash condition 2 of Definition 10.

Definition 12. *Let K be a simplicial complex. A* ***non-deterministic combinatorial vector field*** *V on K is a family $\{V_n |\ n \in \mathbb{N}\}$ of maps*

$$V_n : K_{n-1} \to K_n \cup \{0\}$$

that satisfies $V_n \circ V_{n-1} = 0$.

In the remaining of this section we shall develop a notion of flow of a nondeterministic combinatorial vector field, that is a map $C_n(K) \to C_n(K)$ which takes an n-simplex to its future n-chain (a linear combination of the simplices in very next future).

Remark 1. *The linear combination of simplices indicates nondeterminism in the future evolution. Thus for example $\tau \mapsto \sigma + \beta$ means that the future of τ is σ or β. In fact, the semantics adopted in this paper is such that any flow $\tau \mapsto a\sigma + b\gamma$ for $a, b \in \mathbb{Z} \setminus \{0\}$ indicates that the future of τ is σ or β.*

For simplicial complexes we may use the following definition of a **combinatorial scalar product** $\langle\cdot,\cdot\rangle : K_n \times K_n \to \{0,1\}$, defined by

$$\langle \sigma, \alpha \rangle = \begin{cases} 1 & \text{if} \quad \sigma = \alpha \\ 0 & \text{otherwise} \end{cases}$$

We extend it to the bilinear product $\langle\cdot,\cdot\rangle : C_n(K) \times C_n(K) \to \mathbb{Z}$. In particular

$$\left\langle \sum_j n_j \sigma_j, \sigma_k \right\rangle = n_k.$$

Define $\theta_n : C_n(K) \to C_{n-1}(K)$ by

$$\theta_n(\sigma) = \sum_{i=0}^{n} (-1)^i \left\langle V_n \circ d_i^n(\sigma), \sigma \right\rangle d_i^n \sigma.$$

The map θ takes $\sigma \in K_n$ to a linear combination of the simplices in $V_n^{-1}(\sigma)$, see Figure 5.

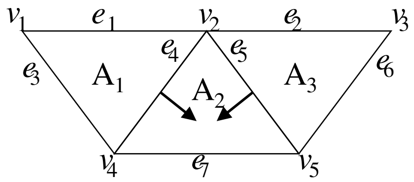


Fig. 5. $\theta_2(A_2) = e_4 - e_5$.

Discrete dynamics of a combinatorial control systems is encapsulated in the following definition of the flow.

Definition 13. *A* ***flow*** *(of a nondeterministic combinatorial vector field) is the map $\Phi_n : C_n(K) \to C_n(K)$ given by*

$$\Phi_n = (\partial_{n+1} - \theta_{n+1}) \circ V_{n+1} + V_n \circ (\partial_n - \theta_n).$$

Example 1. Consider the nondeterministic combinatorial vector field defined in Figure 6.

Firstly, we compute flow starting at e_0:

$$\begin{aligned} \Phi_1(e_0) &= (\partial_2 - \theta_2)V_2(e_0) + V_1(\partial_1 e_0 - \theta_1 e_0) \\ &= (\partial_2 - \theta_2)0 + V_1(v_0 - v_1 - v_0) = -e_4. \end{aligned}$$

The result is the 1-simplex e_4, which corresponds to our expectation, as seen from Figure 6.

Another flow of interest is the one initiated at e_1:

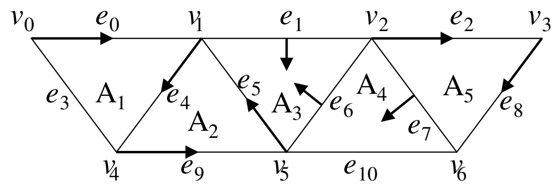


Fig. 6. An example of a simplicial complex with its associated vector field.

$$\begin{aligned}\Phi(e_1) &= (\partial_2 - \theta_2)V_2 + V_1(\partial_1 - \theta_1)\\ &= (\partial_2 - \theta_2)A_3 + V_1(v_1 - v_2)\\ &= e_1 - e_6 + e_5 - e_1 + e_6 - e_2 = e_5 - e_2.\end{aligned}$$

Again it is seen that the resulting flow gives the foreseen adjacent edge, e_1, along with a possible side flow to e_2.

Lastly, we calculate the flow from the 2-simplex A_4:

$$\begin{aligned}\Phi_2(A_4) &= (\partial_3 - \theta_3)V_3 + V_2(\partial_2 - \theta_2)\\ &= (\partial_3 - \theta_3)0 + V_2(e_7 - e_{10} + e_6 - e_7) = A_3.\end{aligned}$$

Again the flow from A_4 to A_3 is found as expected.

The flow Φ_n generates an ***n*-flow line**. An n-simplex $\sigma \in K_n$ belongs to the n-flow line with the initial n-simplex τ if there is $k \in \mathbb{Z}_+$ such that $\langle \sigma, \Phi_n^k(\tau)\rangle \neq 0$. It will be seen below that the flow lines of dimension n and $n-1$ are the only important for control synthesis for combinatorial control systems. It is worth noticing that a flow line born in an n-simplex σ - a source - does not die in a sink, since it is a vertex (0-simplex). It dies in fact in an n-simplex belonging to star of a sink.

Once the combinatorial vector field is in place it is possible to define the equivalence of flow lines, which in the combinatorial setting will be called a V-path.

Definition 14. (Forman [1998]). *A V**-path** of index k is a sequence of length r,*

$$\gamma : \sigma_0^{(k)}, \tau_0^{(k+1)}, \sigma_1^{(k)}, \tau_1^{(k+1)}, ..., \tau_{r-1}^{(k+1)}, \sigma_r^{(k)}, \tag{6}$$

with $\sigma^k \in K_k$, such that for all $i \in \{0, 1, ..., r-1\}$

(1) $\tau_i = V(\sigma_i)$
(2) $\sigma_i \neq \sigma_{i+1} = \Phi(\sigma_i)$

If $\sigma_0 = \sigma_r$ the V-path is called closed. Two closed V-paths, γ, $\tilde{\gamma}$, are equivalent if $\tilde{\gamma}$ can be produced by selecting another starting point of γ.

A V-path is calculated by taking an initial simplex, σ_0, and propagating its flow, i.e.:

$$\sigma_0 \to V(\sigma_0) \to \Phi(\sigma_0) \to V \circ \Phi(\sigma_0) \to \Phi \circ \Phi(\sigma_0) \ldots \tag{7}$$

Moreover, since nondeterminism is allowed in this definition of flow the V-path is allowed to split into more paths, thus resulting in a tree of reachable locations compared to just a single track in the deterministic case.

5. SUPERVISORY CONTROL

Having the definition of simplicial complexes, combinatorial vector fields and combinatorial flow it is now possible to combine all of them in a supervisory control algorithm, which forces the trajectory of the closed-loop system to a target simplex.

Algorithm 1. Algorithm for going from a continuous system to control on a simplicial complex.

(1) *Triangulate manifold.* Firstly the manifold in question is triangulated. This can be done in a number of ways and with larger or smaller discretisation.
(2) *Barycentric linearization.* In order to obtain a piecewise affine system in each simplex the dynamics in each simplex is linearized around its barycenter.
(3) *Calculate controllability.* Calculate all possible exit faces of the given simplex. This results in a maximum combinatorial vector field.
(4) *Find shortest path.* Once the combinatorial vector field is in place it is possible to calculate all possible V-paths and thereby finding the shortest.
(5) *Block undesired branchings.* In order to force the system to take the shortest V-path all branchings not on the path will be blocked.

Remark 2. *Computational wise the greatest challenge with this algorithm is to do a good job in the first step. This goes both for bringing the dimension of the system down before commencing the dividing, and selecting an appropriate number of simplices. This is easily seen since given the first step yields n-simplices, then the complexity of step 2 and 3 are $O(n)$, step 4 is $O(d^n)$(albeit in practice often much lower) and the final step 5 is also $O(n)$.*

This algorithm is obviously not optimal, since it calculates all possible ways to traverse the simplicial complex before returning the shortest one. Thus practically the first point in the algorithm will be performed first. Following this the next 3 points will be performed iteratively. This is done by taking the initial n-simplex, linearize it around its barycenter and calculate all the possible n-simplices reachable from this simplex. This is then repeated for the resulting simplices of the previous operation until the target simplex is reached. In order to avoid loops, any simplices, which previously have been visited are omitted from the new set of simplices to be checked. This operation can be seen as a tree search algorithm, and in order to find the shortest path a breadth-first search algorithm will be preferable compared to a depth-first algorithm.

Once the traversing reaches the target simplex the algorithm is terminated, and the shortest possible V-path from the initial to the target simplex, γ_{op}, is found.

The supervisory control task is now to make sure that γ_{op} is followed, which is ensured by blocking undesired exit faces of simplices with more than one exit face. This is practically achieved by altering the flow lines in the continuous world through control, as described in Section 3.

Remark 3. *It is often desirable to address the problem of avoiding forbidden or unsafe sets. The advantage of the formalism developed in this paper is that there is no combinatorial vector field thus no flow defined on the forbidden simplices. Therefore branches of the search tree hitting such simplices will automatically be abandoned.*

The performance of the supervisory control algorithm will be illustrated in the following example.

Example 2. Consider a combinatorial vector field as depicted in Figure 7. For the sake of simplicity it is assumed that the first 3 steps of Algorithm 1 has been successfully performed.

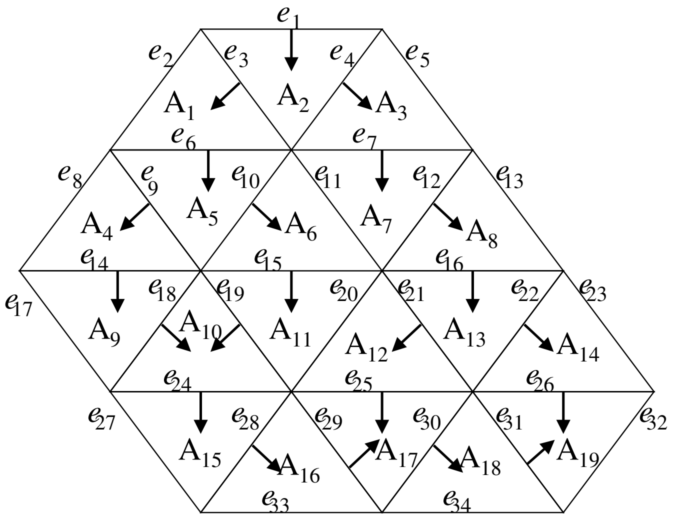


Fig. 7. An example of a simplicial complex with its associated vector field.

The initial 1-simplex is e_1, and the target simplex is e_{32}. The first step is to calculate the V-path for the simplicial complex starting at $\sigma_0 = e_1$ and propagate it according to the strategy described in step 4 of the algorithm, until reaching the target simplex $\sigma_t = e_{32}$. The V-path tree is shown below.[1]

$$\begin{aligned}
\gamma : e_1 \to A_2 &\to e_3 \to A_1 \to e_6 \to A_5 \to e_9 \to A_4 \\
&\qquad\qquad\qquad\qquad\quad\;\; A_5 \to e_{10} \to A_6 \\
A_2 &\to e_4 \to A_3 \to e_7 \to A_7 \to e_{12} \to A_8
\end{aligned}$$

$$\begin{aligned}
A_4 &\to e_{14} \to A_9 \to e_{18} \to A_{10} \to e_{24} \to A_{15} \to e_{28} \\
A_6 &\to e_{15} \to A_{11} \to e_{19} \to A_{10} \to e_{24} \to A_{15} \to e_{28} \\
A_8 &\to e_{16} \to A_{13} \to e_{21} \to A_{12} \to e_{25} \to A_{17} \to e_{30} \\
&\qquad\qquad A_{13} \to e_{22} \to A_{14} \to e_{26} \to A_{18} \to e_{32}
\end{aligned}$$

It is seen that the V-path on the bottom is the minimal path, γ_{op}. It is also clearly seen that there are two 2-simplices, A_2 and A_{13}, which allow for changing to another path, thus in order to control the system to follow γ_{op} the two 1-simplices e_3 and e_{21} must be blocked.

6. CONCLUSIONS

We have treated combinatorics associated to control of a dynamical system defined on a particularly nice simplicial complex - a combinatorial manifold. We have introduced a notion of a nondeterministic combinatorial vector field, which determines the future behavior of the discrete system. The combinatorial vector field gives rise to a flow line that is a path of possible evolutions including all possible mergings and splittings. This information has been used in the algorithm for supervisor synthesis suggested in this paper.

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[1] Note that the tree has been split in two due to space constraints.