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# Homogeneous High-Order Sliding Modes

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**Abstract:** Homogeneity features of dynamic systems are known to provide for a number of general practically important features. In particular, the finite-time convergence is easily proved, and the asymptotic accuracy is readily calculated in the presence of input noises, delays and discrete sampling. General uncertain single-input-single-output regulation problems are only solvable by means of discontinuous control via the so-called high-order sliding modes (HOSM). The homogeneity approach facilitates the design and investigation of new HOSM controllers, featuring such attractive properties as practical continuity of the control in the presence of noises. Robust output-feedback controllers are produced, using robust exact homogeneous differentiators. The asymptotic accuracy of the obtained controllers is the best possible under given circumstances. The dangerous chattering effect is removed by means of a standard procedure. The resulting systems are robust with respect to the presence of unaccounted-for fast stable dynamics of actuators and sensors. Simulation results and applications are presented in the fields of control, signal and image processing.

## 1. INTRODUCTION

Sliding mode control is one of the main tools to cope with heavy uncertainty conditions. The corresponding approach (Utkin, 92; Zinober, 94; Edwards et al., 1998) is based on the exact keeping of a properly chosen constraint by means of high-frequency control switching. Although very robust and accurate, the approach also features certain drawbacks. The standard sliding mode may be implemented only if the relative degree of the constraint is 1, i.e. control has to explicitly appear already in the first total time derivative of the constraint function. Another problem is that the high-frequency control switching may cause dangerous vibrations (chattering effect, Boiko et al., 2005; Fridman, 2001, 2003).

The issues can be settled in a few ways. High-gain control with saturation is used to overcome the chattering effect approximating the sign-function in a narrow boundary layer around the switching manifold (Slotine et al., 1991), the sliding-sector method (Furuta et al., 2000) avoids chattering in control of disturbed linear time-invariant systems. This paper surveys the sliding-mode order approach (Levant, 1993) successfully treating both the chattering and the relative-degree restrictions, while preserving the sliding-mode features and improving the accuracy.

High order sliding mode (HOSM) (Levant, 1993, 2003) actually is a movement on a discontinuity set of a dynamic system understood in Filippov's sense (1988). The sliding order characterizes the dynamics smoothness degree in the vicinity of the mode. Let the task be to make some smooth function $\sigma$ vanish, keeping it at zero afterwards. Then successively differentiating $\sigma$ along trajectories, a discontinuity will be encountered sooner or later in the general case. Thus, sliding modes $\sigma \equiv 0$ may be classified by the number $r$ of the first successive total derivative $\sigma^{(r)}$ which is not a continuous function of the state space variables or does not exist due to some reason, like trajectory nonuniqueness. That number is called the sliding order.

The words "$r$th order sliding" are often abridged to "$r$-sliding". The term "$r$-sliding controller" replaces the longer expression "finite-time-convergent $r$-sliding mode controller".

The standard sliding mode, on which most variable structure systems (VSS) are based, is of the first order ($\dot\sigma$ is discontinuous). While the standard modes feature finite time convergence, convergence to HOSMs may be asymptotic as well. The standard sliding mode precision is proportional to the time interval between the measurements or to the switching delay, while $r$-sliding mode realizations may provide for the sliding precision of up to the $r$th order with respect to sampling intervals and delays (Levant, 1993).

Properly used HOSM practically removes the dangerous chattering effect (Levant 1993, 2007c). One just needs to consider the control derivative as a new control input (Levant, 1993, 2007d; Bartolini, Ferrara et al., 1998).

Asymptotically stable HOSMs arise in many systems with traditional sliding-mode control. In particular, if the relative degree of the constraint is higher than 1, an auxiliary constraint is usually build, being a linear combination of the original constraint and its successive total time derivatives, so that it has the first relative degree (Slotine et al., 1991). Such HOSMs are also deliberately introduced in systems with dynamical sliding modes (Sira-Ramírez, 1993). The limit sliding accuracy asymptotics is the same in that case, as of the standard 1-sliding mode (Slotine et al., 1991). The asymptotic convergence to the constraint inevitably complicates the overall system performance analysis.

While finite-time-convergent arbitrary-order sliding-mode controllers are still theoretically studied (Levant, 2003, 2005; Floquet et al., 2003}, 2-sliding controllers are already successfully implemented for the solution of practical problems (Bartolini, Ferrara et al., 2000, 1998; Bartolini, Pisano et al., 2000, 2003; Ferrara et al., 2000; Levant et al., 2000; Massey et al., 2005, Sira-Ramírez, 2002; Orlov et al., 2003, Krupp et al., 2002; Spurgeon et al., 2002, Shtessel et al., 2003).

Almost all known $r$-sliding controllers possess specific homogeneity called the $r$-sliding homogeneity (Levant, 2005). Thus, new finite-time convergent HOSM controllers are naturally constructed basing on the homogeneity-based approach. The homogeneity makes the convergence proofs of the HOSM controllers standard and provides for the highest possible asymptotic accuracy (Levant, 1993) in the presence of measurement noises, delays and discrete measurements. Thus, with $\tau$ being the sampling interval, the accuracy $\sigma = O(\tau^r)$ is attained (Levant, 2005). These asymptotical features are preserved, when a robust exact homogeneous differentiator of the order $r$ - 1 (Levant, 2003) is applied as a standard part of the homogeneous output-feedback $r$-sliding controller.

Actually $r$-sliding controllers (Levant, 2003, 2005, 2006a) require only the knowledge of the system relative degree $r$. The produced control is a discontinuous function of $\sigma$ and of its real-time-calculated successive derivatives $\dot{\sigma}, ..., \sigma^{(r-1)}$.

The discontinuity set of nested sliding-mode controllers (Levant 2003) is a complicated stratified set with codimension varying in the range from 1 to $r$, which causes certain transient chattering. To avoid it one needs to artificially increase the relative degree. The finite-time-stable exact constraint keeping is lost with alternative controllers developed in (Shtessel et al., 2003; Barbot et al., 2002) for $r$ = 2 and $r$ = 3 respectively.

Quasi-continuous $r$-sliding controller (Levant, 2006a) is a feedback function of $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ being continuous everywhere except the manifold $\sigma = \dot{\sigma} = \ddot{\sigma} = ... = \sigma^{(r-1)} = 0$ of the $r$-sliding mode. In the presence of errors in evaluation of $\sigma$ and its derivatives, these equalities never take place simultaneously with $r > 1$. Therefore, control practically turns to be a continuous function of time.

Simulation demonstrates the practical applicability of the approach in control, signal and image processing.

## 2. PRELIMINARIES

**Definition 1.** A differential inclusion $\dot{x} \in F(x)$ is further called a *Filippov differential inclusion* (Filippov, 1988) if the vector set $F(x)$ is non-empty, closed, convex, locally bounded and upper-semicontinuous. The latter condition means that the maximal distance of the points of $F(x)$ from the set $F(y)$ vanishes when $x \to y$. Solutions are defined as absolutely-continuous functions of time satisfying the inclusion almost everywhere.

Such solutions always exist and have most of the well-known standard properties except the uniqueness (Filippov, 1988).

**Definition 2.** It is said that a differential equation $\dot{x} = f(x)$ with a locally-bounded Lebesgue-measurable right-hand side is understood in the Filippov sense (Filippov, 1988), if it is replaced by a special Filippov differential inclusion $\dot{x} \in F(x)$, where

$$F(x) = \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{\mathrm{co}} f(O_\delta(x) \setminus N) .$$

Here $\mu$ is the Lebesgue measure, $O_\delta(x)$ is the $\delta$-vicinity of $x$, and $\overline{\mathrm{co}}\, M$ denotes the convex closure of $M$.

In the most usual case, when $f$ is continuous almost everywhere, the procedure is to take $F(x)$ being the convex closure of the set of all possible limit values of $f$ at a given point $x$, obtained when its continuity point $y$ tends to $x$. In the general case approximate-continuity (Saks, 1964) points $y$ are taken (one of the equivalent definitions by Filippov (1988)). A solution of $\dot{x} = f(x)$ is defined as a solution of $\dot{x} \in F(x)$. Obviously, values of $f$ on any set of the measure 0 do not influence the Filippov solutions. Note that with continuous $f$ the standard definition is obtained.

In order to better understand the definition consider the case when the number of limit values $f_1, \dots, f_n$ at the point $x$ is finite. Then any possible Filippov velocity has the form $\dot{x} = \lambda_1 f_1 + \dots + \lambda_n f_n$, $\lambda_1 + \dots + \lambda_n = 1$, $\lambda_i \geq 0$, and can be considered as a mean value of the velocity taking on the values $f_i$ during the time share $\lambda_i \Delta t$ of a current infinitesimal time interval $\Delta t$.

**Definition 3.** Consider a discontinuous differential equation $\dot{x} = f(x)$ (Filippov differential inclusion $\dot{x} \in F(x)$) with a smooth output function $\sigma = \sigma(x)$, and let it be understood in the Filippov sense. Then, provided that

1. successive total time derivatives $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ are continuous functions of $x$,
2. the set

$$\sigma = \dot{\sigma} = \ddot{\sigma} = ... = \sigma^{(r-1)} = 0 \qquad (1)$$

   is a non-empty integral set,

3. the Filippov set of admissible velocities at the $r$-sliding points (1) contains more than one vector,

the motion on set (1) is said to exist in $r$-sliding ($r$th-order sliding) mode (Levant, 1993, 2003). Set (1) is called $r$-sliding set. It is said that the sliding order is strictly $r$, if the next derivative $\sigma^{(r)}$ is discontinuous or does not exist as a single-valued function of $x$. The non-autonomous case is reduced to the considered one introducing the fictitious equation $\dot{t} = 1$.

Note that the third requirement here is not standard: it means that set (1) is a discontinuity set of the equation, and it is introduced here only to exclude extraneous cases of integral manifolds of continuous differential equations. The standard sliding mode used in the traditional variable structure systems is of the first order ($\sigma$ is continuous, and $\dot\sigma$ is discontinuous).

The notion of the sliding order appears to be connected with the relative degree notion.

**Definition 4.** A smooth autonomous SISO system $\dot x = a(x) + b(x)u$ with the control $u$ and output $\sigma$ is said to have the relative degree $r$, if the Lie derivatives locally satisfy the conditions (Isidori, 1989)

$$L_b\sigma = L_aL_b\sigma = \ldots = L_a^{r-2}L_b\sigma = 0,\ L_a^{r-1}L_b\sigma \neq 0.$$

It can be shown that the equality of the relative degree to $r$ actually means that the successive total time derivatives $\sigma = \dot\sigma = \ddot\sigma = ... = \sigma^{(r-1)}$ do not depend on control and can be taken as a part of new local coordinates, and $\sigma^{(r)}$ linearly depends on $u$ with the nonzero coefficient $L_a^{r-1}L_b\sigma$. Also here the non-autonomous case is reduced to the autonomous one introducing the fictitious equation $\dot t = 1$.

## 3. OUTPUT REGULATION PROBLEM

### *3.1 Systems nonlinear in control*

First consider an uncertain smooth nonlinear Single-Input Single-Output (SISO) system $\dot x = f(t,x,u)$, $x\in\mathbf{R}^n$, $t$, $u \in \mathbf{R}$ with a smooth output $s(t, x) \in \mathbf{R}$. Let the goal be to make the output $s(t, x)$ to track some real-time-measured smooth signal $s_c(t)$. Introducing a new auxiliary control $v \in \mathbf{R}$, $\dot u = v$, and the output $\sigma(t, x) = s(t, x) - s_c(t)$, obtain a new affine-in-control system $\frac{d}{dt}(x,u)^t = (f(t,x,u),\ 0)^t + (0,1)^t v$ with the control task to make $\sigma(t, x)$ vanish. Therefore, the further consideration is restricted only to systems affine in control.

### *3.2 SISO regulation problem and the idea of its solution*

Consider a dynamic system of the form

$$\dot x = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \tag{2}$$

where $x \in \boldsymbol{R}^n$, $a$, $b$ and $\sigma$: $\boldsymbol{R}^{n+1} \to \boldsymbol{R}$ are unknown smooth functions, $u \in \boldsymbol{R}$, the dimension $n$ might be also uncertain. Only measurements of $\sigma$ are available in real time. The task is to provide in finite time for exactly keeping $\sigma \equiv 0$.

The relative degree $r$ of the system is assumed to be constant and known. In other words, for the first time the control explicitly appears in the $r$th total time derivative of $\sigma$ and

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{3}$$

where $h(t,x) = \sigma^{(r)}|_{u=0}$, $g(t,x) = \frac{\partial}{\partial u}\sigma^{(r)} \neq 0$. It is supposed that for some $K_m$, $K_M$, $C > 0$

$$0 < K_m \le \frac{\partial}{\partial u}\sigma^{(r)} \le K_M, \qquad |\sigma^{(r)}|_{u=0}| \le C, \tag{4}$$

which is always true at least locally. Trajectories of (2) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control $u(t, x)$.

Finite-time stabilization of smooth systems at an equilibrium point by means of continuous control is considered in (Bacciotti et al., 2005; Bhat et al., 2000)). In our case any continuous control

$$u = \varphi(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) \tag{5}$$

providing for $\sigma \equiv 0$, would satisfy the equality $\varphi(0,0, ..., 0) = -h(t,x)/g(t,x)$, whenever (1) holds. Since the problem uncertainty prevents it, *the control has to be discontinuous at least on the set* (1). Hence, the $r$-sliding mode $\sigma = 0$ is to be established.

As follows from (3), (4)

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]\, u\, . \tag{6}$$

The differential inclusion (5), (6) is understood here in the Filippov sense, which means that the right-hand vector set is enlarged at the discontinuity points of (5), in order to satisfy the convexity and semicontinuity conditions from Definition 1. The Filippov procedure from Definition 2 is applied for this aim to the function (5), and the resulting scalar set is substituted for $u$ in (6). The obtained inclusion does not "remember" anything on system (2) except the constants $r$, $C$, $K_m$, $K_M$. Thus, provided (4) holds, the finite-time stabilization of (6) at the origin simultaneously solves the stated problem for all systems (3).

Note that the realization of this plan requires real-time differentiation of the output. The controllers, which are designed in this paper, are *r-sliding homogeneous* (Levant, 2005). The corresponding notion is introduced further.

## 4. HOMOGENEITY, FINITE-TIME STABILITY AND ACCURACY

**Definition 5.** A function $f$: $\mathbf{R}^n \to \mathbf{R}$ (respectively a vector-set field $F(x) \subset \mathbf{R}^n$, $x \in \mathbf{R}^n$, or a vector field $f$: $\mathbf{R}^n \to \mathbf{R}^n$) is called *homogeneous of the degree $q \in \mathbf{R}$ with the dilation*

$$d_\kappa: (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n)$$

(Bacciotti et al., 2001), where $m_1$, ..., $m_n$ are some positive numbers (*weights*), if for any $\kappa > 0$ the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds (respectively $F(x) = \kappa^{-q}d_\kappa^{-1}F(d_\kappa x)$, or $f(x) = \kappa^{-q}d_\kappa^{-1}f(d_\kappa x)$). The non-zero homogeneity degree $q$ of a vector field can always be scaled to $\pm1$ by an appropriate proportional change of the weights $m_1$, ..., $m_n$.

Note that the homogeneity of a vector field $f(x)$ (a vector-set field $F(x)$) can equivalently be defined as the invariance of the differential equation $\dot{x} = f(x)$ (differential inclusion $\dot{x} \in F(x)$) with respect to the combined time-coordinate transformation

$$G_\kappa : (t, x) \mapsto (\kappa^p t, d_\kappa x),$$

where $p$, $p = -q$, might naturally be considered as the weight of $t$. Indeed, the homogeneity condition can be rewritten as

$$\dot{x} \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^p t)} \in F(d_\kappa x).$$

**Examples.** In the following the weights of $x_1$, $x_2$ are 3 and 2 respectively. Then the function $x_1^2 + x_2^3$ is homogeneous of the weight (degree) 6: $(\kappa^3 x_1)^2 + (\kappa^2 x_2)^3 = \kappa^6(x_1^2 + x_2^3)$. The differential inequality $|\dot{x}_1| + \dot{x}_2^{4/3} \le x_1^{4/3} + x_2^2$ corresponds to the homogeneous differential inclusion

$$(\dot{x}_1, \dot{x}_2) \in \{(z_1, z_2): |z_1| + z_2^{4/3} \le x_1^{4/3} + x_2^2\}$$

of the degree +1. The system of differential equations

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^{1/3} - |x_2^{1/2}| \operatorname{sign} x_2 \end{cases} \tag{7}$$

is of the degree -1 and is finite-time stable (Bhat et al., 2000).

**1°.** A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *globally uniformly finite-time stable* at 0, if it is Lyapunov stable and for any $R > 0$ exists $T > 0$ such that any trajectory starting within the disk $\|x\| < R$ stabilizes at zero in the time $T$.

**2°.** A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *globally uniformly asymptotically stable* at 0, if it is Lyapunov stable and for any $R > 0$, $\varepsilon > 0$ exists $T > 0$ such that any trajectory starting within the disk $\|x\| < R$ enters the disk $\|x\| < \varepsilon$ in the time $T$ to stay there forever.

A set $D$ is called *dilation retractable* if $d_\kappa D \subset D$ for any $\kappa \in [0, 1]$. In other words with any its point $x$ it contains the whole line $d_\kappa x$, $\kappa \in [0, 1]$.

**3°.** A homogeneous differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *contractive* if there are 2 compact sets $D_1$, $D_2$ and $T > 0$, such that $D_2$ lies in the interior of $D_1$ and contains the origin; $D_1$ is dilation-retractable; and all trajectories starting at the time 0 within $D_1$ are localized in $D_2$ at the time moment $T$.

**Theorem 1** (Levant, 2005)**.** *Let $\dot{x} \in F(x)$ be a homogeneous Filippov inclusion with a negative homogeneous degree -p, then properties* 1°, 2° *and* 3° *are equivalent and the maximal settling time is a continuous homogeneous function of the initial conditions of the degree p.*

Finite-time stability of homogeneous discontinuous differential equations was also considered in (Orlov, 2005).

**Idea of the proof.** Obviously, both 1° and 2° imply 3°, and 1° implies 2°. Thus, it is enough to prove that 3° implies 1°. All trajectories starting in the set $D_1$ concentrate in a smaller set $D_2$ in time $T$. Applying the homogeneity transformation obtain that the same is true with respect to the sets $d_\kappa D_1$, $d_\kappa D_2$ and the time $\kappa T$ for any $\kappa > 0$. An infinite collapsing chain of embedded regions is now constructed, such that any point belongs to one of the regions, and the resulting convergence time is majored by a geometric series.■

Due to the continuous dependence of solutions of the Filippov inclusion $\dot{x} \in F(x)$ on its graph $\Gamma = \{(x, y)| \ y \in F(x)\}$ (Filippov 1988), the contraction feature 3° is obviously robust with respect to perturbations causing small changes of the inclusion graph in some vicinity of the origin.

**Corollary 1** (Levant 2005)**.** *The global uniform finite-time stability of homogeneous differential equations* (*Filippov inclusions*) *with negative homogeneous degree is robust with respect to locally small homogeneous perturbations*.

Let $\dot{x} \in F(x)$ be a homogeneous Filippov differential inclusion. Consider the case of "noisy measurements" of $x_i$ with the magnitude $\beta_i \tau^{m_i}$, $\beta_i, \tau > 0$,

$$\dot{x} \in F(x_1 + \beta_1[-1, 1]\tau^{m_1}, ..., x_n + \beta_n[-1, 1]\tau^{m_n}).$$

Successively applying the global closure of the right-hand-side graph and the convex closure at each point $x$, obtain some new Filippov differential inclusion $\dot{x} \in F_\tau(x)$.

**Theorem 2** (Levant, 2005)**.** *Let $\dot{x} \in F(x)$ be a globally uniformly finite-time stable homogeneous Filippov inclusion with the homogeneity weights $m_1, ..., m_n$ and the degree $-p < 0$, and let $\tau > 0$. Suppose that a continuous function $x(t)$ be defined for any $t \ge -\tau^p$ and satisfy some initial conditions $x(t) = \xi(t)$, $t \in [-\tau^p, 0]$. Then if $x(t)$ is a solution of the disturbed inclusion*

$$\dot{x}(t) \in F_\tau(x(t + [-\tau^p, 0])), \quad 0 < t < \infty,$$

*the inequalities $|x_i| < \gamma_i \tau^{m_i}$ are established in finite time with some positive constants $\gamma_i$ independent of $\tau$ and $\xi$.*

Note that Theorem 2 covers the cases of retarded or discrete noisy measurements of all, or some of the coordinates, and any mixed cases. In particular, infinitely extendible solutions certainly exist in the case of noisy discrete measurements of some variables or in the constant time-delay case. For example, with small delays of the order of $\tau$ introduced in the right-hand side of (7) the accuracy $x_1 = O(\tau^3)$, $\dot{x}_1 = x_2 = O(\tau^2)$ is obtained. As follows from Corollary 1, with sufficiently small $\varepsilon$ the addition of the term $\varepsilon x_1^{2/3}$ in the first equation of (7) disturbs neither the finite-time stability, nor the above asymptotic accuracy.

## 5. HOMOGENEOUS SLIDING MODES

Suppose that feedback (5) imparts homogeneity properties to the closed-loop inclusion (5), (6). Due to the term $[-C, C]$, the right-hand side of (5) can only have the homogeneity degree 0 with $C \ne 0$. Indeed, with a positive degree the right hand side of (5), (6) approaches zero near the origin, which is not possible with $C \ne 0$. With a negative degree it is not bounded near the origin, which contradicts the local boundedness of $\varphi$. Thus, the homogeneity degree of $\sigma^{(r-1)}$ is to be opposite to the degree of the whole system.

Scaling the system homogeneity degree to -1, achieve that the homogeneity weights of $t$, $\sigma$, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ are 1, $r$, $r$ - 1, ..., 1 respectively. This homogeneity is further called the *r-sliding* *homogeneity*. The inclusion (5), (6) is called *r-sliding homogeneous* if for any $\kappa > 0$ the combined time-coordinate transformation

$$G_\kappa: (t, \sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \mapsto (\kappa t, \kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)}) \quad (8)$$

preserves the closed-loop inclusion (5), (6). Note that the Filippov differential inclusion corresponding to the closed-loop inclusion (5), (6) is also $r$-sliding homogeneous.

Transformation (8) transfers (5), (6) into

$$\frac{d^r(\kappa^r\sigma)}{(d\kappa t)^r} \in [-C, C] + [K_m, K_M]\, \varphi(\kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)}).$$

Hence, (5), (6) is $r$-sliding homogeneous if

$$\varphi(\kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)}) \equiv \varphi(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}). \quad (9)$$

**Definition 6.** Controller (5) is called $r$-sliding homogeneous ($r$th order sliding homogeneous) if (9) holds for any $(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$ and $\kappa > 0$. The corresponding sliding mode is also called homogeneous (if exists).

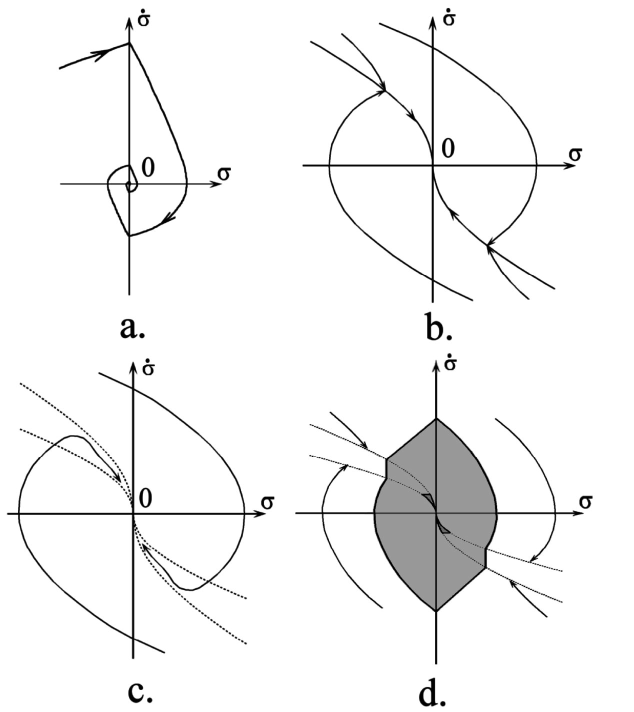



Fig. 1. Convergence of various 2-sliding homogeneous controllers

Such a homogeneous controller is inevitably discontinuous at the origin (0, ..., 0), unless $\varphi$ is a constant function. It is also uniformly bounded, since it is locally bounded and takes on all its values in any vicinity of the origin. Recall that the values of $\varphi$ on any zero-measure set do not affect the corresponding Filippov inclusion.

A controller is called $r$-sliding homogeneous *in the broader sense* if (8) preserves the resulting trajectories of (6). Thus, the sub-optimal 2-sliding controller (Bartolini, Ferrara et al., 1998, Bartolini, Pisano et al., 2003)

$$u = - r_1 \operatorname{sign}(\sigma - \sigma^*/2) + r_2 \operatorname{sign}\sigma^*, \quad r_1 > r_2 > 0,$$

$$2[(r_1 + r_2)K_m - C] > (r_1 - r_2)K_M + C, \quad (r_1 - r_2)K_m > C,$$

is homogeneous, though it does not have the feedback form (5). Here $\sigma^*$ is the value of $\sigma$ detected at the closest time in the past, when $\dot{\sigma}$ was 0. The initial value of $\sigma^*$ is 0.

Almost all known $r$-sliding controllers, $r \geq 2$, are $r$-sliding homogeneous. The only important exception is the terminal 2-sliding controller maintaining 1-sliding mode $\dot{\sigma} + \beta\sigma^\rho \equiv 0$, where $\rho = (2k+1)/(2m+1)$, $\beta > 0$, $k < m$, and $k$, $m$ are natural numbers (Man et al., 1994). Indeed, the homogeneity requires $\rho = 1/2$ and $\sigma \geq 0$.

*5.1 Second order sliding mode controllers*

Let $r = 2$. As follows from the previous Section it is sufficient to construct a 2-sliding-homogeneous contractive controller. Their discrete-sampling versions provide for the accuracy described in Theorem 2, i.e. $\sigma = O(\tau^2)$, $\dot{\sigma} = O(\tau)$. Similarly, the noisy measurements lead to the accuracy $\sigma = O(\varepsilon)$, $\dot{\sigma} = O(\varepsilon^{1/2})$, if the maximal errors of $\sigma$ and $\dot{\sigma}$ sampling are of the order of $\varepsilon$ and $\varepsilon^{1/2}$ respectively.

Design of such 2-sliding controllers is greatly facilitated by the simple geometry of the 2-dimensional phase plane with coordinates $\sigma$, $\dot{\sigma}$ : any smooth curve locally divides the plane in two parts. It is easy to construct any number of such controllers (Levant, 2007a). Only few controllers are presented here.

The twisting controller (Levant, 1993)

$$u = - (r_1 \operatorname{sign}\sigma + r_2 \operatorname{sign}\dot{\sigma}),$$

has the convergence conditions

$$(r_1 + r_2)K_m - C > (r_1 - r_2)K_M + C, \quad (r_1 - r_2) K_m > C.$$

Its typical trajectory in the plane $\sigma$, $\dot{\sigma}$ is shown in Fig. 1a.

A homogeneous form of the controller with prescribed convergence law (Fig. 1b; Levant, 1993)

$$u = - \alpha \operatorname{sign}(\dot{\sigma} + \beta|\sigma|^{1/2}\operatorname{sign}\sigma), \quad \alpha K_m - C > \beta^2/2$$

is a 2-sliding homogeneous analogue of the terminal sliding mode controller originally featuring a singularity at $\sigma = 0$ (Man et al., 1994).

The 2-sliding stability analysis is based on the fact that all the trajectories in the plane $\sigma$, $\dot{\sigma}$ which pass through a given continuity point of $u = \varphi(\sigma, \dot{\sigma})$ are confined between the properly chosen trajectories of the homogeneous differential equations $\ddot{\sigma} = \pm C + K_M\varphi(\sigma, \dot{\sigma})$ and $\ddot{\sigma} = \pm C + K_m\varphi(\sigma, \dot{\sigma})$. These border trajectories cannot be crossed by other paths, if $\varphi$ is locally Lipschitzian, and may be often chosen as boundaries of appropriate dilation-retractable regions (Levant, 2007a). Recall that a region is dilation-retractable iff, with each its point $(\sigma, \dot{\sigma})$, it contains all the points of the parabolic segment $(\kappa^2\sigma, \kappa\dot{\sigma})$, $0 \leq \kappa \leq 1$.

An important class of HOSM controllers comprises recently proposed so-called *quasi-continuous* controllers. Controller

(5) is called *quasi-continuous* (Levant, 2006a), if it can be redefined according to continuity everywhere except the $r$-sliding manifold $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$. Due to always present disturbances and noises, in practice, with the sliding order $r > 1$ the general-case trajectory does never hit the $r$-sliding manifold, for the $r$-sliding condition has the codimension $r$. Hence, the control practically remains continuous function of time all the time. As a result, the chattering is significantly reduced. Following is the 2-sliding controller from such a family of arbitrary-order sliding controllers (Levant, 2006a):

$$u = -\alpha \frac{\dot\sigma + \beta|\sigma|^{1/2}\operatorname{sign}\sigma}{|\dot\sigma| + \beta|\sigma|^{1/2}}, \quad \beta > 0.$$

This control is continuous everywhere except the origin. It vanishes on the parabola $\dot\sigma + \beta|\sigma|^{1/2}\operatorname{sign}\sigma = 0$. With sufficiently large $\alpha$ there are such numbers $\rho_1, \rho_2$, $0 < \rho_1 < \beta < \rho_2$ that all the trajectories enter the region between the curves $\dot\sigma + \rho_i|\sigma|^{1/2}\operatorname{sign}\sigma = 0$ and cannot leave it (Fig. 1c). The contractivity property of the controller is demonstrated in Fig. 1d.

### *5.2 Arbitrary order sliding mode controllers*

Following are two most known $r$-sliding controller families (Levant, 2003, 2006a). The controllers

$$u = -\alpha\, \Psi_{r-1,r}(\sigma, \dot\sigma, ..., \sigma^{(r-1)}),$$

are defined by recursive procedures, have magnitude $\alpha > 0$, and solve the general output regulation problem from Section 3. The parameters of the controllers can be chosen in advance for each relative degree $r$. Only the magnitude $\alpha$ is to be adjusted for any fixed $C$, $K_m$, $K_M$, most conveniently by computer simulation, avoiding complicated and redundantly large estimations. Obviously, $\alpha$ is to be negative with $(\partial/\partial u)\sigma^{(r)} < 0$. In the following $\beta_1, ..., \beta_{r-1} > 0$ are the controller parameters, and $i = 1,..., r-1$.

1. The following procedure defines the "nested" $r$-sliding controller (Levant, 2003), based on a pseudo-nested structure of 1-sliding modes. Let $q > 1$. The controller is built by the following recursive procedure:

$$N_{i,r} = (|\sigma|^{q/r} + |\dot\sigma|^{q/(r-1)} + ... + |\sigma^{(i-1)}|^{q/(r-i+1)})^{(r-i)/q};$$

$$\Psi_{0,r} = \operatorname{sign}\sigma, \quad \Psi_{i,r} = \operatorname{sign}(\sigma^{(i)} + \beta_i N_{i,r}\Psi_{i-1,r});$$

Following are the nested sliding-mode controllers (of the first family) for $r \le 4$ with tested $\beta_i$ and $q$ being the least multiple of 1,..., $r$:

1. $u = -\alpha\operatorname{sign}\sigma$,
2. $u = -\alpha\operatorname{sign}(\dot\sigma + |\sigma|^{1/2}\operatorname{sign}\sigma)$,
3. $u = -\alpha\operatorname{sign}(\ddot\sigma + 2(|\dot\sigma|^3 + |\sigma|^2)^{1/6}\operatorname{sign}(\dot\sigma + |\sigma|^{2/3}\operatorname{sign}\sigma))$,
4. $u = -\alpha\operatorname{sign}\{\dddot\sigma + 3(\ddot\sigma^6 + \dot\sigma^4 + |\sigma|^3)^{1/12}\operatorname{sign}[\ddot\sigma + (\dot\sigma^4 + |\sigma|^3)^{1/6}$ $\operatorname{sign}(\dot\sigma + 0.5|\sigma|^{3/4}\operatorname{sign}\sigma)]\}$.

Though these controllers can be given an intuitive inexact explanation based on recursively nested standard sliding modes, the proper explanation is more complicated (Levant, 2003), since no sliding mode is possible on discontinuous surfaces, and a complicated motion arises around the control discontinuity set.

2. The following procedure defines a family of quasi-continuous controllers (Levant, 2006a):

$$\varphi_{0,r} = \sigma, \; N_{0,r} = |\sigma|, \quad \Psi_{0,r} = \varphi_{0,r}/N_{0,r} = \operatorname{sign}\sigma,$$

$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\Psi_{i-1,r},$$

$$N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}, \quad \Psi_{i,r} = \varphi_{i,r}/N_{i,r}.$$

Following are quasi-continuous controllers with $r \le 4$ and simulation-tested $\beta_i$.

1. $u = -\alpha\operatorname{sign}\sigma$,
2. $u = -\alpha(\dot\sigma + |\sigma|^{1/2}\operatorname{sign}\sigma)/(|\dot\sigma| + |\sigma|^{1/2})$,
3. $u = -\alpha[\ddot\sigma + 2(|\dot\sigma| + |\sigma|^{2/3})^{-1/2}(\dot\sigma + |\sigma|^{2/3}\operatorname{sign}\sigma)]\,/$ $[|\ddot\sigma| + 2(|\dot\sigma| + |\sigma|^{2/3})^{1/2}]$,
4. $\varphi_{3,4} = \dddot\sigma + 3[\ddot\sigma + (|\dot\sigma| + 0.5|\sigma|^{3/4})^{-1/3}(\dot\sigma + 0.5|\sigma|^{3/4}\operatorname{sign}\sigma)]$ $[|\ddot\sigma| + (|\dot\sigma| + 0.5|\sigma|^{3/4})^{2/3}]^{1/2}$,

$$N_{3,4} = |\dddot\sigma| + 3[|\ddot\sigma| + (|\dot\sigma| + 0.5|\sigma|^{3/4})^{2/3}]^{1/2}, \; u = -\alpha\,\varphi_{3,4}/N_{3,4}.$$

It is easy to see that the sets of parameters $\beta_i$ are chosen the same for both families with $r \le 4$. Note that while enlarging $\alpha$ increases the class (4) of systems, to which the controller is applicable, parameters $\beta_i$, are tuned to provide for the needed convergence rate (Levant et al., 2005). The author considers the second family as the best one. In addition to the reduced chattering, another advantage of these controllers is the simplicity of their coefficients' adjustment (Section 7).

**Theorem 3.** *Each representative of the order r of the above two families of arbitrary-order sliding-mode controllers is r-sliding homogeneous and finite-time stabilizing.*

The proof of the Theorem is based on Theorem 1, i.e. on the proof of the contractivity property. Asymptotic accuracies of these controllers are readily obtained from Theorem 2. In particular $\sigma^{(i)} = O(\tau^{r-i})$, $i = 0, 1, \ldots, r-1$, if the measurements are performed with the sampling interval $\tau$.

**Chattering attenuation.** Chattering vibrations are naturally considered dangerous, if their energy cannot be neglected; i.e. if the energy does not vanish with the gradual vanishing of chattering-producing factors (noises, delays, small singular perturbation parameters, etc). Corresponding formal notions were introduced in (Levant, 2007c). The standard chattering attenuation procedure is to consider the control derivative as a new control input, increasing the relative degree (Levant, 1993). It is proved (Levant, 2007c) that the resulting systems are robust with respect to the presence of unaccounted-for fast stable actuators and sensors, and no dangerous chattering is generated neither by such additional dynamics, nor by the presence of noises and delays. That also remains true when the output feedback is constructed, as in the next Section.

That standard procedure was many times successfully applied (Bartolini, Ferrara 1998; Bartolini, Pisano et al., 2003, Levant, et al. 2000, etc), though formally the convergence is only locally ensured in some vicinity of the ($r$ + 1)-sliding mode $\sigma \equiv 0$. Global convergence can be easily obtained in the case of the transition from the relative degree 1 to 2 (Levant, 1993, 2007a); semi-global convergence can be assured with higher relative degrees (Levant et al., 2007d).

## 6. DIFFERENTIATION AND OUTPUT-FEEDBACK CONTROL

### *6.1 Arbitrary order robust exact differentiation*

Any $r$-sliding homogeneous controller can be complemented by an ($r$-1)th order differentiator (Atassi et al., 2000; Bartolini, Pisano et al., 2000; Krupp, et al., 2000; Kobayashi et al., 2002; Yu et al., 1996) producing an output-feedback controller. In order to preserve the demonstrated exactness, finite-time stability and the corresponding asymptotic properties, the natural way is to calculate $\dot\sigma, ..., \sigma^{(r-1)}$ in real time by means of a robust finite-time convergent exact *homogeneous* differentiator (Levant, 1998, 2003). Its application is possible due to the boundedness of $\sigma^{(r)}$ provided by the boundedness of the feedback function $\varphi$ in (5).

Let the input signal $f(t)$ be a function defined on $[0, \infty)$ and consisting of a bounded Lebesgue-measurable noise with unknown features, and of an unknown base signal $f_0(t)$, whose $k$th derivative has a known Lipschitz constant $L > 0$. The problem of finding real-time robust estimations of $\dot f_0(t)$, $\ddot f_0(t)$, ... , $f_0^{(k)}(t)$ being exact in the absence of measurement noises is solved by the differentiator (Levant, 2003)

$$\begin{aligned}
&\dot z_0 = v_0,\ \ v_0 = -\lambda_k L^{1/(k+1)} | z_0 - f(t)|^{k/(k+1)} \operatorname{sign}(z_0 - f(t)) + z_1,\\
&\dot z_1 = v_1,\ \ v_1 = -\lambda_{k-1} L^{1/k} | z_1 - v_0|^{(k-1)/k} \operatorname{sign}(z_1 - v_0) + z_2,\\
&\qquad\qquad \dots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (10)\\
&\dot z_{k-1} = v_{k-1},\ \ v_{k-1} = -\lambda_1 L^{1/2} |z_{k-1} - v_{k-2}|^{1/2} \operatorname{sign}(z_{k-1} - v_{k-2}) + z_k,\\
&\dot z_k = -\lambda_0 L \operatorname{sign}(z_k - v_{k-1}).
\end{aligned}$$

The parameters $\lambda_0, \lambda_1, \dots, \lambda_k > 0$ being properly chosen, the following equalities are true in the absence of input noises after a finite time of the transient process:

$$z_0 = f_0(t); \qquad z_i = v_{i-1} = f_0^{(i)}(t), \quad i = 1, ..., k.$$

Note that the differentiator has a recursive structure. Once the parameters $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$ are properly chosen for the ($k$ - 1)th order differentiator with the Lipschitz constant $L$, only one parameter $\lambda_k$ is needed to be tuned for the $k$th order differentiator with the same Lipschitz constant. The parameter $\lambda_k$ is just to be taken sufficiently large. Any $\lambda_0 > 1$ can be used to start this process. Such differentiator can be used in any feedback, trivially providing for the separation principle (Atassi et al., 2000; Levant, 2005).

**Idea of the proof**. Denote $\sigma_i = (z_i - f^{(i)}(t))/L$. Dividing by $L$ all equations and subtracting $f^{(i+1)}(t)/L$ from both sides of the equation with $\dot z_i$ on the left, $i = 0, \dots, k$, obtain

$$\begin{aligned}
&\dot\sigma_0 = -\lambda_k |\sigma_0|^{k/(k+1)} \operatorname{sign}(\sigma_0) + \sigma_1,\\
&\dot\sigma_1 = -\lambda_{k-1} | \sigma_1 - \dot\sigma_0|^{(k-1)/k} \operatorname{sign}(\sigma_1 - \dot\sigma_0) + \sigma_2,\\
&\qquad \dots\dots\\
&\dot\sigma_{k-1} = -\lambda_1 |\sigma_{k-1} - \dot\sigma_{k-2}|^{1/2} \operatorname{sign}(\sigma_{k-1} - \dot\sigma_{k-2}) + \sigma_k,\\
&\dot\sigma_k \in -\lambda_0 \operatorname{sign}(\sigma_k - \dot\sigma_{k-1}) + [-1, 1].
\end{aligned}$$

where the inclusion $f^{(k+1)}(t)/L \in [-1, 1]$ is used in the last line. This differential inclusion is homogeneous with the homogeneity degree –1 and the weights $k + 1, k, \dots, 1$ of $\sigma_0, \sigma_1, \dots, \sigma_k$ respectively. The finite time convergence of the differentiator follows from the contractivity property of this inclusion (Levant 2003) and Theorem 1.■

A possible choice of the differentiator parameters with $k \le 5$ is $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 8$, $\lambda_5 = 12$ (Levant, 2006a).

Theorem 2 provides for the asymptotic accuracy of the differentiator. Let the measurement noise be any Lebesgue-measurable function with the magnitude not exceeding $\varepsilon$. Then the accuracy $|z_i(t) - f_0^{(i)}(t)| = O(\varepsilon^{(k+1-i)/(k+1)})$ is obtained. That accuracy is shown to be the best possible (Kolmogoroff, 1962; Levant, 1998).

### *6.2 Output-feedback control*

Introducing the above differentiator of the order $r$-1 in the feedback, obtain an output-feedback $r$-sliding controller

$$u = \varphi (z_0, z_1, ..., z_{r-1}), \tag{11}$$

$$\begin{aligned}
&\dot z_0 = v_0,\ v_0 = -\lambda_{r-1} L^{1/r} | z_0 - \sigma|^{(r-1)/r} \operatorname{sign}(z_0 - \sigma) + z_1,\\
&\dot z_1 = v_1,\ v_1 = -\lambda_{r-2} L^{1/(r-1)} | z_1 - v_0|^{(r-2)/(r-1)} \operatorname{sign}(z_1 - v_0) + z_2,\\
&\dots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (12)\\
&\dot z_{r-2} = v_{r-2},\ v_{r-2} = -\lambda_1 L^{1/2} | z_{r-2} - v_{r-3}|^{1/2} \operatorname{sign}(z_{r-2} - v_{r-3}) + z_{r-1},\\
&\dot z_{r-1} = -\lambda_0 L \operatorname{sign}(z_{r-1} - v_{r-2}),
\end{aligned}$$

where $L \ge C + \sup|\varphi|\, K_M$, and parameters $\lambda_i$ of differentiator (12) are chosen in advance (Subsection 6.1).

**Theorem 4.** *Let controller* (5) *be r-sliding homogeneous and finite-time stable, and the parameters of the differentiator* (11) *be properly chosen with respect to the upper bound of* $|\varphi|$. *Then in the absence of measurement noises the output-feedback controller* (11), (12) *provides for the finite-time* *convergence of each trajectory to the r-sliding mode* $\sigma = 0$*; otherwise convergence to a set defined by the inequalities* $|\sigma| < \gamma_0\varepsilon$, $|\dot\sigma| < \gamma_1 \varepsilon^{(r-1)/r}$, ..., $\overset{(r-1)}{\sigma} < \gamma_{r-1}\varepsilon^{1/r}$ *is ensured, where* $\varepsilon$ *is the unknown measurement noise magnitude and* $\gamma_0, \gamma_1, ..., \gamma_{r-1}$ *are some positive constants.*

**Proof.** Denote $s_i = z_i - \overset{(i)}{\sigma}$. Then using $\overset{(i)}{\sigma} \in [-L, L]$ controller (11), (12) can be rewritten as

$$u = -\alpha\, \varphi(s_0+\sigma, s_1+\dot\sigma, ..., s_{r-1}+\overset{(r-1)}{\sigma}), \tag{13}$$

$$\begin{aligned}
\dot s_0 &= -\lambda_{r-1} L^{1/r}|s_0|^{(r-1)/r}\operatorname{sign}(s_0) + s_1,\\
\dot s_1 &= -\lambda_{r-2} L^{1/(r-1)}|s_1 - \dot s_0|^{(r-2)/(r-1)}\operatorname{sign}(s_1 - \dot s_0) + s_2,\\
&\dots \\
\dot s_{r-2} &= -\lambda_1 L^{1/2}|s_{r-2} - \dot s_{r-3}|^{1/2}\operatorname{sign}(s_{r-2} - \dot s_{r-3}) + s_{r-1},\\
\dot s_{r-1} &\in -\lambda_0 L \operatorname{sign}(s_r - \dot s_{r-2}) + [-L, L].
\end{aligned} \tag{14}$$

Solutions of (3), (11), (12) correspond to solutions of the Filippov differential inclusion (6), (13), (14). Assign the weights $r - i$ to $s_i$, $\overset{(i)}{\sigma}$, $i = 0, 1, ..., r - 1$, and obtain a homogeneous differential inclusion (6), (13), (14) of the degree -1. Let the initial conditions belong to some ball in the space $s_i$, $\overset{(i)}{\sigma}$. Due to the finite-time stability of the differentiator part (14) of the inclusion, it collapses in a bounded finite time, and the controller becomes equivalent to (5), which is uniformly finite-time stabilizing by assumption. Due to the boundedness of the control no solution leaves some larger ball till the moment, when $s_0 \equiv \dots \equiv s_{r-1} \equiv 0$ is established. Hence, (6), (13), (14) is also globally uniformly finite-time stable. Theorems 1, 2 finish the proof. ■

In the absence of measurement noises the convergence time is bounded by a continuous function of the initial conditions in the space $\sigma, \dot\sigma, ..., \overset{(r-1)}{\sigma}, s_0, s_1, ..., s_{r-1}$. This function is homogeneous of the weight 1 and vanishes at the origin (Theorem 1).

Let $\sigma$ measurements be carried out with a sampling interval $\tau$, or let them be corrupted by a noise being an unknown bounded Lebesgue-measurable function of time of the magnitude $\varepsilon$, then solutions of (3), (11), (12) are infinitely extendible in time under the assumptions of Section 2, and the following Theorem is a simple consequence of Theorem 2.

**Theorem 5.** *The discrete-measurement version of the controller* (11), (12) *with the sampling interval* $\tau$ *provides in the absence of measurement noises for the inequalities*

$$|\sigma| < \gamma_0\tau^r,\ |\dot\sigma| < \gamma_1\tau^{r-1}, ..., \overset{(r-1)}{\sigma} < \gamma_{r-1}\tau$$

*for some* $\gamma_0, \gamma_1, ..., \gamma_{r-1} > 0$. *In the presence of a measurement noise of the magnitude* $\varepsilon$ *the accuracies*

$$|\sigma| < \delta_0\varepsilon,\ |\dot\sigma| < \delta_1\varepsilon^{(r-1)/r}, ..., \overset{(r-1)}{\sigma} < \delta_{r-1}\varepsilon^{1/r}$$

*are obtained for some* $\delta_0, \delta_1, ..., \delta_{r-1} > 0$.

The asymptotic accuracy provided by Theorem 5 is the best possible with discontinuous $\overset{(r)}{\sigma}$ and discrete sampling (Levant, 1993). A Theorem corresponding to the case of discrete noisy sampling is also easily formulated basing on Theorem 2. Note that the lacking derivatives can be also estimated by means of divided finite differences, providing for robust control with homogeneous sliding modes (Levant, 2007b). The results of this Section are also valid for the sub-optimal controller (Bartolini, Ferrara et al., 1998). Hence, the problem stated in Section 2 is actually solved.

## 7. ADJUSTMENT OF THE CONTROLLERS

### *7.1 Control magnitude adjustment*

Condition (4) is rather restrictive and is mostly only locally fulfilled, which implies only local (or semi-global) applicability of the described approach in practice. Indeed, one needs to take the control magnitude large enough for the whole operational region.

Consider a more general case, when as previously

$$\overset{(r)}{\sigma} = h(t,x) + g(t,x)u,$$

but $h$ might be not bounded, and $g$ might be not separated from zero. Instead, assume that a locally bounded Lebesgue-measurable non-zero function $\Phi(t,x)$ be available, such that for any positive $d$ with sufficiently large $\alpha$ the inequality

$$\alpha\, g(t,x)\Phi(t,x) > d + |h(t,x)|$$

holds for any $t$, $x$. The goal is to make the control magnitude a feedback adjustable function.

It is also assumed that, if $\sigma$ remains bounded, trajectories of (1) are infinitely extendible in time for any Lebesgue-measurable control $u(t, x)$ with bounded quotient $u/\Phi$. This assumption is needed only to avoid finite-time escape. In practice the system is often required to be weakly minimum phase. Note also that actuator presence might in practice prevent effectiveness of any global control due to saturation effects.

For simplicity the full information on the system state is assumed available. In particular, $t$, $x$, $\sigma$ and its $r - 1$ successive derivatives are measured.

Consider the controller

$$u = -\alpha\, \Phi(t,x)\Psi_{r-1,r}(\sigma, \dot\sigma, ..., \overset{(r-1)}{\sigma}), \tag{15}$$

where $\alpha > 0$, and $\Psi_{r-1,r}$ is one of the two $r$-sliding homogeneous controllers introduced in Subsection 5.2.

**Theorem 1** (Levant, 2004). *With properly chosen parameters of the controller* $\Psi_{r-1,r}$ *and sufficiently large* $\alpha > 0$ *controller* (15) *provides for the finite-time establishment of the identity* $\sigma \equiv 0$ *for any initial conditions. Moreover, any increase of the gain function* $\Phi$ *does not interfere with the convergence.*

While the function $\Phi$ can be chosen large to control exploding systems, it is also reasonable to make the function $\Phi$ decrease significantly, when approaching the system operational point, therefore reducing the chattering (Levant et al, 2000).

Note that controller (15) is not homogeneous. It is proved in (Levant, 2006b) that differentiator (10) converges in finite time also with variable parameter $L$, provided the logarithmic derivative $\dot{L}/L$ remains bounded. Such a global-convergence differentiator can be implemented here, possibly resulting in an output feedback.

*7.2 Parameter adjustment*

Controller parameters presented in Section 5 provide for the formal solution of the stated problem. Nevertheless, in practice one often needs to adjust the convergence rate, either to slow it down relaxing the requirements to actuators, or to accelerate it in order to meet some system requirements. Note in that context that redundantly enlarging the magnitude parameter $\alpha$ of controllers from Section 5 does not accelerate the convergence, but only increases the chattering, while its reduction may lead to the convergence loss.

The main procedure is to take the controller

$$u = \lambda^r \alpha \Psi_{r-1,r}(\sigma,\ \dot{\sigma}/\lambda,\ ...,\ \sigma^{(r-1)}/\lambda^{r-1}), \quad \lambda > 0.$$

instead of

$$u = -\alpha\, \Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$$

providing for the approximately $\lambda$ times reduction of the convergence time. Exact formulations (Levant et al., 2006b) are omitted here in order to avoid unnecessary complication.

In the case of quasi-continuous controllers (Section 5) the form of controller is preserved. The new parameters $\tilde{\beta}_1, \ldots, \tilde{\beta}_{r-1}$, $\tilde{\alpha}$ are calculated according to the formulas $\tilde{\beta}_1 = \lambda\beta_1$, $\tilde{\beta}_2 = \lambda^{r/(r-1)}\beta_2$, ..., $\tilde{\beta}_{r-1} = \lambda^{r/2}\beta_{r-1}$, $\tilde{\alpha} = \lambda^r\alpha$. Following are the resulting quasi-continuous controllers with $r \le 4$, simulation-tested $\beta_i$ and a general gain function $\Phi$:

1. $u = -\alpha\Phi \operatorname{sign}\sigma$,

2. $u = -\alpha\Phi\, (\dot{\sigma} + \lambda\, |\sigma|^{1/2} \operatorname{sign}\sigma)/(|\dot{\sigma}| + \lambda|\sigma|^{1/2})$,

3. $u = -\alpha\Phi\, [\ddot{\sigma} + 2\lambda^{3/2}\, (|\dot{\sigma}| + \lambda|\sigma|^{2/3})^{-1/2}(\dot{\sigma} + \lambda|\sigma|^{2/3} \operatorname{sign}\sigma)\,] \,/\, [|\ddot{\sigma}| + 2\lambda^{3/2}\, (|\dot{\sigma}| + \lambda|\sigma|^{2/3})^{1/2}]$,

4. $\varphi_{3,4} = \dddot{\sigma} + 3\lambda^2[\ddot{\sigma} + \lambda^{4/3}(|\dot{\sigma}| + 0.5\lambda|\sigma|^{3/4})^{-1/3}(\dot{\sigma} + 0.5\lambda\ |\sigma|^{3/4} \operatorname{sign}\ \sigma)]\ [|\ddot{\sigma}| + \lambda^{4/3}(|\dot{\sigma}| + 0.5\lambda\, |\sigma|^{3/4})^{2/3}]^{-1/2}$,

   $N_{3,4} = |\dddot{\sigma}| + 3\lambda^2\, [|\ddot{\sigma}| + \lambda^{4/3}(|\dot{\sigma}| + 0.5\lambda\, |\sigma|^{3/4})^{2/3}]^{1/2}$,

   $u = -\alpha\Phi\ \varphi_{3,4} / N_{3,4}$ .

## 8. APPLICATION AND SIMULATION EXAMPLES

Only the main points of the presented results are demonstrated.

*8.1 Control simulation*

Practical application of HOSM control is presented in a lot of papers, only to mention here (Bartolini, Pisano et al., 2003, Edvards et al., 2002, Levant et al., 2000, Massey et al., 2005). Consider a simple kinematic model of car control

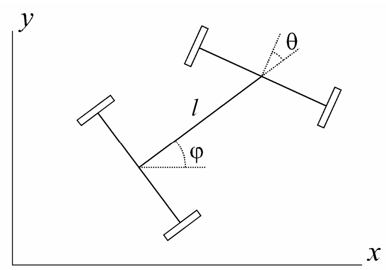



Fig. 2: Kinematic car model

$$\dot{x} = v\cos\varphi, \quad \dot{y} = v\sin\varphi,$$
$$\dot{\varphi} = v/l \tan\theta, \quad \dot{\theta} = u,$$

where $x$ and $y$ are Cartesian coordinates of the rear-axle middle point, $\varphi$ is the orientation angle, $v$ is the longitudinal velocity, $l$ is the length between the two axles and $\theta$ is the steering angle (i.e. the real input) (Fig. 2). The task is to steer the car from a given initial position to the trajectory $y = g(x)$, where $g(x)$ and $y$ are assumed to be available in real time.

Define $\sigma = y - g(x)$. Let $v = const = 10$ m/s, $l = 5$ m, $x = y = \varphi = \theta = 0$ at $t = 0$, $g(x) = 10 \sin(0.05x) + 5$. The relative degree of the system is 3 and the quasi-continuous 3-sliding controller (Subsection 5.2) solves the problem. It was taken $\alpha = 1$, $L = 400$. The resulting output-feedback controller (11), (12) is

$$u = -\,[z_2 + 2\, (|z_1| + |\,z_0|^{2/3})^{-1/2}(z_1 + |\,z_0|^{2/3} \operatorname{sign} z_0\,)] \,/\, [|z_2| + 2\, (|z_1| + |\,z_0|^{2/3})^{1/2}],$$

$$\dot{z}_0 = v_0,\ v_0 = -\,14.7361\, |\,z_0 - \sigma|^{2/3} \operatorname{sign}(z_0 - \sigma) + z_1,$$

$$\dot{z}_1 = v_1,\ \ v_1 = -\,30\, |\,z_1 - v_0|^{1/2} \operatorname{sign}(z_1 - v_0) + z_2,$$

$$\dot{z}_2 = -\,440 \operatorname{sign}(z_2 - v_1).$$

The controller parameter $\alpha$ is convenient to find by simulation. The differentiator parameter $L = 400$ is taken deliberately large, in order to provide for better performance in the presence of measurement errors ($L = 25$ is also sufficient, but is much worse with sampling noises). The control was applied only from $t = 1$, in order to provide some time for the differentiator convergence.

The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics), the sampling step being equal to the integration step $\tau = 10^{-4}$. In the absence of noises the tracking accuracies $|\sigma| \le 5.4{\cdot}10^{-7}$, $|\dot{\sigma}| \le 2.4{\cdot}10^{-4}$, $|\ddot{\sigma}| \le 0.042$ were obtained. With $\tau = 10^{-5}$ the accuracies $|\sigma| \le 5.6{\cdot}10^{-10}$, $|\dot{\sigma}| \le 1.4{\cdot}10^{-5}$, $|\ddot{\sigma}| \le 0.0042$ were attained, which mainly corresponds to the asymptotics stated in Theorem 5. The car trajectory, 3-sliding tracking errors, steering angle θ and its derivative $u$ are shown in Fig. 3a, b, c, d respectively. It is seen from Fig. 3c that the control $u$ remains continuous until the very entrance into the 3-sliding mode. The steering angle θ remains rather smooth and is quite feasible.

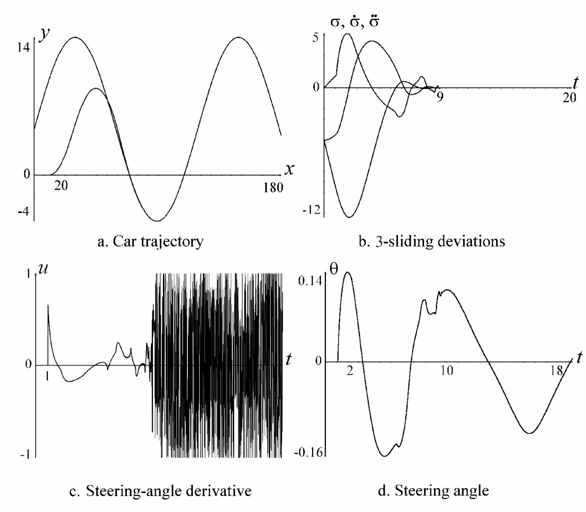



Fig. 3: Quasi-continuous 3-sliding car control [15]

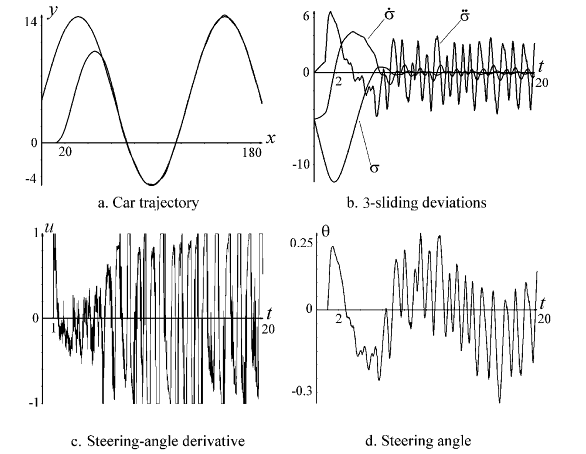



Fig. 4: Performance with the input noise magnitude 0.1m

In the presence of output noise with the magnitude 0.01m the tracking accuracies $|\sigma| \le 0.02$, $|\dot\sigma| \le 0.14$, $|\ddot\sigma| \le 1.3$ were obtained. With the measurement noise of the magnitude 0.1m the accuracies changed to $|\sigma| \le 0.20$, $|\dot\sigma| \le 0.62$, $|\ddot\sigma| \le 2.8$ which corresponds to the asymptotics stated by Theorem 4. The performance of the controller with the measurement error magnitude 0.1m is shown in Fig. 4. It is seen from Fig. 4c that the control $u$ is a continuous function of $t$. The steering angle vibrations have the magnitude of about 7 degrees and the frequency 1, which is also quite feasible. The performance does not significantly change, when the frequency of the noise varies in the range 100 - 100000.

*8.2 Signal processing: real-time differentiation*

Following is the 5th order differentiator:

$$\dot z_0 = v_0,\ v_0 = -8\, L^{1/6}|z_0 - f(t)|^{5/6}\operatorname{sign}(z_0 - f(t)) + z_1,$$

$$\dot z_1 = v_1,\ v_1 = -5\, L^{1/5}|z_1 - v_0|^{4/5}\operatorname{sign}(z_1 - v_0) + z_2,$$

$$\dot z_2 = v_2,\ v_2 = -3\, L^{1/4}|z_2 - v_1|^{3/4}\operatorname{sign}(z_2 - v_1) + z_3,$$

$$\dot z_3 = v_3,\ v_3 = -2\, L^{1/3}|z_3 - v_2|^{2/3}\operatorname{sign}(z_3 - v_2) + z_4,$$

$$\dot z_4 = v_4,\ v_4 = -1.5\, L^{1/2}|z_4 - v_3|^{1/2}\operatorname{sign}(z_4 - v_3) + z_5,$$

$$\dot z_5 = -1.1\, L \operatorname{sign}(z_5 - v_4);\quad f^{(6)} \le L.$$

It is applied with $L = 1$ for the differentiation of the function

$$f(t) = \sin 0.5t + \cos 0.5t,\qquad |f^{(6)}| \le L = 1.$$

The initial values of the differentiator variables are taken zero. In practice it is reasonable to take the initial value of $z_0$ equal to the current sampled value of $f(t)$, significantly shortening the transient. Convergence of the differentiator is demonstrated in Fig. 5. The 5th derivative is not exact due to the software restrictions (number of digits). Higher order differentiation requires special software development.

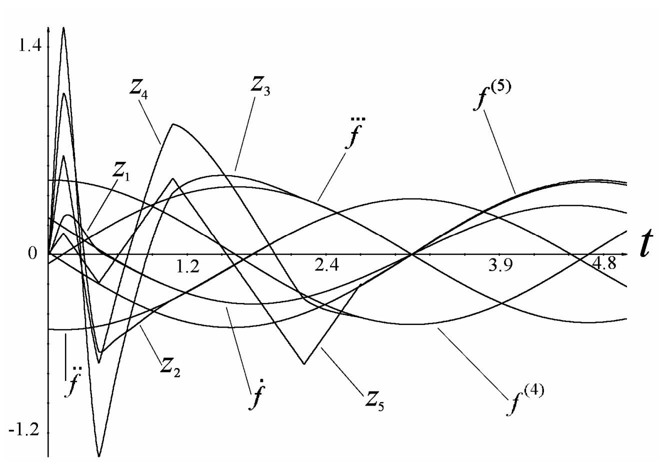
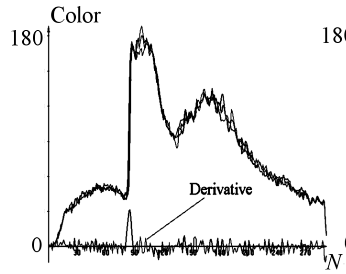


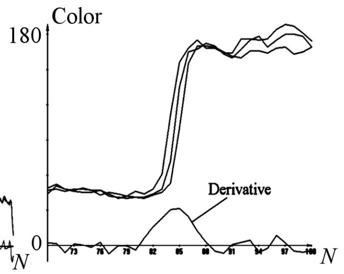
Fig. 5. 5th order differentiation

*8.3 Image processing.*

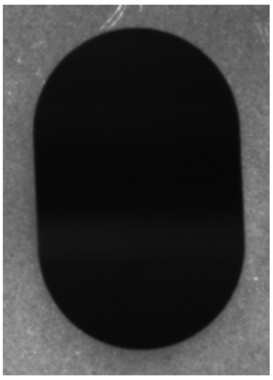
A gray image is represented in computers as a noisy function given on a planar grid, which takes integer values in the range 0 – 255. In particular, 0 and 255 correspond to the black and to the white respectively. An edge point is defined as a point of the maximal gradient. Samples of 3 successive rows from a real gray photo are presented in Fig. 5a together with the results of the first-order differentiation (10) of their arithmetical average, $L = 3$. The differentiation was carried out in both directions, starting from each row end, and the arithmetical average was taken exterminating lags. A zoom of the same graph in a vicinity of an edge point is shown in Fig. 5b. Some results of the edge detection are demonstrated in Fig. 5 c,d.



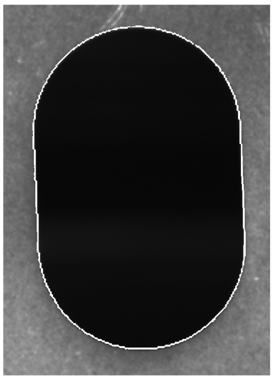

a. Color graph sample (3 rows)




b. Zoom



c. Edge detection source



d. Edge detection result

Fig. 5. Edge detection

These results were obtained by the author in the framework of a practical research project fulfilled by the Institute of Industrial Mathematics (Beer-Sheva, Israel, 2000) for Cognitense Ltd. The simplicity of the differentiator application allows easy tangent line calculation for a curve in an image. It is shown in Fig. 6 how a crack is found and "eliminated" of the edge of a piece given by a photo (the edge was already previously found, and its points were numbered).

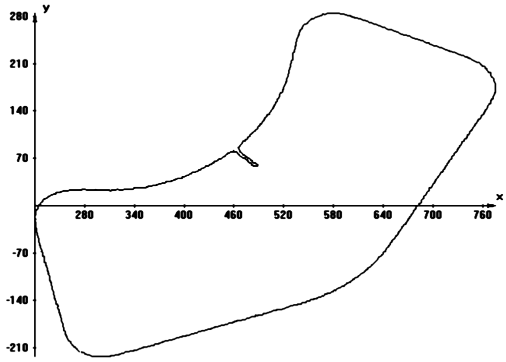
## 9. CONCLUSIONS

Homogeneity features of dynamical systems and differential inclusions greatly simplify the proofs of finite-time stability and provide for the easy calculation of the asymptotical accuracy in the presence of delays and measurement errors.

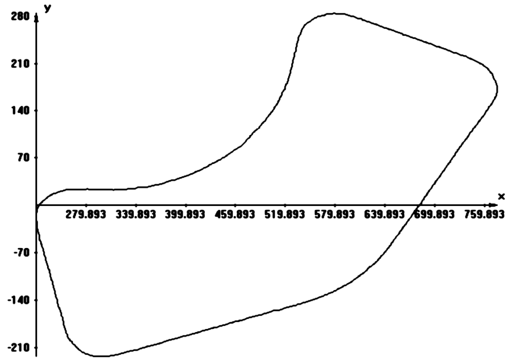
The homogeneity approach provides a convenient effective framework for the design of high-order sliding mode controllers.

High-order sliding mode control provides for effective solution of general SISO problems under uncertainty conditions. Unsolved remain the problems with non-minimum phase dynamics and with undefined relative degree. The general MIMO case under uncertainty conditions surely remains the main challenge for the future research.

Practical image processing applications of the developed differentiator are for the first time demonstrated.




a. Curve with "crack"




b. Smoothed curve

Fig. 6. Smoothing a curve

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