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A Tool for Converting FEM Models into Representations Suitable for Control Synthesis

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Abstract: Finite Element Methods (FEM) play an important role in modelling of complex systems. Models generated from FEM are of very high dimension and are difficult to handle for control system design. In this paper, an algorithm is presented that converts a system generated from FEM into the state space model of an interconnected system, thus vastly reducing the complexity of synthesizing a distributed control strategy. A simple cantilever beam problem is used as an example to illustrate the proposed method. A state space representation of this system is obtained through FEM analysis. This model is converted from lumped state space form to interconnected form. A decentralized controller is then designed using a homotopy approach.

Keywords: Software tool, Interconnected Systems, Finite Element Methods (FEM).

1. INTRODUCTION

Engineering systems in which subsystems interact with each other are of considerable practical interest. With advances in design techniques and computational power complex system models of very high dimensions can be developed, including lumped approximations of partial differential equations (PDEs) — examples include the deflection of beams, plates and membranes, and the temperature distribution of thermally conductive materials. Solutions to these types of problems are possible for very simple systems; for more realistic systems the Finite Element Method (FEM)(Becker, 2004) is a very effective tool for solving such problems numerically (Huebner et al., 2001). Currently FEM is used in structural analysis, heat transfer, fluid mechanics etc.

In FEM a complete system is divided into small elements which are spatially distributed and interact with each other, based on their spatial location. In structural analysis behavior a system is described by the displacement of elements satisfying material laws (constitutive equation) (Becker, 2004). All elements are assembled together and the requirements of continuity and equilibrium are satisfied between neighboring elements. A unique solution can be found for a given boundary conditions.

A state space representation obtained from FEM generally has a large number of states along with large number of inputs and outputs. For such systems modern control design techniques may fail due to the size of the problem. One way of dealing with such systems is to reduce the size of system using standard model order reduction techniques. However, this reduction may eliminate some aspects of the dynamic behavior which may get exited in the closed loop, thus degrading the achieved performance.

In this paper we propose a different approach, which is to decompose the model used for control synthesis into subsystems. Each subsystem has only a small number of states and its own sensor and actuator. This decomposition can facilitate the use of modern controller synthesis techniques to large systems, using the framework of interconnected systems.

A general interconnected systems is composed of N subsystems interconnected with each other. The state space realization of i^{th} subsystem is given by

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + \sum_{j=1, j\neq i}^{N} B_{vij} x_j(t) + B_i w_i(t) \\ z_i(t) &= C_i x_i(t) + D_i w_i(t) \end{aligned} \tag{1}$$

where $x_i \in R^{n_i}$, $w_i \in R^{k_i}$, $z_i \in R^{o_i}$ are the state vector, disturbance inputs and outputs, respectively. The matrices A_i, B_{vij}, B_i, C_i and D_i are constant matrices of appropriate dimensions. Here, the matrix B_{vij} represents the interconnection of the i^{th} subsystem with the j^{th} subsystem. Models in the form of (1) have been extensively studied, and efficient synthesis tools have been proposed to design controllers for such systems. Some earlier work on such systems is presented in (Siljak, 1978), while (Siljak and Zecevic, 2005) presents some new approaches based on linear matrix inequalities. In (Chen et al., 2005) the authors have presented a homotopy based approach to design robust controllers for (1). In (D'Andrea and Dullerud, 2003) the authors have shown that the Kalman-Yaqubovich-Popov lemma can be generalized for a class of interconnected systems with identical subsystems having interconnections only with their near neighbors, which can be considered as a special case of system (1).

In this paper we present a software tool that can decompose an FEM generated model into the form of (1),

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thus making it suitable for control system design. The paper is organized as follows. A general matrix decomposition function is presented in Section 2. This function is applied in Section 3 to a FEM generated system and a complete algorithm to decompose the given system is discussed. In section 4 a uniform cantilever example is used to demonstrate the application of the tool. In section 5 the homotopy approach of (Chen et al., 2005) is used to design decentralized controller for the decomposed system. Concluding remarks and future directions are given in Section 5.

2. THE BASIC FUNCTION

In this section the basic function *decomp*() is introduced. The function decomposes any matrix $G \in R^{p\times q}$ into the block structure

$$\begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & G_{22} & \cdots & G_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{nn} \end{bmatrix}. \tag{2}$$

In the following G_i will represent the i^{th} diagonal block $G(i,i)$, where $G_i \in R^{(f_{hi}\times f_{ji})}$ such that,

$$p = \sum_{i=1}^{n} f_{hi} \tag{3}$$

$$q = \sum_{i=1}^{n} f_{ji} \tag{4}$$

The function requires two further inputs from the user. These are

$$f_h^T = [\, f_{h1} \;\, f_{h2} \;\cdots\; f_{hn} \,]^T \tag{5}$$

$$f_j^T = [\, f_{j1} \;\, f_{j2} \;\cdots\; f_{jn} \,]^T \tag{6}$$

These vectors specify the dimension of each block, and for the general case ($f_{hi} \neq f_{ji}$).

Let us define G_{vi} as

$$G_{vi} = \left[\, G_{(i,1)} \;\cdots\; G_{(i,i-1)} \;,\; G_{(i,i+1)} \;\cdots\; G_{(i,n)} \,\right]$$

In order to extract G_i and G_{vi} from G we need upper and lower row and column indices in terms of the elements of vectors f_h and f_j. Next we will find these indices for G_i and G_{vi_r}, where $r = 1, \ldots, i-1, i+1, \ldots, n$.

2.1 Indices of G_i

Let e_h be the lower row index and e_j be the lower column index of G_i respectively, then these can be found from vectors f_h and f_j as

$$e_h = \sum_{k=1}^{i-1} f_{h_k} + 1 \tag{7}$$

$$e_j = \sum_{k=1}^{i-1} f_{j_k} + 1 \tag{8}$$

Then we have

$$G_i = G(e_{h+}l_i, e_{j+}g_i)$$

, where

$$\begin{aligned} l_i &= 0, 1, \ldots, (f_{hi} - 1) \\ g_i &= 0, 1, \ldots, (f_{ji} - 1) \end{aligned} \tag{9}$$

2.2 Indices of G_{vi_r}

Let $G_{vi_r} = G(l_{vi_r}, g_{vi_r})$, then $l_{vi_r} = e_h + l_i$.

For the case when $r < i$,

$$g_{vi_r} = e_j - \sum_{k=r}^{i-1} f_{jk} + \{0, 1, \ldots, f_{jr} - 1\}$$

and for the case when $r > i$,

$$g_{vi_r} = e_j + \sum_{k=i}^{r-1} f_{jk} + \{0, 1, \ldots, f_{jr} - 1\}$$

The above approach is coded into a MATLAB function $decomp(G, f_h, f_j, i, m)$ where the binary input m can be used to get G_i if $m = 0$ or G_{vi} if $m \neq 0$.

3. APPLICATION TO FEM GENERATED SYSTEMS

A system model generated via FEM is given as

$$M\ddot{q} + C\dot{q} + Kq = F_d + F \tag{10}$$

where q represents the generalized co-ordinates and

$M =$ Mass matrix
$K =$ Stiffness matrix
$C =$ Damping matrix
$F_d =$ Applied nodal forces (external forces)
$F =$ Boundary conditions

These matrices are obtained by combining the equations of motion of each element of the FEM analysis. Following this the continuity constraints between neighboring elements are applied. The above equation can be written in state space form as

$$\begin{aligned} \dot{\tilde{x}} &= \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}}_{\tilde{A}} \tilde{x} + \underbrace{\begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}}_{} F(t) \\ y &= \tilde{C}x + \tilde{D}u \end{aligned} \tag{11}$$

where $x \in R^n$, $u \in R^k$, $y \in R^o$. The state vector in (11) is arranged as

$$\tilde{x}^T = [\, x_1^T \; x_2^T \;\cdots\; x_n^T \; \dot{x}_1^T \; \dot{x}_2^T \;\cdots\; \dot{x}_n^T \,]^T$$

This can be transformed into

$$x^T = [\, x_1^T \; \dot{x}_1^T \; x_2^T \; \dot{x}_2^T \;\cdots\; x_n^T \; \dot{x}_n^T \,]^T \tag{12}$$

Defining a unitary transformation matrix T_r such that $x = T_r\tilde{x}$ we obtain $A = T_r\tilde{A}T_r^{-1}$, $B = T_r\tilde{B}$, $C = \tilde{C}T_r^{-1}$ and $D = \tilde{D}$.

Next we show how to decompose the transformed system into the from (1) with N subsystems.

Let for the i^{th} subsystem $\dot{x}_i, x_i, \in R^{f_{xi}}$, $z_i \in R^{f_{zi}}$ and $w_i \in R^{f_{wi}}$. Define the vectors

$$F_x = (f_{x_1}, \ldots, f_{x_N}) \tag{13}$$
$$F_z = (f_{z_1}, \ldots, f_{z_N}) \tag{14}$$
$$F_w = (f_{w1}, \ldots, f_{wn}) \tag{15}$$

The function *decomp*() can then be applied to A, B, C and D to decompose them to get the state space representation (1). This is expressed in the following algorithm.

3.1 Algorithm

For an FEM generated state space in the from (11)

Step 1 Find the transformation matrix T_r and apply this similarity transformation to get A, B, C, D.
Step 3 Define the vectors F_x, F_z and F_w.
Step 2 Obtain the other matrices by
- $A_i = decomp(A, F_x, F_x, i, 0)$;
- $B_{vi} = decomp(A, F_x, F_x, i, 1)$;
- $B_i = decomp(B, F_x, F_w, i, 0)$;
- $C_i = decomp(C, F_z, F_x, i, 0)$;
- $D_i = decomp(D, F_z, F_w, i, 0)$;

4. EXAMPLE: CANTILEVER BEAM

This section briefly presents the FEM treatment of a uniform cantilever (Juang and Phan, 2004).

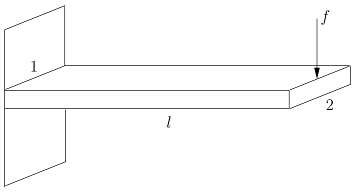

Fig. 1. Beam cantilever

4.1 FEM of single cantilever beam element

First consider a beam with clamped boundary condition at one end and free boundary condition at the other as shown in Fig 1. In order to illustrate the generation of an FEM model, a single element with two nodes of a clamped-free uniform beam is taken.

Let an impulse f act downwards at the node 2 as shown in the Fig 1. This force results in a displacement $d(x,t)$ and an angular rotation, $\theta = \frac{\partial d(x,t)}{\partial x}$. The equation of motion can be written as

$$\frac{\rho \partial^2 d(x,t)}{\partial t^2} + EM_I \frac{\partial^4 d(x,t)}{\partial x^4} = f \tag{16}$$

Here ρ, E, M_I are density, Young modulus of elasticity and second moment of inertia, respectively. The behavior of displacement and rotation can be approximated in terms of interpolation functions as

$$d(x,t) = [\, N_1(x) \;\; N_2(x) \;\; N_3(x) \;\; N_4(x) \,] \begin{bmatrix} d_1(t) \\ \theta_1(t) \\ d_2(t) \\ \theta_2(t) \end{bmatrix}$$

where the subscripts 1 and 2 denote the first and second node of the element. The $N_i(x)$ are the interpolation functions of the corresponding variable, which can be approximated as

$$N_k(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \;\; \forall \, k = 1,2,3,4$$

Values of these coefficients can be determined by applying geometric boundary conditions on d and θ at node 1 and 2. Let us define

$$N^T(x) = [\, N_1(x) \;\; N_2(x) \;\; N_3(x) \;\; N_4(x) \,]$$

Substituting in (16) we obtain

$$\rho N^T(x) \frac{\partial^2 q(t)}{\partial t^2} + EM_I \frac{\partial^4 N^T(x) q(t)}{\partial x^4} = f(x,t)$$

where $q^T = [d_1, \theta_1, d_2, \theta_2]^T$. Solving this equation results in

$$\left[\underbrace{\int_0^l \rho N(x) N^T(x) dx}_{M} \right] \frac{\partial^2 q(t)}{\partial t^2} + \left[\underbrace{\int_0^l EM_I \frac{\partial^2 N(x)}{\partial x^2} \frac{\partial^2 N^T(x)}{\partial x^2}}_{K} \right] q(t)$$
$$= \underbrace{\int_0^l N(x) f(x,t) dx}_{F_d} + F \tag{17}$$

which is equivalent to

$$M\ddot{q} + Kq = F_d + F$$

F in terms of shear force f and moment m is given as

$$F = \begin{bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \end{bmatrix} = \begin{bmatrix} \left. EM_I \frac{\partial^3 d(x,t)}{\partial x^3} \right|_{x=0} \\ \left. EM_I \frac{\partial^2 d(x,t)}{\partial x^2} \right|_{x=0} \\ \left. EM_I \frac{\partial^3 d(x,t)}{\partial x^3} \right|_{x=l} \\ \left. EM_I \frac{\partial^2 d(x,t)}{\partial x^2} \right|_{l=0} \end{bmatrix}$$
$$= \begin{bmatrix} f \\ -fl \\ f \\ 0 \end{bmatrix}$$

For the above interpolation function the following local mass and stiffness matrices can be calculated:

$$M = \rho \begin{bmatrix} \frac{13l}{35} & \frac{11l^2}{210} & \frac{9l}{70} & \frac{-13l^2}{420} \\ \frac{11l^2}{210} & \frac{l^3}{105} & \frac{13l^2}{420} & \frac{-l^2}{140} \\ \frac{9l}{70} & \frac{13l^2}{420} & \frac{13l}{35} & \frac{-11l^2}{210} \\ \frac{-13l^2}{420} & \frac{-l^2}{140} & \frac{-11l^2}{210} & \frac{l^3}{105} \end{bmatrix}$$

$$K = EM_I \begin{bmatrix} \frac{12}{l^3} & \frac{6}{l^2} & \frac{-12}{l^3} & \frac{6}{l^2} \\ \frac{6}{l^2} & \frac{4}{l} & \frac{-6}{l^2} & \frac{2}{l} \\ \frac{-12}{l^3} & \frac{-6}{l^2} & \frac{12}{l^2} & \frac{-6}{l^2} \\ \frac{6}{l^2} & \frac{2}{l} & \frac{-6}{l^2} & \frac{4}{l} \end{bmatrix}$$

4.2 Generalization to cantilever with a large number of elements

The above expressions for K and M represent local elements in FEM which are assembled in a global mass and stiffness matrix to generate lumped matrices. The global equation for the whole assembly can be obtained by combining the matrix contribution of all individual elements, such that coefficients belonging to common nodes are added together.

Let us define

$$k^{(i)} = \begin{bmatrix} k_{11}^{(i)} & k_{12}^{(i)} \\ k_{21}^{(i)} & k_{22}^{(i)} \end{bmatrix}$$

which is the stiffness matrix of i^{th} element, then

$$K = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & & & \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & & & \\ & & \ddots & & \\ & & & k_{22}^{(i-1)} + k_{11}^{(i)} & k_{12}^{(i)} \\ & & & k_{21}^{(i)} & k_{22}^{(i)} + k_{11}^{(i+1)} \\ & & & & & \ddots \end{bmatrix}$$

Similarly the global mass matrix M can also be found.

These global mass and stiffness matrices are used in (11) to generate a state space representation.

In this paper a cantilever beam consisting of 10 nodes, as shown in Fig 2, is considered. Each node has four states (d, θ, $\dot{d}$, $\dot{\theta}$), and the overall system has five inputs and outputs at every second node also shown in Fig 2. The inputs are the forces, and the outputs are the displacements d of these elements.

It is desired to convert the model into an interconnected system with five subsystems, each consisting of two nodes (shown by dashed boxes in Fig 2). Thus we have $f_{xi} = 8$ and $f_{zi}, f_{wi} = 1\,\forall\, i = 1, \ldots, 5$. The numerical values of the physical parameters used for the simulation are listed in Table 1.

By applying the tool presented here to this lumped system, an interconnected system is created. Both lumped and interconnected systems are simulated. The resulting displacement and angle of the last node are shown in Fig 3 and 4. It is observed that the error between lumped and interconnected system remain below the numerical precision, which shows that the decomposition retains all the dynamics of the system. Note that each subsystem has only 8 states, as compared to a total of 40 states.

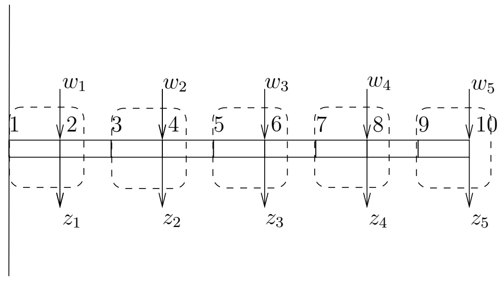

Fig. 2. Beam cantilever

Table 1. Physical parameters of the cantilever example

Field	Values
Area of crossection	$0.1963\times10^{-4}\ m^2$
Length	$1\ m$
Length of element	$0.1\ m$
Modulus of Elasticity	$207\times10^9\ \frac{N}{m^2}$

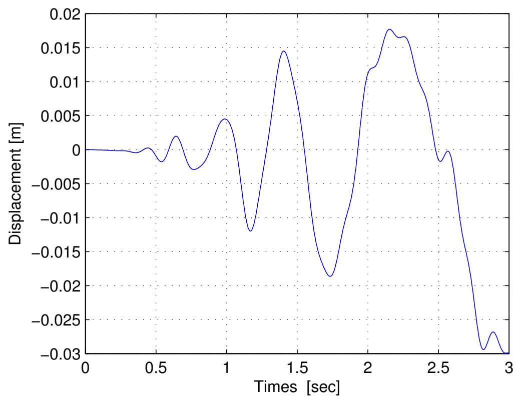

Fig. 3. Displacement (d) of the 10^{th} node

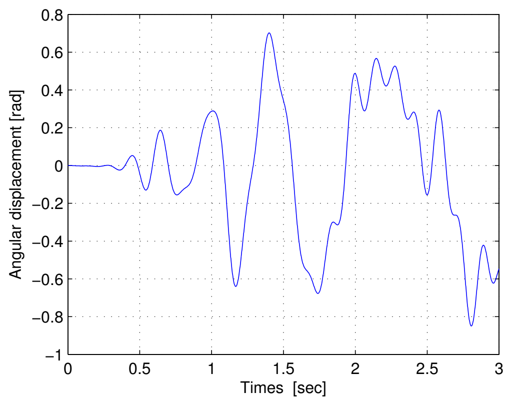

Fig. 4. Angular displacement (θ) of the 10^{th} node

5. CONTROLLER SYNTHESIS

The active vibration control of a cantilever can be considered as an input disturbance rejection problem as shown in Figure 5, where, W_1 and W_2 are the weighting filters. Let the state space of the weighted generalized plant be given as,

$$\begin{aligned} \dot{x} &= Ax + Bw + Bu \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{18}$$

where, d,z_1,z_2 are the disturbance,inputs and outputs acting on every node. In the cantilever example we have considered state feedback case and d include both force and torque disturbances acting on each node. Using the

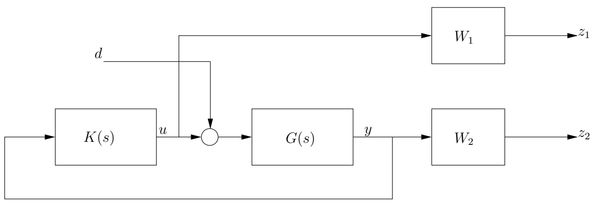

Fig. 5. Generalized plant used for loopshaping.

approach presented in section 2 the lumped system (18) is decomposed, where dynamics of the i^{th} subsystem can be written as

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + B_{1i}w_i(t) + B_{2i}u_i(t) + \sum_{j=1,j\neq i}^{N} A_{ij}x_j(t) \\ z_i(t) &= C_{1i}x_i(t) + D_{12}u_i(t) \\ y_i(t) &= C_{2i}x_i(t) + D_{21i}w_i(t) \\ &\forall i = 1, 2, ...N \end{aligned} \tag{19}$$

$x_i \in R^{n_i}$, $w_i \in R^{r_i}$, $u_i \in R^{m_i}$, $z_i \in R^{(l_i+m_i)}$ and $y_i \in R^{l_i}$ are the states, disturbance input , controlled input, controlled output and measured output of the i^{th} subsystem. The matrices A_{ii} , B_{1i} , B_{2i} , C_{1i} , C_{2i} , D_{12i} and D_{21i} are of appropriate dimensions found by *decomp*().

A strictly proper i^{th} output feedback controller is defined by

$$\begin{aligned} \dot{x}_{ci} &= A_{ci}x_{ci} + B_{ci}y_i \\ u_i &= C_{ci}x_{ci} \\ &\forall i = 1, 2, \ldots, N \end{aligned} \tag{20}$$

¿From (19) and (20) the closed loop system from w to z is given by

$$\begin{aligned} \dot{x}_{cl} &= A_{cl}x_{cl} + B_{cl}w \\ z &= C_{cl}x_{cl} \end{aligned} \tag{21}$$

where

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A & B_2C_c \\ B_cC_2 & A_c \end{bmatrix} \qquad B_{cl} = \begin{bmatrix} B_1 \\ B_cD_{21} \end{bmatrix} \\ C_{cl} &= [C_1 \;\; D_{12}C_c] \end{aligned} \tag{22}$$

Decentralized control of the system (18) requires that $A_c = diag\{A_{c1}, \ldots A_{cN}\}$, $B_c = diag\{B_{c1}, \ldots B_{cN}\}$, $C_c = diag\{C_{c1}, \ldots, C_{cN}\}$.

In order to obtain a decentralized control law one needs to impose structural constraints on the bounded real lemma. In (Chen et al., 2005) it has been proposed that for a given constant $\gamma > 0$, the system (18) is stabilizable with the disturbance attenuation level γ by a decentralized controller (20), if there exist positive definite block diagonal matrices X, Y and block-diagonal matrices F, L, Q such that $T < 0$, where

$$T = \begin{bmatrix} J_{11} & J_{21}^T & B_1 & XC_1 \\ J_{21} & J_{22} & YB_1 + LD_{21} & C_1^T \\ B_1^T & B_1^TY + D_{21}^TL^T & -\gamma I & 0 \\ C_1X + D_{12}F & C_1 & 0 & -\gamma I \end{bmatrix} \tag{23}$$

and

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0$$

Positive definite block diagonal matrices X and Y and diagonal matrices F , L , Q are obtained by solving the above bilinear matrix inequality (BMI). Block diagonal coefficient matrices are then given by

$$A_c = V^{-1}QU^{-T}, \quad B_c = V^{-1}L, \quad C_c = FU^{-T} \tag{24}$$

where

$$\begin{aligned} J_{11} &= AX + XA^T + B_2F + F^TB^T, \\ J_{21} &= A^T + YAX + LC_2X + YB_2F + Q, \\ J_{22} &= YA + A^TY + LC_2 + C_2^TL^T. \end{aligned}$$

The size of sub-matrices in the block diagonal are compatible with the dimensions of the state, input, and output vectors of the subsystems. The approach presented in (Chen et al., 2005) involves fixing a group of variables to convert it into an LMI. First the variables L, and Y and then the variables X, F are fixed. Solving these two LMIs alternately is then a way of finding a solution. In the proposed approach the path from centralized controller to decentralized controller is divided into M steps and the structural constraints are gradually applied, by defining a real number λ such that $\lambda = \frac{k}{M}$, where k is gradually increased from 0 to M. This defines a matrix function H as

$$H(X, Y, F, L, Q, \lambda) = T(X, Y, F, L, (1-\lambda)Q_F + \lambda Q) < 0$$

This is same as (23) except that J_{21} is replaced by $A^T + YAX + LC_2X + YB_2F + (1-\lambda)Q_F + \lambda Q$. Then

$$H(X, Y, F, L, Q, \lambda) = \begin{cases} T(X, Y, F, L, Q_F), \; \lambda = 0 \\ T(X, Y, F, L, Q), \;\; \lambda = 1 \end{cases}$$

By using the approach of (Chen et al., 2005) a state feedback decentralized controller for the active vibration control of flexible beam is designed for $W_1 = 1$ and $W_2 = 0.01$. The frequency domain closed-loop response between the force input, linear displacement and the force disturbance for the last node are as shown in Figure 6 and 7.

These figures show that the open loop system will exhibit an oscillatory response whereas in the closed loop system the disturbance is rejected, while the control gain remains in reasonable limits. One point which must be kept in mind is that for simplicity these frequency domain results are generated from the state space representation without imposing the appropriate boundary conditions.

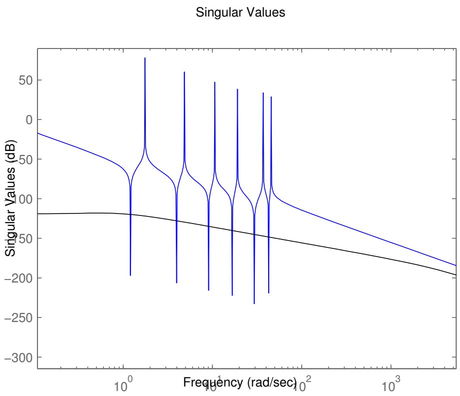

Fig. 6. Frequency response of the open-loop system (upper curve) and the closed-loop system (lower curve)

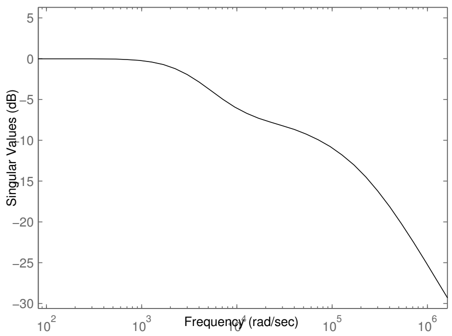

Fig. 7. Control sensitivity

6. CONCLUSION

The main idea presented in this paper is that instead of reducing the model order of a high order large system one can decompose the overall system into small subsystems, where each of these subsystems interacts with its neighboring elements. This approach is suitable for FEM generated models since these models are often of very high order and are generated from a combination of small elements. A simple yet efficient tool is presented here to carry out such a decomposition. This allows the application of analysis and synthesis tools for large interconnected systems, and enables the designer to synthesize controllers for such systems in a systematic way.

As an illustrative example, the tool presented in this paper is applied to a cantilever model. Results shows that the time responses of the original system and the one decomposed into interconnected form give identical results, up to numerical precision. Furthermore a previously presented approach of designing decentralized controller is applied to the decomposed system to demonstrate that by using the decomposition one can systematically design a distributed control strategy for FEM-generated systems without reducing the model order.

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